Computer Science

COMPSCI 372 S2 C – Exercise Sheet 4
Sample Solution

Q1: Let $\mathbf{M} = \begin{pmatrix} 1 & 5 & 3 \\ 3 & 0 & 2 \end{pmatrix}$, $\mathbf{v}=\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\mathbf{u}=\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Answers:

1. $5\mathbf{v} - 3\mathbf{u} = 5\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 5-6 \\ 15-0 \end{pmatrix} = \begin{pmatrix} -1 \\ 15 \end{pmatrix}$

2. $\mathbf{v} \bullet \mathbf{u} = \mathbf{v}^T\mathbf{u} = \begin{pmatrix} 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 1*2 + 3*0 = 2$

3. $\mathbf{u}\mathbf{v}^T = \begin{pmatrix} 2 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 3 \end{pmatrix} = \begin{pmatrix} 2*1 & 2*3 \\ 0*1 & 0*3 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 0 & 0 \end{pmatrix}$

4. $|\mathbf{v}| = \sqrt{1^2 + 3^2} = \sqrt{10}$

5. $\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \left(|\hat{\mathbf{v}}| = \sqrt{\left(\frac{1}{\sqrt{10}}\right)^2 + \left(\frac{3}{\sqrt{10}}\right)^2} = \sqrt{\frac{1}{10} + \frac{9}{10}} = 1 \right)$

6. $\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 3 \\ 5 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 5 & 3 \\ 3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1*1+3*3 & 1*5+3*0 & 1*3+3*2 \\ 5*1+0*3 & 5*5+0*0 & 5*3+0*2 \\ 3*1+2*3 & 3*5+2*0 & 3*3+2*2 \end{pmatrix} = \begin{pmatrix} 10 & 5 & 9 \\ 5 & 25 & 15 \\ 9 & 15 & 13 \end{pmatrix}$

7. $\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 5 & 3 \\ 3 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 5 & 0 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1*1+5*5+3*3 & 1*3+5*0+3*2 \\ 3*1+0*5+2*3 & 3*3+0*0+2*2 \end{pmatrix} = \begin{pmatrix} 35 & 9 \\ 9 & 13 \end{pmatrix}$

8. $\cos\varphi = \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{2}{2*\sqrt{10}} = \sqrt{0.1} \quad \Rightarrow \quad \varphi = \cos^{-1}\sqrt{0.1} = 71.57^{\circ}$

9. The x-axis has the direction $w=(1,0)^T$:

$$\cos\varphi = \frac{\mathbf{w} \bullet \mathbf{v}}{|\mathbf{w}||\mathbf{v}|} = \frac{1}{1*\sqrt{10}} = \sqrt{0.1} \quad \Rightarrow \quad \varphi = \cos^{-1}\sqrt{0.1} = 71.57^{\circ}$$

Q2: Given a matrix $\mathbf{M}$ the matrix $\mathbf{M}^{-1}$ is called the inverse of $\mathbf{M}$ if and only if $\mathbf{M}^{-1}\mathbf{M} = \mathbf{M}\mathbf{M}^{-1}=\mathbf{I}$ where $\mathbf{I}$ is the identity matrix (i.e. $\mathbf{I}$ is the matrix where all diagonal elements are 1 and all off-diagonal elements are zero).

For $\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ the inverse is computed by $\mathbf{M}^{-1} = \frac{1}{|\mathbf{M}|}\begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$

(a) Show that $\mathbf{M}^{-1}\mathbf{M}=\mathbf{M}\mathbf{M}^{-1}=\mathbf{I}$.

$$\mathbf{M}\mathbf{M}^{-1} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\frac{1}{|\mathbf{M}|}\begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} = \frac{1}{m_{11}m_{22}-m_{12}m_{21}}\begin{pmatrix} m_{11}m_{22}-m_{12}m_{21} & -m_{11}m_{12}+m_{12}m_{11} \\ m_{21}m_{22}-m_{22}m_{21} & -m_{12}m_{21}+m_{11}m_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{M}^{-1}\mathbf{M} = \frac{1}{|\mathbf{M}|}\begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \frac{1}{m_{11}m_{22}-m_{12}m_{21}}\begin{pmatrix} m_{22}m_{11}-m_{12}m_{21} & m_{22}m_{12}-m_{12}m_{22} \\ -m_{21}m_{11}+m_{11}m_{21} & -m_{21}m_{12}+m_{11}m_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) Let $\mathbf{S} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$, $\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\mathbf{H} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

Compute

1. $\mathbf{S}^{-1} = \frac{1}{2*5}\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/5 \end{pmatrix}$

   i.e. the inverse of a scale matrix is a matrix which scales by the reciprocal values.

2. $\mathbf{R}^{-1} = \frac{1}{\cos\theta\cos\theta+\sin\theta\sin\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$

   since $\cos^2\theta+\sin^2\theta = 1, \cos\theta=\cos(-\theta)$ and $\sin\theta=-\sin(-\theta)$.

   i.e. the inverse of a rotation matrix is a matrix which performs a rotation by the same angle in the opposite direction.

3. $\mathbf{H}^{-1} = \frac{1}{1*1}\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$

   i.e. the inverse of a shear matrix is a matrix which shears by the corresponding negative values.

Q3: Given are two vectors $\mathbf{a}=\begin{pmatrix}5\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}1\\3\end{pmatrix}$ with a common origin. Find all vectors orthogonal to **a**. Decompose **b** into two components $\mathbf{b}_a$ *and* $\mathbf{b}_{a^\perp}$ parallel and perpendicular to **a**, respectively.

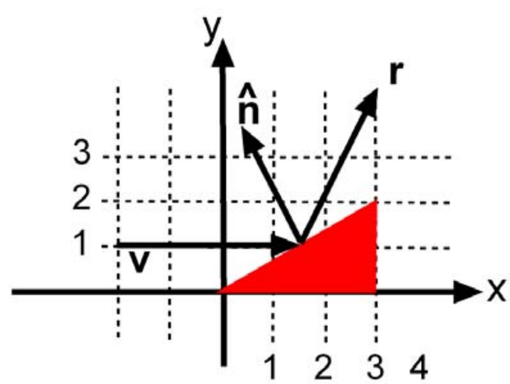
$$\mathbf{b}_a = \frac{\mathbf{b}\bullet\mathbf{a}}{\mathbf{a}\bullet\mathbf{a}}\mathbf{a} = \frac{5+6}{25+4}\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}55/29\\22/29\end{pmatrix} \approx \begin{pmatrix}1.897\\0.759\end{pmatrix}$$

$$\mathbf{b}_{a^\perp} = \mathbf{b}-\mathbf{b}_a = \begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}55/29\\22/29\end{pmatrix} = \begin{pmatrix}-26/29\\65/29\end{pmatrix} \approx \begin{pmatrix}-0.897\\2.241\end{pmatrix}$$

All vectors orthogonal to **a** are given by: $\mathbf{v}_{\text{orthogonal\_to\_}\mathbf{a}} = k\begin{pmatrix}-2\\5\end{pmatrix}$ where $k \neq 0$

Q4: Given is a triangle with the corners $(0,0)^T$, $(3,0)^T$ and $(3,2)^T$ made out of a reflective material. A light ray originates at the point $(-2,1)^T$ and travels in the direction $(1,0)^T$. Compute the direction of the light ray after hitting the triangle.

Drawing a diagram of the scene reveals that the light ray first hits the triangle edge connecting the vertices $(0,0)^T$ and $(3,2)^T$ (alternatively we could test all edges in order to find out which edge is hit first)



The hit edge of the triangle has the direction $\mathbf{d}=\begin{pmatrix}3\\2\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$

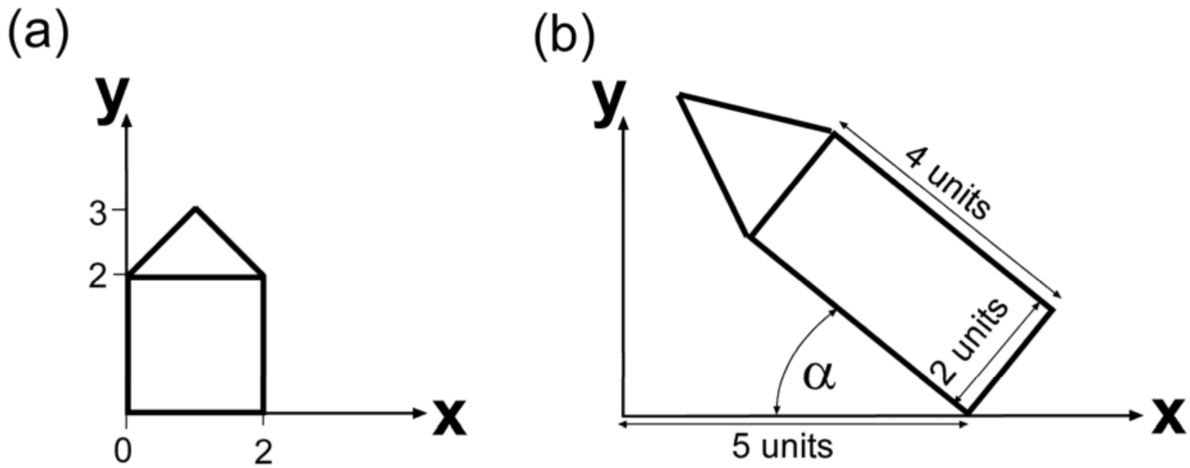
Hence its normal is $\mathbf{n}=\begin{pmatrix}-2\\3\end{pmatrix}$ and its unit normal is $\hat{\mathbf{n}}=\begin{pmatrix}-2/\sqrt{13}\\3/\sqrt{13}\end{pmatrix}$

The direction of the reflected ray is therefore

$$\mathbf{r}=\mathbf{v}-2(\mathbf{v}\bullet\hat{\mathbf{n}})\,\hat{\mathbf{n}} = \begin{pmatrix}1\\0\end{pmatrix} - 2\left(1*\frac{-2}{\sqrt{13}}+0*\frac{3}{\sqrt{13}}\right)\begin{pmatrix}-2/\sqrt{13}\\3/\sqrt{13}\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}+\frac{4}{\sqrt{13}}\begin{pmatrix}-2/\sqrt{13}\\3/\sqrt{13}\end{pmatrix} = \begin{pmatrix}5/13\\12/13\end{pmatrix}$$

Solution to example 3 (slide 36) from chapter 5 of the lecture notes

Given is the 2D scene in part (a) of the image below. Write down the homogeneous 2D transformation matrix **M**, which transforms the object shown in (a) into the object in part (b) of the image. You are allowed to write the transformation matrix as a product of simpler matrices (i.e. you are not required to multiply the matrices).



Answer: In order to get from (a) to (b) we first have to scale the object by a factor of 2 in y-direction, then rotate it by (90-α) in anticlockwise direction, and then translate it by 5 units in x-direction. Hence the final transformation matrix is:

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(90-\alpha) & -\sin(90-\alpha) & 0 \\ \sin(90-\alpha) & \cos(90-\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$