5. 2D Geometry

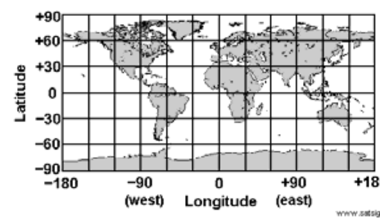
In order to design and render complex scenes we require techniques for transforming points and vectors. Points are used to represent OpenGL primitives (`glVertex`) and vectors are used to represent surface normals (necessary for computing the illumination at a point).

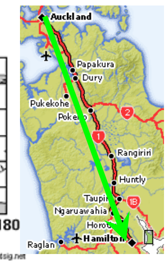
5.1. Points and Vectors

- A ***point*** is a position in space, e.g. Auckland
- A ***vector*** represents a displacement – a *difference* between two points.
- The only way to represent a point is with reference to the origin of a coordinate system. The vector from the origin of the coordinate system to the point is the ***position vector*** of the point.

Example: Describe where Hamilton is!

- 120km to the south-south-west of Auckland
- 37.43S Latitude, 175.19E Longitude





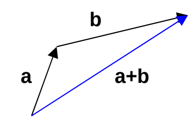
Points and Vectors (cont'd)

- Vectors are represented as 2-tuples (2D) or 3-tuples (3D) in a coordinate system.
- We denote the components of a vector **v** in 2D with v_1 and v_2 and of a vector **u** in 3D with u_1, u_2 and u_3:
- We denote vectors with small bold letters and points with capital letters, e.g. **p** is the position vector of the point P.

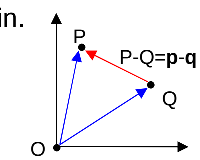
$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

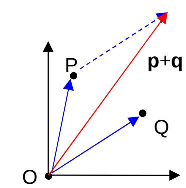
Points and Vectors (cont'd)

- Operations on vectors
  - Can add, subtract and scale vectors



- Operations on points
  - Subtracting one point from another gives a vector (the displacement between these points)
  - Can **NOT** add two points (what is Auckland + Hamilton??)
  - But we can add and subtract their position vectors w.r.t. some origin.





Basic Operations on Vectors

- Addition
  - Represents the combined displacement
  - Implement by adding components

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}u_1\\u_2\end{pmatrix}+\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}u_1+v_1\\u_2+v_2\end{pmatrix}$$

- Scaling
  - i.e. multiplication by a scalar
  - Defined such that **v** + **v** = 2**v**
  - Implement by multiplying all components by the scalar.

$$s\ \mathbf{u}=s\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}s\ u_1\\s\ u_2\end{pmatrix}$$

- Subtraction
  - Addition of a negated vector, i.e. one in opposite direction.
  - Implement by subtracting components.

$$\mathbf{u}-\mathbf{v}=\begin{pmatrix}u_1\\u_2\end{pmatrix}-\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}u_1-v_1\\u_2-v_2\end{pmatrix}$$

- The magnitude of a vector
  - i.e. its "length" (2-norm).

$$|\mathbf{u}|=\sqrt{u_1^{\ 2}+u_2^{\ 2}}\quad,\ |s\mathbf{u}|=|s||\mathbf{u}|$$

- Normalisation
  - The process of creating a unit vector (length 1)
  - Scale by reciprocal of magnitude:

$$\hat{\mathbf{u}}=\frac{\mathbf{u}}{|\mathbf{u}|}$$

Basic Operations on Matrices

- Dimension of a matrix
  - A $m\times n$ matrix is a matrix has m rows and n columns.

$$\mathbf{R}=\begin{pmatrix}r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\end{pmatrix}$$

row

column

Example of a 2 ×3 matrix

- Addition/Subtraction
  - Implement by adding/subtracting components.

$$\mathbf{M}\pm\mathbf{N}=\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}\pm\begin{pmatrix}n_{11} & n_{12}\\ n_{21} & n_{22}\end{pmatrix}=\begin{pmatrix}m_{11}\pm n_{11} & m_{12}\pm n_{12}\\ m_{21}\pm n_{21} & m_{22}\pm n_{22}\end{pmatrix}$$

- Scaling
  - Implement by multiplying all components by the scalar.

$$s\mathbf{M}=\begin{pmatrix}s\ m_{11} & s\ m_{12}\\ s\ m_{21} & s\ m_{22}\end{pmatrix}$$

Basic Operations on Matrices (cont'd)

- Transpose (indicated by ^{T}) of a matrix **M**
  - Swap m_{ij} and m_{ji} for all i,j.

$$\mathbf{M}=\begin{pmatrix}m_{11} & m_{12} & m_{13}\\ m_{21} & m_{22} & m_{23}\end{pmatrix}\Rightarrow\mathbf{M}^T=\begin{pmatrix}m_{11} & m_{21}\\ m_{12} & m_{22}\\ m_{13} & m_{23}\end{pmatrix}$$

Algebraic rules for transposition:

$(\mathbf{M}^T)^T=\mathbf{M}$

$(s\mathbf{M})^T=s(\mathbf{M}^T)$

$(\mathbf{M}+\mathbf{N})^T=\mathbf{M}^T+\mathbf{N}^T$

$(\mathbf{MN})^T=\mathbf{N}^T\mathbf{M}^T$

Basic Operations on Matrices (cont'd)

- Determinant *det* **M** (also written | **M** |) of a matrix **M**
  - For a 2x2 matrix:

$$\begin{vmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{vmatrix}=m_{11}m_{22}-m_{12}m_{21}$$

- Inverse $\mathbf{M}^{-1}$ of a matrix **M**
  - For a 2x2 matrix:

$$\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}^{-1}=\frac{1}{m_{11}m_{22}-m_{12}m_{21}}\begin{pmatrix}m_{22} & -m_{12}\\ -m_{21} & m_{11}\end{pmatrix}$$

Exercise: Prove that $\mathbf{M}^{-1}$ is the inverse of **M**, i.e. show $\mathbf{M}^{-1}\mathbf{M}=\mathbf{M}\mathbf{M}^{-1}=\mathbf{I}$

Basic Operations on Vectors and Matrices

- The transpose of a vector
  - Transpose of a row vector is a column vector and vice versa

$$\mathbf{u}=\begin{pmatrix}u_1\\u_2\end{pmatrix}\Rightarrow\mathbf{u}^T=\begin{pmatrix}u_1 & u_2\end{pmatrix}$$

- The dot product (scalar product)
  - Symmetry: **a•b=b•a**
  - Linearity: **(a+b)•c=a•c+b•c**
  - Homogeneity: (s**a**)•**b**=s(**a•b**)
  - $|b|^2$= **b•b**

$$\mathbf{u}\bullet\mathbf{v}=\begin{pmatrix}u_1\\u_2\end{pmatrix}\bullet\begin{pmatrix}v_1\\v_2\end{pmatrix}=u_1v_1+u_2v_2=\mathbf{u}^T\mathbf{v}$$

- Matrix multiplication
  - Multiplying an $l \times m$ and $m \times n$ matrix gives an $l \times n$ matrix with the elements $a_{ij}=b_{i1}c_{1j}+\ldots+b_{im}c_{mj}=\sum_{k=1}^{m}b_{ik}c_{kj}$ [Note: a_{ij}= row_i•column_j]

$$\mathbf{A}=\mathbf{B}\,\mathbf{C}=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}=\begin{pmatrix}b_{11}c_{11}+b_{12}c_{21} & b_{11}c_{12}+b_{12}c_{22}\\ b_{21}c_{11}+b_{22}c_{21} & b_{21}c_{12}+b_{22}c_{22}\end{pmatrix}$$

5.2 Applications of the Scalar Product (Dot Product)

- Angle between two vectors
- Projection of a vector
- Distance of a point to a line
- Reflections
- Area of a triangle

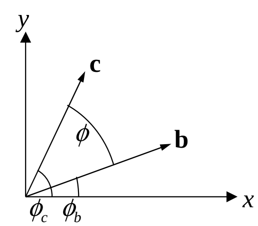
The Angle between Two Vectors

- The most important application of the dot product is to find the angle between two vectors (or two intersecting lines).

$$\mathbf{b}=\begin{pmatrix}|b|\cos\phi_b\\|b|\sin\phi_b\end{pmatrix},\quad \mathbf{c}=\begin{pmatrix}|c|\cos\phi_c\\|c|\sin\phi_c\end{pmatrix}$$

hence

$$\begin{aligned}\mathbf{b}\bullet\mathbf{c}&=|b||c|\cos\phi_b\cos\phi_c+|b||c|\sin\phi_b\sin\phi_c\\&=|b||c|\cos(\phi_c-\phi_b)\\&=|b||c|\cos\varphi\end{aligned}$$



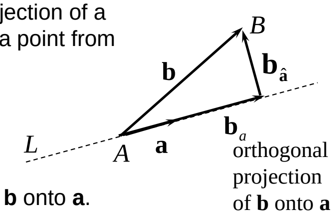
Two non-zero vectors **b** and **c** with common start point are

less than	90^o apart	if **b•c**>0	
exactly	90^o apart	if **b•c**=0	[**b** and **c** are *orthogonal* (*perpendicular*)]
more than	90^o apart	if **b•c**<0	

Projection of a Vector

- In many applications we must compute the projection of a vector onto another vector and the distance of a point from a line:

Let L be a line through A in the direction of **a**.
Let **b** be the vector from A to a point B.



We want to find $\mathbf{b}_\mathbf{a}$ the orthogonal projection of **b** onto **a**.

$$\mathbf{b}=\mathbf{b}_\mathbf{a}+\mathbf{b}_{\hat{\mathbf{a}}}=k\mathbf{a}+\mathbf{b}_{\hat{\mathbf{a}}}\text{ for some }k$$

$$\mathbf{b}\bullet\mathbf{a}=(k\mathbf{a}+\mathbf{b}_{\hat{\mathbf{a}}})\bullet\mathbf{a}=k\mathbf{a}\bullet\mathbf{a}+\mathbf{b}_{\hat{\mathbf{a}}}\bullet\mathbf{a}=k(\mathbf{a}\bullet\mathbf{a})$$

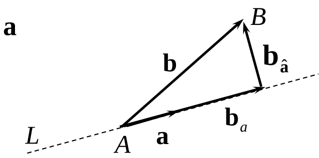
Note: at this point we don't know the value for $\mathbf{b}_{\hat{\mathbf{a}}}=\mathbf{b}-\mathbf{b}_\mathbf{a}$ but we know it is perpendicular to **a**.

$$\Rightarrow k=\frac{\mathbf{b}\bullet\mathbf{a}}{\mathbf{a}\bullet\mathbf{a}}$$

$$\Rightarrow \mathbf{b}_\mathbf{a}=\frac{\mathbf{b}\bullet\mathbf{a}}{\mathbf{a}\bullet\mathbf{a}}\mathbf{a}$$

Distance of a Point to a Line

- In the previous slide we computed $\mathbf{b}_{\mathbf{a}} = \dfrac{\mathbf{b} \bullet \mathbf{a}}{\mathbf{a} \bullet \mathbf{a}} \mathbf{a}$



hence the distance of B to the line L is

$$\left|\mathbf{b}_{\hat{\mathbf{a}}}\right| = \left|\mathbf{b} - \mathbf{b}_{\mathbf{a}}\right| = \left|\mathbf{b} - \frac{\mathbf{b} \bullet \mathbf{a}}{\mathbf{a} \bullet \mathbf{a}} \mathbf{a}\right|$$

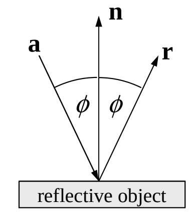
Reflections

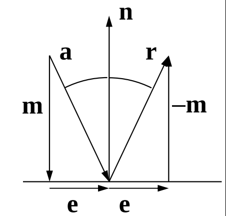
- Ray tracing is a popular rendering algorithm which displays a scene by tracing rays from the eye through each pixel of the screen into the scene (i.e. trace a light ray hitting the eye backwards → see 2nd part of this course!). If the scene contains reflective objects such as mirrors it is necessary to compute for a ray with direction **a** its reflection **r**.
  Let **n** be the surface normal at the point where the ray hits the object:

$$\mathbf{m} = \frac{\mathbf{a} \bullet \mathbf{n}}{\mathbf{n} \bullet \mathbf{n}} \mathbf{n} = \frac{\mathbf{a} \bullet \mathbf{n}}{|\mathbf{n}|^2} \mathbf{n} = (\mathbf{a} \bullet \hat{\mathbf{n}}) \hat{\mathbf{n}}$$

$$\Rightarrow \mathbf{r} = \mathbf{e} - \mathbf{m} = (\mathbf{a} - \mathbf{m}) - \mathbf{m} = \mathbf{a} - 2\mathbf{m}$$

$$= \mathbf{a} - 2(\mathbf{a} \bullet \hat{\mathbf{n}}) \hat{\mathbf{n}}$$

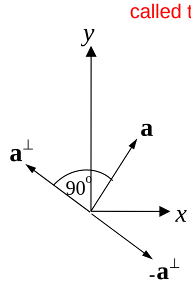




Orthogonal Vectors in 2D

- Let **a**=$(a_1, a_2)^T$ then in 2D we can find two vectors perpendicular to it
  - **a**$^\perp$=$(-a_2, a_1)^T$ (note **a**$^\perp$**• a=0**)
  - **-a**$^\perp$=$(a_2, -a_1)^T$ (note **-a**$^\perp$**• a=0**)

In the textbook **a**$^\perp$ is called the "Perp" vector



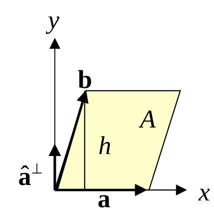
The Area of a Triangle (in 2D)

- The area of a parallelogram:

$$A = |\mathbf{a}| h$$

$$h = \left|\mathbf{b}_{\mathbf{a}^\perp}\right| = \left|\frac{\mathbf{a}^\perp \bullet \mathbf{b}}{\mathbf{a}^\perp \bullet \mathbf{a}^\perp} \mathbf{a}^\perp\right| = \frac{\left|\mathbf{a}^\perp \bullet \mathbf{b}\right|}{|\mathbf{a}|}$$

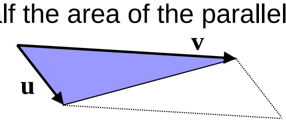
$$A = |\mathbf{a}| h = |\mathbf{a}| \frac{\left|\mathbf{a}^\perp \bullet \mathbf{b}\right|}{|\mathbf{a}|} = \left|\mathbf{a}^\perp \bullet \mathbf{b}\right|$$



NOTE: These formulas are only valid in 2D!!

Area of a triangle:

- Area is half the area of the parallelogram formed by two of its edges

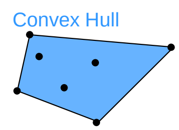


$$A = \frac{1}{2} \left|\mathbf{u}^\perp \bullet \mathbf{v}\right|$$

5.3 Convex and Concave Objects

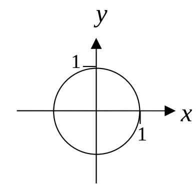
- A *Convex Polygon* is a polygon where any line connecting any pair of vertices lies entirely within the polygon (this is equivalent with: all interior angles between neighbouring edges are smaller or equal to 180 degree). If a polygon is not convex it is called concave.

- The *Convex Hull* of a set of points is the smallest convex set containing the points. [i.e. smallest convex polygon containing the points]



5.4 Implicit Curves

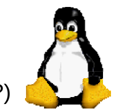
- Implicit curves
  - A 2D curve can be defined as the set of points $p=(x,y)^T$ fulfilling the mathematical equation $f(x,y)=0$.



Example:

$x^2+y^2-1=0$

defines a unit circle centred at the origin



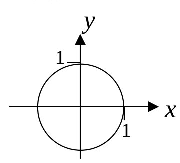
Disadvantages:
- Modelling is non-intuitive (e.g. how to draw a penguin?)
- Difficult to draw: have to find a set of points fulfilling the equation (hard!) and connect them by line segments.

Advantages:
- Easy to compute normal **n** at a point $(x_0,y_0)^T$:

$$\mathbf{n} = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right)^T \Bigg|_{(x_0,y_y)}$$

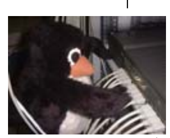
5.5 Parametric Curves

- Parametric curves
  - A 2D curve defined by a set of points $\mathbf{p}(t)=(x(t), y(t))^T$ where $x(t)$ and $y(t)$ are functions of the parameter t (often called the "speed").
  - Have to specify the parameter interval $[t_{min}, t_{max}]$ for t. It's a good idea to specify curve such that $[t_{min}, t_{max}]=[0,1]$.



Example: $p(t) = \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix}$, $t \in [0,1]$

defines a unit circle centred at the origin.



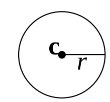
Disadvantages:
- Modelling is still non-intuitive (e.g. how to draw a penguin?)

Advantages
- Can compute tangent at a point by the derivative of the components $\mathbf{p}'(t) = \left(\frac{dx}{dt} \quad \frac{dy}{dt} \right)^T$
- Easy to "splice" curve segments together.
- Can draw curve by computing points on the curve and connecting them by line segments.

Parametric Curves (cont'd)

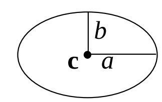
- Examples for parametric curves (in all cases $t \in [0,1]$):

Curve with centre **c** and radius r:



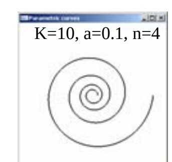
$$p(t) = \begin{pmatrix} c_1 + r\cos(2\pi t) \\ c_2 + r\sin(2\pi t) \end{pmatrix}$$

Ellipsoid with centre **c,** long axis a and short axis b:



$$p(t) = \begin{pmatrix} c_1 + a\cos(2\pi t) \\ c_2 + b\sin(2\pi t) \end{pmatrix}$$

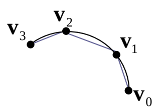
Logarithmic spiral with centre **c** and n revolutions:



$$p(t) = \begin{pmatrix} c_1 + f(\theta)\cos(\theta) \\ c_2 + f(\theta)\sin(\theta) \end{pmatrix} \text{ where } f(\theta) = Ke^{a\theta} \text{ , } \theta = 2\pi nt$$

Parametric Curves (cont'd)

- How to draw a parametric curve?
  - Compute (n+1) points $\mathbf{v}_i = \mathbf{p}\left(\frac{i}{n}\right)$ for $i = 0,...,n$ on the curve
  - Connect the points with line segments



G:\...\ParametricCurve.h

```
// A virtual class for a 2D parametric curve
// No instances of this class can be constructed
class CParametricCurve
{
public:
    CParametricCurve(){}
    virtual ~CParametricCurve(){}
    void draw();            // draws curve as a line strip
    virtual void computePointOnCurve(float t, float& x, float& y)=0;
                            // computes curve point p(t)=(x(t), y(t))
protected:
    float vertices[n+1][2];
    void init();            // computes n vertices on the curve
};
```


Parametric Curves (cont'd)

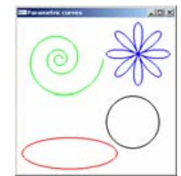
G:\...\ParametricCurve.h

```
class CParametricCircle:public CParametricCurve
{
public:
    CParametricCircle();
    CParametricCircle(float centreX, float centreY, float radius);
    virtual ~CParametricCircle(){}
    void computePointOnCurve(float t, float& x, float& y);
private:
    float cx,cy;    // x-coordinate and y-coordinate of the centre
    float r;        // radius
};
```

```
void CParametricCurve::init(){              // compute (n+1) points on the curve
  for(int i=0;i<=n;i++)
    computePointOnCurve((float) i/(float) n,vertices[i][0],vertices[i][1]);}

void CParametricCurve::draw(){              // draw line segments
  glBegin(GL_LINE_STRIP);
    for(int i=0;i<=n;i++) glVertex2fv(vertices[i]);
  glEnd();}

void CParametricCircle::computePointOnCurve(float t, float& x, float& y){
  x=cx+r*cos(t*2.0*PI);
  y=cy+r*sin(t*2.0*PI);}
```



5.6 2D Affine Transformations

Affine Transformations transform a pair of parallel straight lines to another pair of parallel straight lines and preserve ratios of distances. Assume for now that the transformations apply only to *points* but with an origin and an underlying vector space defined.

Examples of affine transformations:

- Scaling about Origin
  - For any point **p**= (p_1, p_2)^T, scale p_1 by factor s_1, p_2 by factor s_2.

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

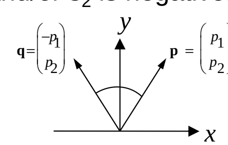
i.e. $\mathbf{q} = \mathbf{M}_{scale}\mathbf{p}$

where $\mathbf{M}_{scale} = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$

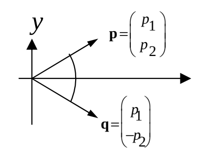
- Translation ("movement")
  - Add a vector **t** to all points in the scene, i.e. **q** = **p** + **t**

2D affine transformations (cont'd)

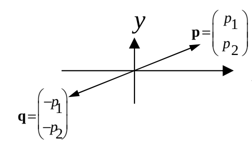
- Note that "scaling" includes "reflection" if s_1 and/or s_2 is negative:
  - Reflection at the y-axis: $\mathbf{q} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{p}$



  - Reflection at the x-axis $\mathbf{q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{p}$

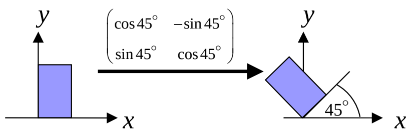


  - Reflection at the origin $\mathbf{q} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{p}$



2D affine transformations (cont'd)

- Rotation about origin by an angle θ (right-handed i.e. anticlockwise)
  - Proof: UDOO

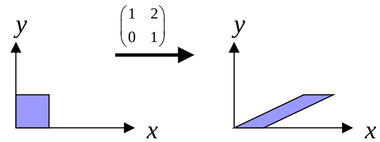


$$\begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix}$$

$$\mathbf{q} = \mathbf{M}_{rotate}\mathbf{p}$$

$$\text{where } \mathbf{M}_{rotate} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- Shearing



$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{q} = \mathbf{M}_{shear}\mathbf{p}$$

$$\text{where } \mathbf{M}_{shear} = \begin{pmatrix} 1 & xShear \\ yShear & 1 \end{pmatrix}$$

Some properties of affine transformations (both 2D & 3D)

- Straight lines are preserved
- Parallel lines remain parallel
- Proportional distances are preserved
- Any closed *area* in 2D or *volume* in 3D is multiplied by | det M | (unchanged by translation)
- Any arbitrary affine transformation can be represented as a sequence of shearing, scaling, rotation and translation
- **Affine transformations do not in general commute (i.e. $\mathbf{T_1T_2 \neq T_2T_1}$)**
- Transformations *are* associative, i.e. $\mathbf{T_1(T_2T_3) = (T_1T_2)\,T_3}$

5.7 2D Homogeneous Coordinates

Translation is a nuisance - don't have a matrix representation for it.

So we introduce *homogeneous coordinates* as a way of "unifying" the representation of translation with the other transformations.

- The idea
- Geometric interpretation
- Converting from HC to ordinary coordinates
- Composition of transformations

The idea

Represent the ordinary 2D point $(x, y)^T$ as a homogeneous coordinate point $(x, y, 1)^T$.

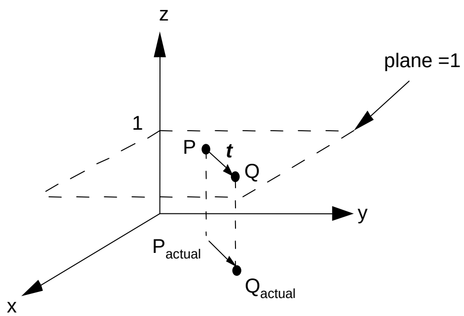
Then can do translation by:

$$\begin{pmatrix} q_x \\ q_y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

and the other transformations by

$$\begin{pmatrix} q_x \\ q_y \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$$

Geometric interpretation



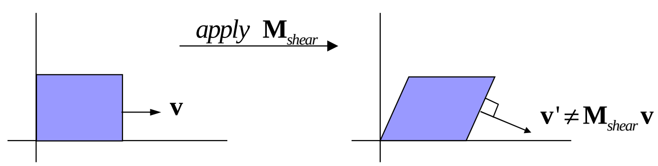
Can see that the translation in 2D is implemented as a shear in 3D.

Converting from HC to Ordinary Coordinates

- More generally, we regard all homogenous coordinate points $(w\,p_1, w\,p_2, w)^T$, $w \neq 0$, as representing the same ordinary coordinate point $(p_1, p_2)^T$.
- Hence, in general, the homogeneous coordinate point $(a,b,c)^T$ converts to the ordinary coordinate point $(a/c, b/c)^T$.
- With all the transformations so far, c will equal 1, but we will see a couple of examples later where this is not the case.

5.8 Notes & Examples

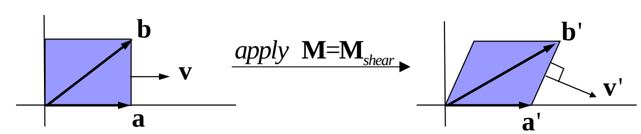
Be careful when transforming vectors. ***Doesn't work for position vectors or surface normals*** (i.e. vectors perpendicular to given surfaces).



$$\mathbf{v}=\begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{M}_{\text{shear}}=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix} \Rightarrow \mathbf{M}_{\text{shear}}\mathbf{v}=\begin{pmatrix}1\\0\end{pmatrix}\neq \mathbf{v}'$$

Transforming Vectors (cont'd)

- How do we find the transformed surface normal **v**' ?



$\mathbf{v}$ is perpendicular to $(\mathbf{b}-\mathbf{a}) \Leftrightarrow (\mathbf{b}-\mathbf{a})\bullet\mathbf{v}=0 \Leftrightarrow (\mathbf{b}-\mathbf{a})^T\mathbf{v}=0$

$\mathbf{v}'$ must be perpendicular to $(\mathbf{b}'-\mathbf{a}') \Leftrightarrow (\mathbf{b}'-\mathbf{a}')\bullet\mathbf{v}'=0 \Leftrightarrow (\mathbf{b}'-\mathbf{a}')^T\mathbf{v}'=0$

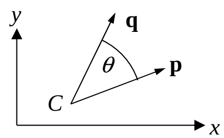
$(\mathbf{b}'-\mathbf{a}')^T\mathbf{v}'=(\mathbf{Mb}-\mathbf{Ma})^T\mathbf{v}'=\left(\mathbf{M}(\mathbf{b}-\mathbf{a})\right)^T\mathbf{v}'=(\mathbf{b}-\mathbf{a})^T\mathbf{M}^T\mathbf{v}'$

Choose $\mathbf{v}'=\left(\mathbf{M}^T\right)^{-1}\mathbf{v}$ then $(\mathbf{b}'-\mathbf{a}')^T\mathbf{v}'=(\mathbf{b}-\mathbf{a})^T\mathbf{M}^T\left(\mathbf{M}^T\right)^{-1}\mathbf{v}=(\mathbf{b}-\mathbf{a})\bullet\mathbf{v}=0$

$$\boxed{\mathbf{v}'=\left(\mathbf{M}^T\right)^{-1}\mathbf{v}=\left(\mathbf{M}^{-1}\right)^T\mathbf{v}\ \text{for any surface normal } \mathbf{v} \text{ and linear transf. } \mathbf{M}}$$

Composition of transformations

- With homogeneous coordinates, it's now much easier to compose multiple transformations into a single one.
- Consider for example the problem of rotating some object about its centre point C.
  - translate the whole object so that its centre is at the origin, rotate about the origin, and then translate back.
  - Hence, transformation is $(q_1 \quad q_2 \quad 1)^T = \mathbf{M}(p_1 \quad p_2 \quad 1)^T$



where

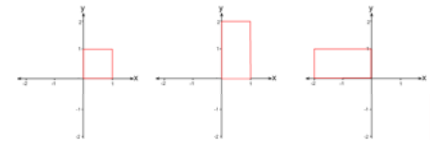
$$\mathbf{M} = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \\ 0 & 0 & 1 \end{pmatrix}$$

  - Can multiply the three component matrices to get the composite transformation matrix M, and then apply M to all points in the object.
  - UDOO: Work out **M** – show that it is equivalent to a rotation of θ followed by a single (different) translation.

Example 1

- In general affine transformations do not commute:
  - First scale by (1,2), then rotate 90º

$$\mathbf{M} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

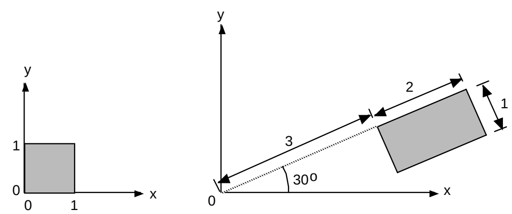


  - First rotate 90º then scale by (1,2)

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 2

- Find the homogenous coordinate transformation matrix that transforms the figure on the left to the figure on the right



- Often easier to do these backwards, then take inverse. In this case, starting with figure on right:
  - Rotate -30º , shift by (-3,1) , scale by (1/2, 1)
- Hence required transformation from right to left is:
  - R(30) T(3,-1) S(2,1)
  - Easy to convert to HC matrix expression [UDOO]

Example 3

- Given is the 2D scene in part (a) of the image below. Write down the **homogeneous 2D transformation matrix M**, which transforms the object shown in (a) into the object in part (b) of the image. You are allowed to write the transformation matrix as a product of simpler matrices (i.e. you are not required to multiply the matrices).