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Foundations of Artificial Intelligence

—Fuzzy Set Theory—

Hans W. Guesgen
Computer Science Department

THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

Fuzzy Sets

- A fuzzy subset $\tilde{A}$ of a domain D is a set of ordered pairs, $\langle d, \mu_{\tilde{A}}(d) \rangle$, where $d \in D$ and $\mu_{\tilde{A}} : D \rightarrow [0, 1]$ is the membership function of $\tilde{A}$.

- The membership function replaces the characteristic function of a classical subset $A \subseteq D$.

- If the range of $\mu_{\tilde{A}}$ is $\{0, 1\}$, $\tilde{A}$ is nonfuzzy and $\mu_{\tilde{A}}(d)$ is identical with the characteristic function of a nonfuzzy set.

Examples of Fuzzy Sets

- Real numbers considerably larger than 10:

$$\tilde{A} = \{\langle d, \mu_{\tilde{A}}(d)\rangle \mid d \in \Re\} \text{ with } \mu_{\tilde{A}}(d) = \begin{cases} 0 & \text{for } d \leq 10 \\ \frac{1}{1+\frac{1}{(d-10)^2}} & \text{for } d > 10 \end{cases}$$

- Real numbers close to 10:

$$\tilde{A} = \{\langle d, \mu_{\tilde{A}}(d)\rangle \mid d \in \Re\} \text{ with } \mu_{\tilde{A}}(d) = \frac{1}{1+(d-10)^2}$$

(Strong) α-Level Sets

- Let $\tilde{A}$ be a fuzzy subset in D, then the (crisp) set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α-level set of $\tilde{A}$:

$$A_\alpha = \{d \in D \mid \mu_{\tilde{A}}(d) \geq \alpha\}$$

- If the degree of the elements is greater than α, the set is called the strong α-level set of $\tilde{A}$:

$$A_{\overline{\alpha}} = \{d \in D \mid \mu_{\tilde{A}}(d) > \alpha\}$$

Basic Operations on Fuzzy Sets

- The membership function $\mu_{\tilde{C}}(d)$ of the intersection $\tilde{C} = \tilde{A} \cap \tilde{B}$ is pointwise defined by $\mu_{\tilde{C}}(d) = \min\{\mu_{\tilde{A}}(d), \mu_{\tilde{B}}(d)\}$.

- The membership function $\mu_{\tilde{C}}(d)$ of the union $\tilde{C} = \tilde{A} \cup \tilde{B}$ is pointwise defined by $\mu_{\tilde{C}}(d) = \max\{\mu_{\tilde{A}}(d), \mu_{\tilde{B}}(d)\}$.

- The membership function $\mu_{\tilde{A}^c}(d)$ of the complement $\tilde{A}^c$ of a fuzzy set $\tilde{A}$ is pointwise defined by $\mu_{\tilde{A}^c}(d) = 1 - \mu_{\tilde{A}}(d)$.

Generalization of Intersection: t-Norms

t-norms are two-valued functions from $[0, 1] \times [0, 1]$ into $[0, 1]$ that satisfy the following conditions:

- $t(0, 0) = 0$
  $t(\mu_{\tilde{A}}(d), 1) = t(1, \mu_{\tilde{A}}(d)) = \mu_{\tilde{A}}(d), \quad d \in D$
- $t(\mu_{\tilde{A}}(d), \mu_{\tilde{B}}(d)) \leq t(\mu_{\tilde{U}}(d), \mu_{\tilde{V}}(d))$
  if $\mu_{\tilde{A}}(d) \leq \mu_{\tilde{U}}(d)$ and $\mu_{\tilde{B}}(d) \leq \mu_{\tilde{V}}(d)$ (monotonicity)
- $t(\mu_{\tilde{A}}(d), \mu_{\tilde{B}}(d)) = t(\mu_{\tilde{B}}(d), \mu_{\tilde{A}}(d))$ (commutativity)
- $t(\mu_{\tilde{A}}(d), t(\mu_{\tilde{B}}(d), \mu_{\tilde{C}}(d))) =$
  $t(t(\mu_{\tilde{A}}(d), \mu_{\tilde{B}}(d)), \mu_{\tilde{C}}(d))$ (associativity)

Generalization of Union: s-Norms

s-norms (or t-conorms) are two-valued functions from $[0, 1] \times [0, 1]$ into $[0, 1]$ that satisfy the following conditions:

- $s(1, 1) = 1$
  $s(\mu_{\tilde{A}}(d), 0) = s(0, \mu_{\tilde{A}}(d)) = \mu_{\tilde{A}}(d), \quad d \in D$
- $s(\mu_{\tilde{A}}(d), \mu_{\tilde{B}}(d)) \leq s(\mu_{\tilde{U}}(d), \mu_{\tilde{V}}(d))$
  if $\mu_{\tilde{A}}(d) \leq \mu_{\tilde{U}}(d)$ and $\mu_{\tilde{B}}(d) \leq \mu_{\tilde{V}}(d)$ (monotonicity)
- $s(\mu_{\tilde{A}}(d), \mu_{\tilde{B}}(d)) = s(\mu_{\tilde{B}}(d), \mu_{\tilde{A}}(d))$ (commutativity)
- $s(\mu_{\tilde{A}}(d), s(\mu_{\tilde{B}}(d), \mu_{\tilde{C}}(d))) =$
  $s(s(\mu_{\tilde{A}}(d), \mu_{\tilde{B}}(d)), \mu_{\tilde{C}}(d))$ (associativity)

Examples of t-Norms and s-Norms

- $\min\{\mu_{\tilde{A}}(d),\ \mu_{\tilde{B}}(d)\}$ *(minimum)*
  $\max\{\mu_{\tilde{A}}(d),\ \mu_{\tilde{B}}(d)\}$ *(maximum)*

- $\mu_{\tilde{A}}(d) \cdot \mu_{\tilde{B}}(d)$ *(algebraic product)*
  $\mu_{\tilde{A}}(d) + \mu_{\tilde{B}}(d) - \mu_{\tilde{A}}(d) \cdot \mu_{\tilde{B}}(d)$ *(algebraic sum)*

- $\max\{0,\ \mu_{\tilde{A}}(d) + \mu_{\tilde{B}}(d) - 1\}$ *(bounded difference)*
  $\min\{1,\ \mu_{\tilde{A}}(d) + \mu_{\tilde{B}}(d)\}$ *(bounded sum)*