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Liouville Numbers, Borel Normality and Algorithmic Randomness

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Abstract

We present a systematic comparison between Liouville, computable, Borel normal and Martin-Löf random numbers. The nine non-empty combinations, all small in measure or category, are illustrated with concrete examples. The sets of Liouville numbers and Martin-Löf random numbers are disjoint, thus showing that the irrationality exponent is not a measure of randomness. Finally, we construct the first computable set of correlations appearing in every Martin-Löf random number, but not in all numbers.

1 Introduction

Let α be a real number. The *irrationality exponent* of α, $\mu(\alpha)$, is the supremum of all reals μ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\mu}$ has infinitely many solutions $p/q \in \mathbb{Q}$. A number with infinite irrationality exponent is called a *Liouville number*. In detail, α is a Liouville number if it is irrational and for every positive integer n, there exist integers p_n and q_n with $q_n > 1$ such that

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^n}.$$

In 1844 Liouville found the first transcendental number by constructing the "classical" Liouville number $\sum_{i=1}^{\infty} b^{-i!}$. Numbers which are not Liouville share the following computable pattern (1).

Fact 1.1. *If $\mathbf{x} = x_1x_2\cdots$ is the b-ary expansion of an irrational and not Liouville number α, then there exists an integer $k > 1$ such that the sequence $\mathbf{x}$ satisfies the following* *computable set of correlations: for every $m \geq 1$:*

$$x_{m+1}x_{m+2}\cdots x_{m+k}\cdots x_{mk} \neq 0^{m(k-1)}. \tag{1}$$

Proof. The number α is irrational and not Liouville, hence there exists an integer $k > 1$ such that for all integers p and q with $q > 1$ we have

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^k}. \tag{2}$$

Assume that there exists an integer $m \geq 1$ such that (1) is false and take $p = x_1 b^{m-1} + x_2 b^{m-2} + \cdots + x_m$ and $q = b^m$. Because α is irrational we have:

$$\left|\alpha - \frac{p}{q}\right| = |\alpha - 0.x_1 \cdots x_{mk}| = \sum_{i=km+1}^{\infty} \frac{x_i}{b^i} < \sum_{i=km+1}^{\infty} \frac{1}{b^i} \leq \frac{1}{(b^m)^k},$$

which contradicts (2). □

In [5] J. Borwein observed that, due to the fact that Liouville numbers have infinite irrationality exponent, random numbers should be Liouville numbers. This intuition seems to be corroborated with the fact that non Liouville numbers satisfy the computable pattern (1), which suggests that they might not be random; for example, π, which is not random (because it is computable) is not Liouville [23] and its irrationality exponent does not exceed 7.6304. However, the above intuitions conflict with two well-known facts: on the one hand, random numbers have (Lebesgue) measure one and Liouville numbers have measure zero, and on the other hand, the set of random numbers is of first Baire category and the set of Liouville numbers is co-meagre [29, 8]. So, there are random numbers which are non Liouville and Liouville numbers which are not random. Clearly, this phenomenon needs to be further investigated.

2 Computable and normal numbers

We now introduce four classes of numbers to be compared with the class of Liouville numbers $\mathcal{L}$: the computable numbers $\mathcal{C}$, the (Borel) normal numbers $\mathcal{N}$ and the Martin-Löf random numbers $\mathcal{M}$.

The set $A_b = \{0, 1, \ldots, b-1\}$, where b a positive integer, is called the *b–base*; the elements of A_b are called b–digits. The *b-ary expansion* of the real α is the infinite sequence $x_1 x_2 \cdots$ with $x_i \in \{0, 1, \ldots, b-1\}, b > 1$, such that $\alpha = \sum_{i \geq 1} x_i \cdot b^{-i}$. If α is rational then we choose the infinite sequence ending up in zeroes.

A real α is *computable* if its b-ary expansion is computable, that is, there is a computable function f such that $f(n) = x_n$, for all $n \geq 1$. A *(Borel) normal number in base* b is a real number whose infinite b-ary expansion is uniformly distributed, i.e. each b–digit has the same natural density $1/b$, every string of two b–digits has the same natural density b^{-2}, and, in general, every string of k digits has the same natural density b^{-k}. If the base is clear we will simply say that the number is normal. Normality was introduced by Borel [4] as a model of randomness. Champernowne's number [15] is normal in base 10, but computable, so clearly not random. A *(Borel) absolutely normal* number is a real which is normal in every base. There exist computable absolutely normal numbers (cf. [2, 3, 17]), so even absolute normality is only a necessary condition for randomness.

3 Algorithmic random numbers

How to define mathematically randomness? As in case of normality we discuss randomness of real numbers in terms of their b-ary expansions. Intuition suggests that a "random sequence" should be *typical*, that is, it should belong to any "reasonable" majority. A natural way to model typicalness can be obtained by isolating the set of all sequences having "all verifiable" properties true from the point of view of classical probability theory, i.e. all properties which are satisfied with "probability one" with respect to Lebesgue probability measure (induced by the unbiased discrete probability). Formally, the unbiased discrete probability on A_b defined by the function $h(\{\sigma\}) = b^{-1}$, for every $\sigma \in A_b$ induces the product Lebesgue measure λ on the set of all Borel subsets of the set of all sequences. In what follows measure will refer to λ.

If $x = x_1 x_2 \ldots x_n$ is a string of length n, then the cylinder induced by x, $[x]$, which is the set of all sequences starting with x, has the probability $\lambda([x]) = b^{-n}$. This number can be interpreted as "the probability that a sequence $\mathbf{y} = y_1 y_2 \ldots y_n \ldots$ has the first element $y_1 = x_1$, the second element $y_2 = x_2, \ldots$, the nth element $y_n = x_n$". Independence means that the probability of an event of the form "$y_i = x_i$" does not depend upon the probability of the event "$y_j = x_j, j \neq i$". Every open set, i.e. a union of cylinders, is measurable. A set of sequences S is a *null set* in case for every real $\varepsilon > 0$ there exists an open set containing S with measure less than ε. For instance, every enumerable set of sequences, in particular, the set of computable sequences, is a null set. For more details see [8].

A property P of sequences is *true almost everywhere* if the set of sequences not having the property P is a null set. The main example of such a property is the famous *Law of Large Numbers* discovered by Borel (also known to Jakob Bernoulli around 1700): For every binary sequence $\mathbf{x}$, $\lim_{n\to\infty} S_n(\mathbf{x})/n$ exists almost everywhere in the sense of measure and has the value $1/2$; $S_n(\mathbf{x}) = x_1 + x_2 + \cdots + x_n$.

It is clear that a sequence satisfying a property false almost everywhere is very "particular". Accordingly, it is tempting to say that a sequence $\mathbf{x}$ is "random" if it satisfies every property true almost everywhere. Unfortunately this definition is vacuous because we can define, for every sequence $\mathbf{x}$ the property $P_{\mathbf{x}}$ by

> A sequence $\mathbf{y}$ satisfies $P_{\mathbf{x}}$ if for every $n \geq 1$ there exists an $m \geq n$ such that $x_m \neq y_m$.

Every $P_{\mathbf{x}}$ is true almost everywhere and $\mathbf{x}$ does not have property $P_{\mathbf{x}}$. Accordingly, *no sequence can verify all properties true almost everywhere.* Note that the above argument is well corroborated with results in Ramsey theory which show patterns in *every* sequence [31].

This rather disappointing result which shows that mathematically there is no "true randomness" can be mitigated by considering not *all* properties true almost everywhere, but only a *countable set* of such properties. The "larger" the chosen class of properties is, the "more random" will be the sequences satisfying those properties. Which properties should be considered? The statistical practice and the philosophical intuition suggest to consider classes of "computable properties". By "constructivising" the notion of null set in the most "liberal" computable way, Martin-Löf [25] obtained arguably the most natural (and famous) definition of algorithmic randomness. A *constructive open set* is an open set that is the union of the sequence of cylinders determined by a computably enumerable sequence of strings. A computably enumerable sequence $(O_i)_{i\geq 1}$ of constructive open sets such that $\lambda(O_i) \leq b^{-i}/(b-1)$, for every $i > 0$, determines a unique G_δ set S of constructive measure zero, namely the intersection of all sets O_i; such a set S is called a *constructive null set.* In contrast with the case of "classical" null sets, Martin-Löf proved that *the union of all constructive null sets is a (maximal) constructive null set.* A constructive null set is a "smaller" null set, so a set of constructive measure one is "larger" than a measure one set.

A sequence is *Martin-Löf random* if it is not contained in any constructive null set, that is, if it is not contained in the maximal constructive null set. As a consequence, constructively, with probability one, every sequence is Martin-Löf random. Randomness is a relative property and, consequently, there are many other classes of algorithmic random numbers, some smaller, other larger than Martin-Löf random numbers, cf. [8, 19]. Later in this paper we will briefly consider the larger class of finite-state incompressible reals [13].

Next we present a few useful results using the plain Kolmogorov complexity K [8, 19]. Recall that the plain complexity (Kolmogorov) of a string $w \in A_b^*$ with respect to a partially defined computable function $\varphi : A_b^* \to A_b^*$ is $K_\varphi(w) = \inf\{|p| : \varphi(p) = w\}$. It is well-known that there is a universal partially computable function $U : A_b^* \to A_b^*$ such that

$$K_U(w) \leq K_\varphi(w) + c_\varphi,$$

holds for all strings $w \in A_b^*$. Here the constant c_φ depends only on U and φ, but not on the particular string. We will denote the complexity K_U simply by K.

For the sequence $\mathbf{x}$ we denote by $\mathbf{x} \upharpoonright n$ its prefix of length n.

Fact 3.1 ([19])**.** (a) *Let* $\mathbf{x}$ *be the* b*-ary expansion of a Martin-Löf random number. Then* $\liminf_{n\to\infty} K(\mathbf{x} \upharpoonright n)/n = 1$*.* (b) *If* $\mathbf{x}$ *is the* b*-ary expansion of a computable number then* $\limsup_{n\to\infty} K(\mathbf{x} \upharpoonright n)/n = 0$*.*

Using a result of Kolmogorov [22] this fact can, to a certain extent, be reversed.

Lemma 3.1 ([32, Corollary 9])**.** *Let* $\mathbf{x}$ *be the* b*-ary expansion of a number. If* $\liminf_{n\to\infty} K(\mathbf{x} \upharpoonright n)/n = 1$*, then* $\mathbf{x}$ *is normal in base* b*.*

Since the value of $\liminf_{n\to\infty} K(\mathbf{x} \upharpoonright n)/n$ is independent of the chosen base (see e.g. [32]), we need no relativisation to certain base in Lemma 3.1.

The following result appears in the proof of Lemma 10 of [32]. It provides a sufficient condition for expansions of non Liouville numbers.

Lemma 3.2. *There is a computable function* $\psi : A_b^* \to A_b^*$ *such that* $\liminf_{n\to\infty} K_\psi(\mathbf{x} \upharpoonright n)/n = 0$ *for every* b*-ary expansion* $\mathbf{x}$ *of a Liouville number.*

Combining Fact 3.1 and Lemma 3.2 we obtain the following.

Corollary 3.1. *The sets of Liouville numbers and Martin-Löf random numbers are disjoint.*

Next we generalise an idea of [28, Lemma 1] to construct Liouville numbers of a certain shape (including normal ones). Maillet [24] sketched, without proof, a similar construction as in our Lemma 3.3 (see also [30, Kapitel 1]).

To this end let the length of a finite or infinite string η over A_b be $|\eta|$; the jth letter ($j = 1, \ldots, |\eta|$) of η is denoted by $\eta(j)$. If $w \in A_b^*$ and $i \geq 0$ is an integer, then w^i is the concatenation $ww \ldots w$ (i times) and w^ω is the infinite concatenation $ww \ldots w \ldots$. Using finitely or infinitely many strings $w_i \in A_b^*$ we can construct b-ary expansions of real numbers.

Lemma 3.3. *Let* $(w_i)_{i\in\mathbb{N}}$ *be a family of non-empty strings* $w_i \in A_b^*$*,* $f : \mathbb{N} \to \mathbb{N} \setminus \{0\}$*, and* $n_i = \sum_{j=0}^{i} f(i) \cdot |w_i|$*. If* $\liminf_{i\to\infty} \frac{n_{i-1}+|w_i|}{n_i} = 0$*, then* $\mathbf{x} = \prod_{j=0}^{\infty} w_j^{f(j)}$ *is the* b*-ary expansion of a rational or a Liouville number.*

Proof. First observe that $n_i = \left|\prod_{j=0}^{i} w_j^{f(j)}\right|$. Next, choose $i, n \in \mathbb{N}$ such that $(n_{i-1} + |w_i|) \cdot n < n_i - 1$ and consider the b-ary expansion $\mathbf{y}_i = \prod_{j=0}^{i-1} w_j^{f(j)} \cdot w_i^\omega$. Then $\mathbf{y}_i$ is the b-ary expansion of the rational number

$$\frac{a_i}{b^{n_{i-1}}} + \frac{a'_i}{b^{n_{i-1}}(b^{|w_i|} - 1)},$$

with $a_i = \sum_{j=1}^{n_{i-1}} \mathbf{y}_i(j) \cdot b^{n_{i-1}-j}$ and $a'_i = \sum_{j=1}^{|w_i|} w_i(j) \cdot b^{|w_i|-j}$, thus of the form p_i/q_i with the denominator $q_i = b^{n_{i-1}}(b^{|w_i|} - 1) < b^{n_{i-1}+|w_i|}$.

By construction, the b-ary expansions $\mathbf{x}$ and $\mathbf{y}_i$ have a common prefix of length n_i. Thus the real number $\alpha = 0.\mathbf{x}$ satisfies

$$\left|\alpha - \frac{p_i}{q_i}\right| \leq b^{-(n_i-1)} \leq (b^{n_{i-1}+|w_i|})^{-n} < q_i^{-n}.$$

$\square$

In [28] normal Liouville numbers have been constructed using $w_i = B(2, i)$ and $f(i) = i^i$ (see Eq. (3) below). The above construction with $w_{2i} = 0$, $w_{2i+1} = 1$, $f(2i) = i!$ and $f(2i + 1) = 1$ gives the "classical" Liouville number with the 2-ary expansion $\prod_{j=0}^{\infty} 0^{j!}1$.

4 Relations between $\mathcal{L}, \mathcal{C}, \mathcal{N}$, and $\mathcal{M}$

First, how large are the classes $\mathcal{L}, \mathcal{C}, \mathcal{N}, \mathcal{M}$ from the points of view of measure and category (in Baire sense, cf. [29])? While $\mathcal{C}$ is countable, all the other classes have the cardinality of the continuum. $\mathcal{L}$ is a dense G_δ-set (hence co-meagre), measure zero set [29, 6]. $\mathcal{N}$ and $\mathcal{M}$ are constructive measure one [25], but constructively meagre in the Cantor space [29, 9] (a constructive meagre set is a meagre set covered by computably enumerable union of computably enumerable nowhere dense subsets; a constructive meagre set is "smaller" than a meagre set). In [10] it is shown that $\mathcal{M}$ is co-meagre for a suitably chosen metric topology.

Second, some relations are easy to see or are known. Liouville's "classical" number is computable, but not normal in base 10. Every Martin-Löf random number is absolutely normal (see [7] or Lemma 3.1 and Fact 3.1) and incomputable [8, 19]. Normal numbers may be incomputable; for example, Martin-Löf random numbers. Champernowne number is computable and normal in base 10, but not absolutely normal (in general, normality in base a implies normality in base b if and only if a is a power of b, [20]); for computable normal numbers see [2, 3, 17].

As mentioned above, $\mathcal{M} \subset \mathcal{N}$, and, clearly $\mathcal{C} \cap \mathcal{M} = \emptyset$. Moreover, from Liouville's construction it follows that $\mathcal{C} \cap \mathcal{L} \neq \emptyset$ [2]. Also, $\mathcal{C} \cap \mathcal{N} \neq \emptyset$ and for cardinality reasons $\mathcal{L} \not\subseteq \mathcal{C}$ and $\mathcal{N} \not\subseteq \mathcal{C}$.

We study all possible combinations between the four classes of numbers considered. We denote by $\bar{S}$ the complement of the set S. Out of 16 possible combinations the following seven sets are empty: $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}} \cap \mathcal{M}$, $\bar{\mathcal{L}} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \mathcal{M}$, $\mathcal{L} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}} \cap \mathcal{M}$, $\mathcal{L} \cap \mathcal{C} \cap \bar{\mathcal{N}} \cap \mathcal{M}$ (all because $\mathcal{M} \subset \mathcal{N}$), $\bar{\mathcal{L}} \cap \mathcal{C} \cap \mathcal{N} \cap \mathcal{M}$, $\mathcal{L} \cap \mathcal{C} \cap \mathcal{N} \cap \mathcal{M}$ (both because $\mathcal{C} \cap \mathcal{M} = \emptyset$), and $\mathcal{L} \cap \bar{\mathcal{C}} \cap \mathcal{N} \cap \mathcal{M}$ (because of Corollary 3.1).

Next we show that all other 9 intersections are non-empty, but they are all "small" in measure or/and category.

To prove the first two results we will use *de Bruijn words* over A_b of order $r \geq 1$ which are strings w of length $b^r + r - 1$ such that any string of length r occurs as a substring of w (exactly once). It is well-known that de Bruijn words of any order and for every A_b exist, and have an explicit construction [18, 33]. For example, 00110 and 0001011100 are binary de Bruijn strings of orders 2 and 3 respectively.

Note that de Bruijn words are derived in a circular way, hence their prefix of length $r-1$ coincides with the suffix of length $r-1$. Denote by $B(b, r)$ the prefix of length 2^r of a de Bruijn string of order r. The examples of binary de Bruijn words of orders 2 and 3 previously presented are derived from the strings $B(2, 2) = 0011$ and $B(2, 3) = 00010111$, respectively. Thus the string $B(b, r) \cdot B'(b, r)$, where $B'(b, r)$ is the length $r - 1$ prefix of $B(b, r)$, contains every b-ary string of length string r exactly once as a substring. For definiteness, we agree here on the fact that $B(b, r)$ starts with r zeroes and ends on a symbol different from 0. Thus $B(b, r)$ is not a prefix of $B(b, r + 1)$.

According to [27, 28] every sequence of the form

$$\mathbf{x}_f = \prod_{i=1}^{\infty} B(i)^{f(i)} = B(b,1)^{f(1)} B(b,2)^{f(2)} \cdots B(b,i)^{f(i)} \cdots \tag{3}$$

is normal in base b provided the function $f : \mathbb{N} \to \mathbb{N}$ is increasing and satisfies the condition $f(i) \geq i^i$, for all $i \geq 1$. If, moreover, the family $\big(B(b,i)\big)_{i \in \mathbb{N}}$ and f satisfy the hypothesis of Lemma 3.3 the real $\alpha_f = 0.\mathbf{x}_f$ is a Liouville number.

4.1 $\mathcal{L} \cap \mathcal{C} \cap \mathcal{N} \cap \bar{\mathcal{M}}$

Let $f(i) = i^i$. Then f is a computable function and thus α_f is also computable, normal and a Liouville number, thus, in view of Fact 3.1 and Lemma 3.2, not Martin-Löf random.

4.2 $\mathcal{L} \cap \bar{\mathcal{C}} \cap \mathcal{N} \cap \bar{\mathcal{M}}$

In [21] incomputable disjunctive (a sequence is disjunctive in case every string appears in it) Liouville numbers have been constructed. These numbers are not normal, hence not Martin-Löf random.

If $i^i + 1 \geq f(i) \geq i^i$ is an incomputable function, then α_f is an incomputable normal Liouville number which by Fact 3.1 and Lemma 3.2 is not Martin-Löf random.

4.3 $\mathcal{L} \cap \mathcal{C} \cap \bar{\mathcal{N}}$

Liouville "classical" number $\sum_{i=1}^{\infty} b^{-i!}$ is computable, hence not Martin-Löf random, and not normal in base b. This set is countable because of $\mathcal{C}$.

4.4 $\mathcal{L} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}}$

The construction of Lemma 3.3 with $w_{2i} = 0$, $w_{2i+1} = 1$, $f(2i+1) = 1$ and $f(2i) \geq i!$, where the function $f : \mathbb{N} \to \mathbb{N}$ is incomputable yields a non-computable Liouville number with the expansion $\prod_{j=0}^{\infty} 0^{f(2j)}1$ which is not normal in base b.

4.5 $\bar{\mathcal{L}} \cap \mathcal{C} \cap \mathcal{N}$

Any Stoneham number $F(1/2) = \sum_{i=1}^{\infty} 2^{-k^i} \cdot k^{-i}$ (where $k \in \mathbb{N}$ is odd , $k \geq 3$) is computable, normal in base 2 (but not in base 6, see [1]), and, by [16, Theorem 1], has irrationality exponent $\mu(F(1/2)) = k$, thus, is not Liouville.

4.6 $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \bar{\mathcal{N}}$

Let $\alpha = 0.x_1x_2 \ldots x_n \ldots$, $x_i \in A_b$ be Martin-Löf random (given by a b-ary expansion) and let $\beta = 0.\mathbf{y}$, where $\mathbf{y} = x_1 00 x_2 00 \ldots x_n 00 \ldots$. Then β is not normal in base b because it contains at least 2/3 more zeroes than ones. It is not computable, for otherwise α would be computable. Finally, since $\liminf_{n\to\infty} K(\mathbf{y} \upharpoonright n)/n = 1/3$ (actually β is 1/3–Martin-Löf random in the sense of [14]), Lemma 3.2 shows that β is not a Liouville number.

4.7 $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \mathcal{M}$

Here $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \mathcal{M} = \mathcal{M} \neq \emptyset$ follows from the obvious relation $\mathcal{C} \cap \mathcal{M} = \emptyset$ and Corollary 3.1.

4.8 $\bar{\mathcal{L}} \cap \mathcal{C} \cap \bar{\mathcal{N}}$

Every rational number is computable but neither Liouville nor normal.

4.9 $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \mathcal{N} \cap \bar{\mathcal{M}}$

Let $\alpha = 0.x_1x_2 \ldots x_n \ldots$ be a Martin-Löf random real (given by a 2-ary expansion) and let $\mathbf{y}(2^i) = 0$ and $\mathbf{y}(j) = x_j$, otherwise. Then $\liminf_{n\to\infty} K(\mathbf{y} \upharpoonright n)/n = 1$ (see [26, Example 4.1]) but $\beta = 0.\mathbf{y}$ is not Martin-Löf random (use the Martin-Löf constructive null set given by the open sets $O_i = \{\beta \mid \beta(2^j) = 0, \text{ for } j = 0, \ldots, i\}$). Now Fact 3.1, Lemma 3.1 and 3.2 show that β is normal, not computable and not a Liouville number.

All sets included in $\mathcal{C}$ are countable. The set $\bar{\mathcal{L}} \cap \bar{\mathcal{C}} \cap \mathcal{M} = \mathcal{M}$ has constructive measure one, but is constructive meagre [9]. The remaining non-empty sets have all constructive measure zero.

5 Computable correlations in Martin-Löf random numbers

We can now come back to the phenomenon discussed at the end of Section 1. Replacing a Turing machine by a finite transducer we can define the finite-state complexity of strings [11, 12] denoted by C_S; this complexity depends on the computable enumeration S of the set of finite transducers. A sequence $\mathbf{x}$ is *C_S–incompressible* if $\liminf_n C_S(\mathbf{x} \upharpoonright n)/n = 1$, cf. [13].

Theorem 5.1. *There is a computable enumeration S such that every finite-state C_S–incompressible sequence satisfies the computable correlations* (1)*.*

Proof. Proposition 4.1 of [13] and Lemma 3.2 prove that there is a computable enumeration $S[\psi]$ such that $C_{S[\psi]}(w) \leq K_\psi(w)+2$, for all $w \in A_b^*$. Thus finite-state $C_{S[\psi]}$–incompressible numbers are not Liouville by Lemma 3.2, so in view of Fact 1.1 every finite-state $C_{S[\psi]}$–incompressible number satisfies (1). $\square$

So, for some S, finite-state C_S–incompressible numbers are normal and incomputable. Every Martin-Löf random sequence is C_S–incompressible, but the converse implication is not true [13, Proposition 5.1].

Corollary 5.1. *Every Martin-Löf random number (sequence) satisfies the pattern* (1)*, but there exist non Martin-Löf random numbers satisfying* (1)*.*

In view of Corollary 5.1 the computable correlations (1) do not correspond to a constructive null set. This fact is interesting because (1) is not a Ramsey type of correlation, typically incomputable, but appearing in every sequence: correlations (1) are computable, but they appear only in some sequences, including all Martin-Löf random sequences. The existence of Liouville numbers which are not Martin-Löf random is less surprising. These facts show that the irrationality exponent is not a measure of randomness.

The number π is not Liouville [23]. Obviously, π is not Martin-Löf random due to its computability; the property of π not to be Martin-Löf random cannot be excluded because π has (1). An interesting *open question is to find a set of correlations in π which are not* *related to computability and exclude its Martin-Löf randomness, i.e. a constructive null set not related to computability which contains π.*

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References

[1] D. H. Bailey, R. E. Crandall. Random generators and normal numbers, *Experiment. Math.* 11 (2002), no. 4 (2003), 527–546.

[2] V. Becher, S. Figueira. An example of a computable absolutely normal number, *Theoretical Computer Science* 270 (2002), 947–958.

[3] V. Becher, P. Heiber, T. A. Slaman. A polynomial-time algorithm for computing absolutely normal numbers, *Information and Computation* 232 (2013), 1–9.

[4] É. Borel. Les probabilités dénombrables et leurs applications arithmétiques, *Rendiconti del Circolo Matematico di Palermo* 27 (1909), 247–271.

[5] J. Borwein. Email personal communication to C. S. Calude, 25 September 2013.

[6] J. Borwein, A. van der Poorten, J. Shallit, W. Zudilin. *Neverending Fractions. An Introduction to Continued Fractions*, Cambridge University Press, Cambridge, to appear.

[7] C. Calude. Borel normality and algorithmic randomness, in G. Rozenberg, A. Salomaa (eds.). *Developments in Language Theory*, World Scientific, Singapore, 1994, 113–129.

[8] C. S. Calude. *Information and Randomness—An Algorithmic Perspective*, 2nd ed., Springer, Berlin, 2002.

[9] C. Calude, I. Chiţescu. Random sequences: some topological and measure-theoretical properties, *An. Univ. Bucureşti Mat.-Inf.* 2 (1988), 27–32.

[10] C. S. Calude, S. Marcus, L. Staiger. A topological characterization of random sequences, *Inform. Process. Lett.* 88 (2003), 245–250.

[11] C. S. Calude, K. Salomaa, T. K. Roblot. Finite state complexity, *Theoretical Computer Science* 412 (2011), 5668–5677.

[12] C. S. Calude, K. Salomaa, T. K. Roblot. State-size hierarchy for FS-complexity, *International Journal of Foundations of Computer Science* 23, 1 (2012) 37–50.

[13] C. S. Calude, L. Staiger, F. Stephan. Finite State Incompressible Infinite Sequences, *CDMTCS Research Report* 443, 2013, 25 pp.

[14] C. S. Calude, L. Staiger, S. A. Terwijn. On partial randomness, *Annals of Applied and Pure Logic* 138 (2006), 20–30.

[15] D. G. Champernowne. The construction of decimals normal in the scale of ten, *Journal of the London Mathematical Society* 8 (1933), 254–260.

[16] M. Coons. Irrationality measures for some normal numbers, manuscript, 30 October 2013, 2 pp.

[17] S. Figueira, A. Nies. Feasible Analysis, Randomness, and Base Invariance, *CDMTCS Research Report* 446 (2013), 21 pp.

[18] N. de Bruijn. A combinatorial problem, *Proc. Kon. Nederl. Akad. Wetensch.* 49, 758–764 (1946).

[19] R. Downey, D. Hirschfeldt. *Algorithmic Randomness and Complexity*, Springer, Heidelberg, 2010.

[20] P. Hertling. Simply normal numbers to different bases, *J. UCS* 8, 2 (2002), 235–242.

[21] H. Jürgensen, G. Thierrin. Some structural properties of ω-languages, *13th Nat. School with Internat. Participation "Applications of Mathematics in Technology"*, Sofia, 1988, 56–63.

[22] A. N. Kolmogorov. Logical basis for information theory and probability theory, *IEEE Trans. Inform. Theory* 14 (1968), 662–664.

[23] K. Mahler. On the approximation of π, *Nederl. Akad. Wetensch. Proc. Ser. A. 56/Indagationes Math.* 15 (1953), 30–42.

[24] E. Maillet. Sur les nombres quasi-rationnels et les fractions arithmétiques ordinaires ou continues quasi-périodiques, *C. R. Acad. Sci.* 138 (1904), 410–411.

[25] P. Martin-Löf. The definition of random sequences, *Inform. and Control* 9 (1966), 602-619.

[26] J. Mielke. Refined bounds on Kolmogorov complexity for ω-languages. *Electr. Notes Theor. Comput. Sci.* 221 (2008), 181–189.

[27] S. Nandakumar. Personal communication to L. Staiger during the Dagstuhl-Seminar 12021, January 2012.

[28] S. Nandakumar, S. K. Vangapelli. Normality and finite-state dimension of Liouville numbers, `arXiv:1204.4104v1 [cs.IT]`, 2012.

[29] J. C. Oxtoby. *Measure and Category*, 2nd ed., Springer-Verlag, Berlin, 1980.

[30] T. Schneider. *Einführung in die transzendenten Zahlen*, Springer-Verlag, Berlin 1957.

[31] A. Soifer. Ramsey theory before Ramsey, prehistory and early history: An essay, in A. Soifer (ed.), *Ramsey Theory: Yesterday, Today, and Tomorrow*, Springer, Progress in Mathematics 285, Berlin, 2011, 1–26.

[32] L. Staiger. The Kolmogorov complexity of real numbers, Theoret. Comput. Sci.,284, (2002) 455–466.

[33] J. H. van Lint, R. M. Wilson, *A Course in Combinatorics,* Cambridge University Press, Cambridge, 1993.