CDMTCS Research Report Series

News from New Zealand (15)

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Group-Theoretic Methods for Designing Networks

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News from New Zealand

1. The joint DMTCS'99 and CATS'99 conference, which is a part of the Australasian Computer Science Week (ACSW'99), will be held in Auckland on 18-21 January 1999. For more information see `http://www.cs.auckland.ac.nz/CDMTCS/docs/cats99cfp.html`.

2. New CDMTCS Research Reports[1]:

76. C. Martin-Vide, G. Păun, G. Rosenberg and A. Salomaa. X-Families: An Approach to the Study of Families of Syntactically Similar Languages
77. P. Hertling. A Lower Bound for Range Enclosure in Interval Arithmetic (updated)
78. R. Laue. Halvings on Small Point Sets
79. P. Hertling and K. Weihrauch. Randomness Spaces
80. M.J. Dinneen, G. Pritchard and M.C. Wilson. Degree- and Time- Constrained Broadcast Networks
81. A. Arslanov. On Hypersimple Sets and Chaitin Complexity

3. Recent CDMTCS visitors: Prof. M. Amos (U. of Liverpool, UK), Prof. I. Antoniou (Solvay Institute, Belgium), Prof. A. Ekert (Oxford, UK), Prof. H.J. Kimble (Caltech, USA), Prof. A. Gibbons (U. of Liverpool, UK), Prof. K. Gustafson (U. of Colorado, USA), Prof. T. Knight (MIT, USA), Prof. S. Lloyd (MIT, USA), Prof. H. Matsueda (Kochi U., Japan), Prof. C. Moore (Santa Fe Institute, USA), Prof. Gh. Păun (Institute of Mathematics, Romania), Prof. J. Reif (Duke U. , USA), Prof. A. Salomaa (U. of Turku, Finland), Prof. G. Sussman (MIT, USA).

4. The design of large interconnection newtworks and multi-processor configurations is an area of great importance for computer science. I have invited my colleague Mike Dinneen to survey some results obtained recently by an interdisciplinary group at the University of Auckland: they show in a convincing way the importance of theory in developing new paradigms of computation. Mike's contribution will be attached to this column.

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[1]At `http://www.cs.auckland.ac.nz/CDMTCS/docs/pubs.html`.

Group-Theoretic Methods for Designing Networks

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1 Introduction

This short note surveys the use of group theory in the design of large interconnection networks and multi-processor configurations. Five local research staff (P.R. Hafner, G. Pritchard, M.C. Wilson, G. Zakeri and myself), with the assistance of a few graduate students, from the University of Auckland[2] have been working in this area. I want to mention two well-studied design problems that we are working on (and have made substantial progress) in the hope that other people may be interested.

Several fundamental design problems that deal with the topology of networks have emerged [10, 11]. We observed that many of the "best-known" constructions for these design problems are based on Cayley graphs, which is our main group-theoretic design tool.

A basic constraint in many network design problems is a bound on the maximum node degree that is imposed by cost and fundamental engineering limitations. That is, network nodes can have at most a fixed number of communication lines connected to other nodes. At the same time, efficient network communications are crucial for many applications.

We view multi-processor configurations and interconnection networks in terms of graph theory where the vertices represent processors or nodes, and the edges represent connecting wires or communication lines. Two basic design problems for which we have successfully designed several best-known constructions are the following.

1. *The Degree/Diameter Problem* (e.g. [1, 8]). Provide constructions of the largest possible networks satisfying bounds on maximum node degree and diameter. The diameter measures the maximum communication delay between any two nodes in a network. If each node can communicate simultaneously with all of its neighbors then the diameter also gives the maximum time needed to flood a message throughout the network.

2. *The Degree/Broadcast-Time Problem* (e.g. [3]). Provide constructions of the largest possible networks satisfying bounds on maximum node degree and broadcast time. In these networks a node can communicate with only one of its neighbors at a time. Under this restriction the broadcast time is the maximum time needed for any node to disseminate (in a point-to-point fashion) a message throughout the network.

Generally a network's diameter is smaller than its broadcast time. This is intuitively clear since communications are one to many (for diameter) verses one to one (for broadcast time). Figure 1 shows the distinction between these two communication concepts in a simple ring architecture.

[2]This research group has members from four different departments within the University: Computer Science, Mathematics, Engineering Science (Operations Research) and Statistics.

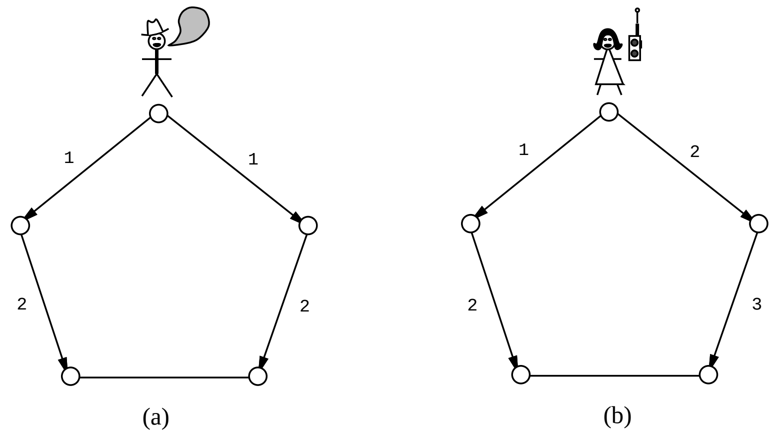

Figure 1: A comparison between (a) diameter and (b) broadcast time.

2 The Cayley Graph Model

Our research uses Cayley graphs, defined below, for designing large networks that satisfy various practical constraints, such as bounded degree and maximum communication time.

Let G be a finite group, S a subset of G which generates G and does not contain the identity. The *Cayley graph of G with respect to S* is the directed graph whose vertices are the elements of G and whose edge set is $E = \{(x, y) \mid y = xs$ for some $s \in S$ and $x, y \in G\}$. If S is closed under inverses, i.e. $S = S \cup S^{-1}$, then $(x, y) \in E$ if and only if $(y, x) \in E$. In this case the edges can be considered as undirected. A few examples of Cayley graphs are given in the following sections.

There are many advantages of using group theory in the design of connected systems. For one thing, our approach yields networks with the nice property of node symmetry. This allows message routing schemes to be node independent. For massive parallel-processors symmetry is a valuable, natural and useful organizational tool for meeting the difficult challenges of coordinating large number of computational units. Many of the developed (or proposed) parallel processor architectures are *node symmetric* (also called vertex transitive). In addition, most (!) node symmetric connected systems are (implicitly) based on Cayley (group) graphs. The few exceptions can be represented as Cayley coset graphs [13]. Other advantages of networks designed using algebraic structures may include: (1) line symmetry, (2) hierarchical structure, and/or (3) high fault tolerance.

3 Degree/Diameter Examples

A very simple example of a Cayley graph is the graph $<Z_{13}, \{1, 5, 8, 12\}>$ shown in Figure 2. Here Z_{13} is the cyclic group of integers modulo 13 (under addition). Somewhat surprisingly, this graph is the largest-known vertex transitive graph with maximum degree 4 and diameter 2.

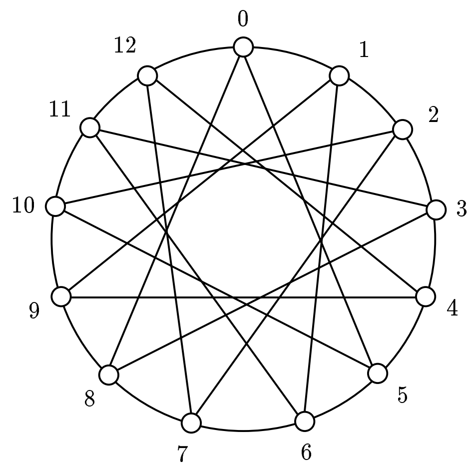

Figure 2: A 4-regular Cayley graph of order 13 and diameter 2.

Note that it is rare to find "good" degree/diameter networks designed via commutative groups. In general, these types of Cayley graphs do not yield graphs with low diameter. This is because we desire a "fat" communication tree (i.e., for every two generators g_1 and g_2 we prefer that the Cayley graph vertices $g_1 * g_2$ and $g_2 * g_1$ be different). In fact, one family of successful non-commutative (non-Abelian) groups (for the Cayley graph model) are based on semi-direct products of cyclic groups. This idea was first presented in [4] and has been applied by many others (e.g., see [14]) for finding the largest-known (degree, diameter) graphs.

Our second example is based on a very recent collaborative study with Los Alamos National Laboratory (USA). Paul Hafner and myself (Auckland) are working with Vance Faber and Dean Prichard (LANL) on the practicality of using a small dense Cayley graph that was discovered in [15]. Los Alamos has recently obtained funding to *build* the network shown in Figure 3. We note that this vertex-transitive (6,2) graph of order 32 has the same number of vertices as the non-symmetric (6,2) graph that is listed in the current (degree, diameter) record book [2].

It took us several days to find this nice circuit layout from the Cayley graph presentation. This drawing illustrates a 4-dimensional layout where pairs of (black/white) vertices connected by generator f form the points of the standard 4-cube of order 16. The connections for the fourth dimension (generator d) and the fifth generator e are not shown; here generator e simply connects antipodal points of the space. (E.g., (bc, ea) and (af, ea) are edges.)

Before the LANL team begins the fabricating process for this network we need to understand some of the properties of this particular Cayley graph. Fortuneately the algebraic structure does help (e.g. using a result of [12] confirmed that the bisection bandwidth is 32.)

The following figure presents the Cayley graph on the free group $G<a,b,c,d,e,f>$ with respect to these relations: $a^2, b^2, c^2, d^2, e^2, f^2, (af)^2, (bf)^2, (cf)^2, (df)^2, (ef)^2$, $ababf$, $cbade$, $dbaec$, $ebacd$, $acacf$, $ecadb$.

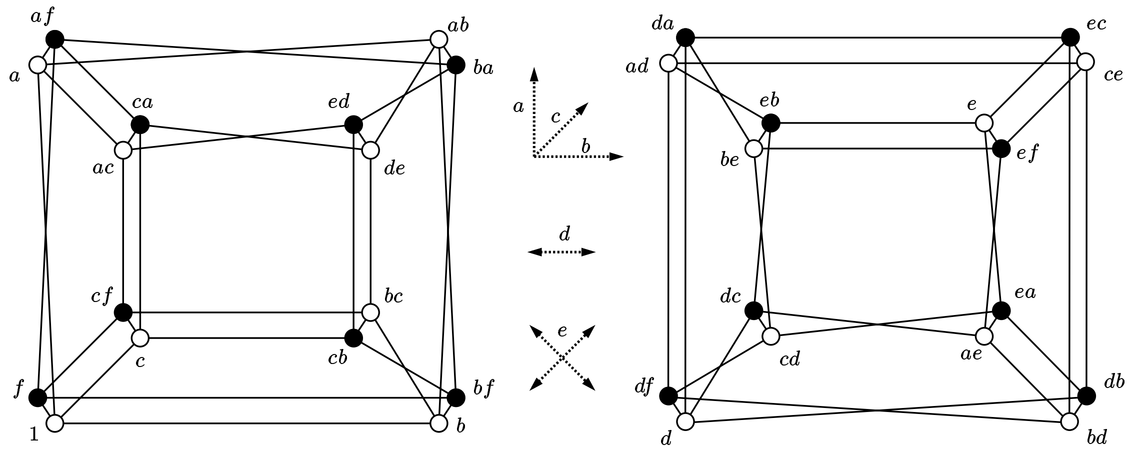

Figure 3: A twisted 4-cube, a (6,2) graph of order 32.

4 Broadcast Network Examples

Farley introduced the idea of broadcasting in networks [9]. He proposed the problem of finding networks such that one can successfully broadcast in time $\lceil \log(n) \rceil$, where n is the number of nodes. This is a natural question since in any broadcasting protocol the number of informed nodes can at most double at each time step. One definition of an *optimal broadcast network* is one with the least number of edges such that broadcasting can be carried out in time $\lceil \log(n) \rceil$ from any originator.

Currently there are only two infinite families of optimal broadcast graphs. These families can be expressed in terms of Cayley graphs. Each of these is also optimal for the Degree/Broadacst Problem (i.e., the graphs are also the largest possible with respect to maximum degree and broadcast time constraints).

The well-known hypercubes (e.g., see the graph on the left of Figure 4) was the first known family. The hypercube Q_n is represented as a Cayley graph using the Abelian group $(Z_2)^n$ with generators $\{e_i \mid 1 \le i \le n\}$ where $e_i = (\overbrace{0,\ldots,0}^{i-1}, 1, \overbrace{0,\ldots,0}^{n-i})$.

A set of recently discovered dihedral Cayley graphs (see [3]) is another infinite family of optimal broadcast graphs of maximum degree Δ and broadcast time $T = \Delta + 1$. Each of these Cayley graphs is based on the dihedral group $D_{2^{\Delta-1}-1} = \langle a, b \mid a^2 = b^{2^{\Delta}-1} = (ab)^2 = 1 \rangle$, with respect to generators $\{ab^{2^i-1} \mid 0 \le i \le \Delta - 1\}$. One broadcast protocol is indicated in Figure 4 for the (3,4) dihedral Cayley graph (which is the well-known Heawood graph) by labeling the edges with the transmission times. Note that the dihedral group D_n can be viewed as the group of rotations and flips of an n-gon. The edges in our dihedral Cayley graphs are defined by the "flip" (involution) generators.

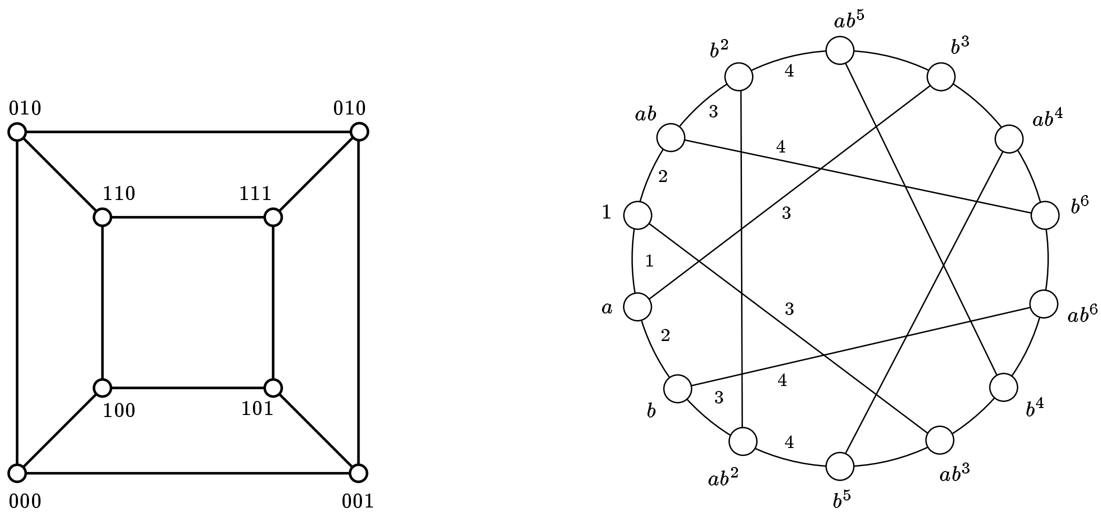

Figure 4: Examples of optimal $(3, 3)$ and $(3, 4)$ broadcast graphs.

For both of these families a broadcast protocol exists which is as simple as possible. Specifically, there is an ordering $s_0 < s_1 < \ldots < s_{\Delta-1}$ of the set of generators S such that at time step i, vertex x sends to vertex xs_j, where $0 \le j \le \Delta - 1$ and $j \equiv i \bmod \Delta$. In other words, at a given time step all transmissions are in a fixed "dimension", and these dimensions cycle through the elements of S.

Recent work done at Auckland on the broadcast problem is presented in the papers [5, 6, 7]. We are currently exploring how to efficiently compute/estimate broadcast times of arbitrary and specific families of graphs (the general broadcast problem is $\mathcal{NP}$-complete). One of our students, H. Wang, is studying the degree/broadcast time problem when restricted to planar networks.

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