CDMTCS
Research
Report
Series

Randomness in Physics: Five Questions, Some Answers

Karl Svozil
Vienna University of Technology

CDMTCS-364
May 2009

Centre for Discrete Mathematics and
Theoretical Computer Science

Randomness in Physics: Five Questions, Some Answers*

Karl Svozil†

Institute for Theoretical Physics, Vienna University of Technology, Wiedner Hauptstraße 8-10/136, A-1040 Vienna, Austria

Abstract

Despite provable unknowables in recursion theory, indeterminism and randomness in physics is confined to conventions, subjective beliefs and preliminary evidence. The history of the issue is very briefly reviewed, and answers to five questions raised by Zenil are presented.

PACS numbers: 01.65.+g,01.70.+w,02.10.-v,02.50.Ey,05.45.-a

Keywords: Indeterminism, stochasticity, randomness in physics, halting problem, induction

* Hector Zenil posed the following five questions: "Why were you initially drawn to the study of computation and randomness?", "What have we learned?", "What don't we know (yet)?", "What are the most important open problems in the field?", "What are the prospects for progress?", at URL `http://www.mathrix.org/experimentalAIT/RandomnessBook.htm`, accessed on May 1st, 2009.

†Electronic address: `svozil@tuwien.ac.at`; URL: `http://tph.tuwien.ac.at/~svozil`

Why should the universe we experience with our senses and brains be "(un)lawful?" Indeed, the "unreasonable effectiveness of mathematics in the natural sciences" [1] appears mind-boggling and tantamount to our (non)existence. Beyond belief, there do not seem to exist *a priori* answers to such questions which would be forced upon us, say maybe by consistency constraints. But then, why should consistency and logic be considered *sacrosanct*?

In view of the opaqueness of the issues, a fierce controversy between supporters and opponents of a "clockwork universe" *versus* "cosmic chaos" has developed from antiquity onwards — cf., e.g., Aristotle's comments on the Pythagoreans in *Physics*, as well as Epicurus' *Letter to Menoeceus*. Indeed, for the sake of purported truth, many varieties of conceivable mixtures of determinism and chance have been claimed and repudiated.

The author has argued elsewhere [2] that there are many emotional reasons (not) to believe in a(n) (in)deterministic universe: does it not appear frightening to be imprisoned by remorseless, relentless predetermination; and, equally frightening, to accept one's fate as being contingent on total arbitrariness and chance? What merits and what efforts appear worthy at these extreme positions, which also unmask freedom, self-determination and human dignity as an idealistic illusion?

In order to disentangle the scientific discussion of topics such as (in)determinism, or realism *versus* idealism, from emotional overtones and possible bias, it might not be totally unreasonable to allow oneself the contemplative strategy of *evenly-suspended attention* outlined by Freud [3]: Nature is thereby treated as a "client-patient," and whatever comes up is accepted "as is," without any immediate emphasis or judgment[1].

In more recent history, the European Enlightenment (illuminating also wide areas across the oceans) has brought about the belief of total causality and almost unlimited predictability, control, and manipulative capabilities. Subsequently, the *principle of sufficient reason* came under pressure at two independent fronts: Poincare's discovery of instabilities in classical many-body motion [4] is now considered as a precursor to *deterministic chaos*, in which the information "held" in the initial value "unfolds" through a deterministic process. Note that, with probability one, an arbitrary real number (representing the initial value) "grabbed" from the "continuum urn" (facilitated by the

[1] In Ref. [3], Freud admonishes analysts to be aware of the dangers caused by "… temptations to project, what [[the analyst]] in dull self-perception recognizes as the peculiarities of his own personality, as generally valid theory into science …" (In German: *"Er wird leicht in die Versuchung geraten, was er in dumpfer Selbstwahrnehmung von den Eigentümlichkeiten seiner eigenen Person erkennt, als allgemeingültige Theorie in die Wissenschaft hinauszuprojizieren … ."*)

axiom of choice) is random in the sense of algorithmic information theory [5–7]; i.e., in terms of algorithmic incompressibility as well as of the equivalent statistical tests. Indeed, if one encodes universal computation into a system on n bodies, then by reduction to the halting problem of recursion theory [8–13], certain observables become provable unknowable [14].

A second attack against determinism erupted through the development of quantum theory. Despite fierce resistance of Einstein[2], Schrödinger and De Brogli, Born expressed the new quantum *canon*, repeated by the "mainstream" ever after [16], as follows (cf. Ref. [17, p. 866], English translation in [18, p. 54])[3]:

> "From the standpoint of our quantum mechanics, there is no quantity which in any individual case causally fixes the consequence of the collision; but also experimentally we have so far no reason to believe that there are some inner properties of the atom which condition a definite outcome for the collision. Ought we to hope later to discover such properties [[...]] and determine them in individual cases? Or ought we to believe that the agreement of theory and experiment — as to the impossibility of prescribing conditions for a causal evolution — is a pre-established harmony founded on the nonexistence of such conditions? I myself am inclined to give up determinism in the world of atoms."

More specifically, Born offers a mixture of (in)determinism: while postulating a probabilistic behavior of individual particles, he accepts a deterministic evolution of the wave function (cf. [19, p. 804], English translation in [20, p. 302])[4]:

> "The motion of particles conforms to the laws of probability, but the probability itself

[2] In a letter to Born, dated December 12th, 1926 [15, p. 113], Einstein expressed his conviction, "In any case I am convinced that he [[the Old One]] does not throw dice." (In German: *"Jedenfalls bin ich überzeugt, dass der [[Alte]] nicht würfelt."*)

[3] *"Vom Standpunkt unserer Quantenmechanik gibt es keine Größe, die im* Einzelfalle *den Effekts eines Stoßes kausal festlegt; aber auch in der Erfahrung haben wir keinen Anhaltspunkt dafür, daß es innere Eigenschaften der Atome gibt, die einen bestimmten Stoßerfolg bedingen. Sollen wir hoffen, später solche Eigenschaften [[...]] zu entdecken und im Einzelfalle zu bestimmen? Oder sollen wir glauben, dass die Übereinstimmung von Theorie und Erfahrung in der Unfähigkeit, Bedingungen für den kausalen Ablauf anzugeben, eine prästabilisierte Harmonie ist, die auf der Nichtexistenz solcher Bedingungen beruht? Ich selber neige dazu,die Determiniertheit in der atomaren Welt aufzugeben."*

[4] *"Die Bewegung der Partikel folgt Wahrscheinlichkeitsgesetzen, die Wahrscheinlichkeit selbst aber breitet sich im Einklang mit dem Kausalgesetz aus. [Das heißt, daß die Kenntnis des Zustandes in allen Punkten in einem Augenblick die Verteilung des Zustandes zu allen späteren Zeiten festlegt.]"*

> is propagated in accordance with the law of causality. [This means that knowledge of a state in all points in a given time determines the distribution of the state at all later times.]"

In addition to the indeterminism associated with outcomes of the measurements of single quanta, there appear to be at least two other types of quantum unknowables. One is complementarity, as first expressed by Pauli [21, p. 7]. A third type of quantum indeterminism was discovered by studying quantum probabilities, in particular the consequences of Gleason's theorem [22]: whereas the classical probabilities can be constructed by the convex sum of all two-valued measures associated with classical truth tables, the elementary yes–no propositions in quantum mechanics associated with projectors in three- or higher-dimensional Hilbert spaces do not allow any two-valued measures [23, 24]. One of the consequences thereof is the impossibility of a consistent co-existence of the outcomes of all conceivable quantum observables (under the *noncontextuality* assumption [25] that measurement outcomes are identical if they "overlap").

Parallel to these developments in physics, Gödel [26] put an end to finitistic speculations in mathematics about possibilities to encode all mathematical truth in a finite system of rules. The recursion theoretic, formal unknowables exhibit a novel feature: they present *provable* unknowables in the fixed axiomatic system in which they are derived. (Note that incompleteness and undecidability exist always relative to the particular formal system or model of universal computation.) From ancient times onwards, individuals and societies have been confronted with a pandemonium of unpredictable behaviors and occurrences in their environments, sometimes resulting in catastrophes. Often these phenomena were interpreted as "God's Will." In more rationalistic times, one could pretend without presenting a formal proof that certain unpredictable behaviors are in principle deterministic, although the phenomena cannot be predicted "for various practical purposes." Now provable unknowables make a difference by being immune to these kinds of speculation. The halting problem in particular demonstrates the impossibility to predict the behavior of deterministic systems in general; it also solves the induction (rule inference) problem to the negative.

In order to be able to fully appreciate the impact of recursion theoretic undecidability on physics [27–29], let us sketch an algorithmic proof of the undecidability of the halting problem; i.e., the decision problem of whether or not a program p (on a finite input) finishes running (or will reach a particular halting state) or will run forever. The proof method will use a *reductio ad absurdum*; i.e., we assume the existence of a *halting algorithm* $h(p)$ deciding the halting problem of p, as well as some trivial manipulations; thereby being able to derive a complete contradiction, the only consistent alternative being the nonexistence of any such halting algorithm. For the sake of contradiction, consider an agent $q(p)$ accepting as input an arbitrary program (code) p. Suppose further that it is able to consult a halting algorithm $h(p)$, thereby producing the *opposite* behavior of p: whenever p halts, q "steers itself" into the halting mode; conversely, whenever p does not halt, q forces itself to halt. A complete contradiction results from q's behavior on itself, because whenever $q(q)$ detects (through $h(q)$) that it halts, it is supposed not to halt; conversely if $q(q)$ detects that it does not halt, it is supposed to halt. Finally, since all other steps in this "diagonal argument" with the exception of h are trivial, in order to avoid inconsistencies, no program can have the capacity to consult a halting algorithm for arbitrary programs.

In physics, analogous arguments embedding a universal computer into a physical substrate yield provable undecidable observables *via reduction to the halting problem* [30]. Note that this argument does not mean that predictions are provable impossible for certain special cases; that would be clearly misleading and absurd! A more quantitative picture arises if we study the potential growth of "complexity" of deterministic systems in terms of their maximal capability to "grow" before reaching a halting state through the Busy Beaver function [31–34]. Another consequence is the recursive unsolvability of the general induction (or rule inference [35–39]) problem for deterministic systems. As an immediate consequence of these findings it follows that no general algorithmic rule or operational method [40] exists which could "extract" some rather general law from a (coded) sequence. (Note again that it still may be possible to extract laws from "low-complex" sequences; possibly with some intuition and additional information.) Nor can there be certainty that some sequence denominated "random" is not generated by a compression algorithm which makes it formally nonrandom [6]; a fact well known in recursion theory but hardly absorbed by the physics community. Thereby, to quote *Shakespeare's Prospero,* any claims of absolute ("ontological") randomness decay into "thin air." Of course, one could still vastly restrict the domain of possible laws and *define* a source to be random if it "performs well" with respect to the associated, very limited collection of statistical tests, a strategy adapted by the *Swiss Federal Office of Metrology*[5].

Despite the formal findings reviewed above, which suggest that claims of absolute indetermi-

[5] Cf. the *Certificate of Conformity No 151-04255*, available from URL `http://www.idquantique.com/products/files/CC_151-04255.pdf`, accessed on May 4th, 2009.

nacy cannot be proven but represent subjective beliefs, their predominance in the physics community can be understood, or rather motivated, by the obvious inability to predict physical events, such as the outcomes of certain quantum measurements, deterministically. Why this effective incapacity to predict individual outcomes or time series of measurement data should be different from other "classical" statistical sources of randomness — even when complementarity and value indefiniteness is taken into account — remains an open question, at least from a formal point of view.

For the sake of explicit demonstration, let us consider a particular method of generation of a sequence from single quantum outcomes [41] by combination of source and beam splitter [42–50]. Ideally (to employ quantum complementarity as well as quantum value indefiniteness), a system allowing three or more outcomes is prepared to be in a particular pure state "contained" in a certain context (maximal observable [51] or block [52, 53]), and then measured "along" a different context not containing the observable corresponding to that pure state. All outcomes except two are discarded [6, 54], and the two remaining outcomes are mapped onto the symbols "0" and "1," respectively. The concatenation and normalization [55–59] of subsequent recordings of these encoded outcomes yield an "absolutely random sequence" relative to the unprovable axiomatic assumption of quantum randomness. Since all such operational physical sequences are finite, algorithmic information theory [6] applies to them in a limited, finite sense. Particular care should be given to the difficulties in associating an algorithmic information measure to "nontrivial" sequences of finite length.

In the author's conviction, the postulate of quantum randomness as well as physical randomness emerging from the continuum will be maintained by the community of physicists at large unless somebody comes up with evidence to the contrary. This opportunistic interpretation of the phenomena appears reasonable if and only if researchers are aware of the tentativeness and conventionality of their assumptions.

Acknowledgements

The author gratefully acknowledges discussions with Cristian Calude, as well as the kind hospitality of the *Centre for Discrete Mathematics and Theoretical Computer Science (CDMTCS)* of the *Department of Computer Science at The University of Auckland.* This work was also supported by *The Department for International Relations* of the *Vienna University of Technology.*

References

[1] E. P. Wigner, "The unreasonable effectiveness of mathematics in the natural sciences. Richard Courant Lecture delivered at New York University, May 11, 1959," Communications on Pure and Applied Mathematics **13**, 1–14 (1960).
http://dx.doi.org/10.1002/cpa.3160130102

[2] K. Svozil, "Science at the crossroad between randomness and determinism," in *Millennium III*, C. Calude and K. Svozil, eds. (Black Sea University Foundation in colaboration with the Romanian Academy of Sciences, Bucharest, Romania, 2002), pp. 73–84.

[3] S. Freud, "Ratschläge für den Arzt bei der psychoanalytischen Behandlung," in *Gesammelte Werke. Chronologisch geordnet. Achter Band. Werke aus den Jahren 1909–1913*, A. Freud, E. Bibring, W. Hoffer, E. Kris, and O. Isakower, eds., pp. 376–387 (1999).

[4] F. Diacu, "The Solution of the N-body Problem," The Mathematical Intelligencer **18**, 66–70 (1996).
http://dx.doi.org/10.1007/BF03024313

[5] P. Martin-Löf, "The definition of random sequences," Information and Control **9**, 602–619 (1966).
http://dx.doi.org/10.1016/0030-4018(87)90271-9

[6] C. Calude, *Information and Randomness—An Algorithmic Perspective* (Springer, Berlin, 2002), 2nd edn.

[7] G. J. Chaitin, "Algorithmic Information Theory," IBM Journal of Research and Development **21**, 350–359, 496 (1977), reprinted in [60].

[8] H. Rogers, Jr., *Theory of Recursive Functions and Effective Computability* (MacGraw-Hill, New York, 1967).

[9] M. Davis, *The Undecidable. Basic Papers on Undecidable, Unsolvable Problems and Computable Functions* (Raven Press, Hewlett, N.Y., 1965).

[10] J. Barwise, *Handbook of Mathematical Logic* (North-Holland, Amsterdam, 1978).

[11] H. Enderton, *A Mathematical Introduction to Logic* (Academic Press, San Diego, 2001), second edn.

[12] P. Odifreddi, *Classical Recursion Theory, Vol. 1* (North-Holland, Amsterdam, 1989).

[13] G. S. Boolos, J. P. Burgess, and R. C. Jeffrey, *Computability and Logic* (Cambridge University Press, Cambridge, 2007), fifth edn.

[14] K. Svozil, "Omega and the time evolution of the N-body problem," in *Randomness and Complexity, from Leibniz to Chaitin*, C. S. Calude, ed., pp. 231–236 (2007), eprint arXiv:physics/0703031.

http://arxiv.org/abs/physics/0703031

[15] M. Born, *Physics in my generation* (Springer Verlag, New York, 1969), 2nd edn.

[16] A. Zeilinger, “The message of the quantum,” Nature **438**, 743 (2005).
http://dx.doi.org/10.1038/438743a

[17] M. Born, “Zur Quantenmechanik der Stoßvorgänge,” Zeitschrift für Physik **37**, 863–867 (1926).
http://dx.doi.org/10.1007/BF01397477

[18] J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement* (Princeton University Press, Princeton, 1983).

[19] M. Born, “Quantenmechanik der Stoßvorgänge,” Zeitschrift für Physik **38**, 803–827 (1926).
http://dx.doi.org/10.1007/BF01397184

[20] M. Jammer, *The Conceptual Development of Quantum Mechanics. 2nd edition. The History of Modern Physics, 1800-1950; v. 12* (American Institute of Physics, New York, 1989).

[21] W. Pauli, “Die allgemeinen Prinzipien der Wellenmechanik,” in *Handbuch der Physik. Band V, Teil 1. Prinzipien der Quantentheorie I*, S. Flügge, ed. (Springer, Berlin, Göttingen and Heidelberg, 1958), pp. 1–168.

[22] A. M. Gleason, “Measures on the closed subspaces of a Hilbert space,” Journal of Mathematics and Mechanics (now Indiana University Mathematics Journal) **6**, 885–893 (1957).
http://dx.doi.org/10.1512/iumj.1957.6.56050

[23] E. Specker, “Die Logik nicht gleichzeitig entscheidbarer Aussagen,” Dialectica **14**, 239–246 (1960), reprinted in [61, pp. 175–182]; English translation: *The logic of propositions which are not simultaneously decidable*, reprinted in [62, pp. 135-140].
http://dx.doi.org/10.1111/j.1746-8361.1960.tb00422.x

[24] S. Kochen and E. P. Specker, “The Problem of Hidden Variables in Quantum Mechanics,” Journal of Mathematics and Mechanics (now Indiana University Mathematics Journal) **17**, 59–87 (1967), reprinted in [61, pp. 235–263].
http://dx.doi.org/10.1512/iumj.1968.17.17004

[25] J. S. Bell, “On the Problem of hidden variables in quantum mechanics,” Reviews of Modern Physics **38**, 447–452 (1966), reprinted in [63, pp. 1-13].
http://dx.doi.org/10.1103/RevModPhys.38.447

[26] K. Gödel, “Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme,” Monatshefte für Mathematik und Physik **38**, 173–198 (1931), English translation in [64], and in [9].

[27] J. L. Casti and J. F. Traub, *On Limits* (Santa Fe Institute, Santa Fe, NM, 1994), report 94-10-056.
http://www.santafe.edu/research/publications/workingpapers/94-10-056.pdf

[28] J. L. Casti and A. Karlquist, *Boundaries and Barriers. On the Limits to Scientific Knowledge* (Addison-Wesley, Reading, MA, 1996).

[29] K. Svozil, "Physical Unknowables," in *Horizons of Truth*, M. B. *et al*, ed. (Cambridge University Press, Cambridge, UK, 2009), p. in print.
http://arxiv.org/abs/physics/0701163

[30] K. Svozil, *Randomness & Undecidability in Physics* (World Scientific, Singapore, 1993).

[31] T. Rado, "On Non-Computable Functions," The Bell System Technical Journal **XLI(41)**, 877–884 (1962).

[32] G. J. Chaitin, "Information-theoretic limitations of formal systems," Journal of the Association of Computing Machinery **21**, 403–424 (1974), reprinted in [60].
http://www.cs.auckland.ac.nz/CDMTCS/chaitin/acm74.pdf

[33] A. K. Dewdney, "Computer Recreations: A Computer Trap for the Busy Beaver, the Hardest-Working Turing Machine," Scientific American **251**, 19–23 (1984).

[34] A. H. Brady, "The Busy Beaver Game and the Meaning of Life," in *The Universal Turing Machine. A Half-Century Survey*, R. Herken, ed. (Kammerer und Unverzagt, Hamburg, 1988), p. 259.

[35] M. E. Gold, "Language identification in the limit," Information and Control **10**, 447–474 (1967).
http://dx.doi.org/10.1016/S0019-9958(67)91165-5

[36] L. Blum and M. Blum, "Toward a mathematical theory of inductive inference," Information and Control **28**, 125–155 (1975).

[37] D. Angluin and C. H. Smith, "A Survey of Inductive Inference: Theory and Methods," Computing Surveys **15**, 237–269 (1983).

[38] L. M. Adleman and M. Blum, "Inductive Inference and Unsolvability," The Journal of Symbolic Logic **56**, 891–900 (1991).
http://dx.doi.org/10.2307/2275058

[39] M. Li and P. M. B. Vitányi, "Inductive reasoning and Kolmogorov complexity," Journal of Computer and System Science **44**, 343–384 (1992).
http://dx.doi.org/10.1016/0022-0000(92)90026-F

[40] P. W. Bridgman, *Reflections of a Physicist* (Philosophical Library, New York, 1950).
http://www.archive.org/details/reflectionsofaph031333mbp

[41] C. S. Calude and K. Svozil, “Quantum Randomness and Value Indefiniteness,” Advanced Science Letters **1**, 165168 (2008), eprint arXiv:quant-ph/0611029.
http://dx.doi.org/10.1166/asl.2008.016

[42] K. Svozil, “The quantum coin toss—Testing microphysical undecidability,” Physics Letters A **143**, 433–437 (1990).
http://dx.doi.org/10.1016/0375-9601(90)90408-G

[43] J. G. Rarity, M. P. C. Owens, and P. R. Tapster, “Quantum Random-number Generation and Key Sharing,” Journal of Modern Optics **41**, 2435–2444 (1994).
http://dx.doi.org/10.1080/09500349414552281

[44] T. Jennewein, U. Achleitner, G. Weihs, H. Weinfurter, and A. Zeilinger, “A Fast and Compact Quantum Random Number Generator,” Review of Scientific Instruments **71**, 1675–1680 (2000).
http://dx.doi.org/10.1063/1.1150518

[45] A. Stefanov, N. Gisin, O. Guinnard, L. Guinnard, and H. Zbinden, “Optical quantum random number generator,” Journal of Modern Optics **47**, 595–598 (2000).
http://dx.doi.org/10.1080/095003400147908

[46] V. Jacques, E. Wu, F. Grosshans, F. Treussart, P. Grangier, A. Aspect, and J.-F. Roch, “Experimental Realization of Wheeler’s Delayed-Choice Gedanken Experiment,” Science **315**, 966–968 (2007).
http://dx.doi.org/10.1126/science.1136303

[47] M. Hai-Qiang, W. Su-Mei, Z. Da, C. Jun-Tao, J. Ling-Ling, H. Yan-Xue, and W. Ling-An, “A Random Number Generator Based on Quantum Entangled Photon Pairs,” Chinese Physics Letters **21**, 1961–1964 (2004).
http://dx.doi.org/0256-307X/21/1961

[48] M. Fiorentino, C. Santori, S. M. Spillane, R. G. Beausoleil, and W. J. Munro, “Secure self-calibrating quantum random-bit generator,” Physical Review A (Atomic, Molecular, and Optical Physics) **75**, 032 334 (2007).
http://dx.doi.org/10.1103/PhysRevA.75.032334

[49] P. X. Wang, G. L. Long, and Y. S. Li, “Scheme for a quantum random number generator,” Journal of Applied Physics **100**, 056 107 (2006).
http://dx.doi.org/10.1063/1.2338830

[50] K. Svozil, “Three criteria for quantum random number generators based on beam splitters,” (2009), eprint arXiv:0903.2744.

http://arxiv.org/abs/0903.2744

[51] K. Svozil, "Contexts in quantum, classical and partition logic," in *Handbook of Quantum Logic and Quantum Structures*, K. Engesser, D. M. Gabbay, and D. Lehmann, eds. (Elsevier, Amsterdam, 2008), pp. 551–586.
http://arxiv.org/abs/quant-ph/0609209

[52] J. R. Greechie, "Orthomodular lattices admitting no states," Journal of Combinatorial Theory **10**, 119–132 (1971).

[53] G. Kalmbach, *Orthomodular Lattices* (Academic Press, New York, 1983).

[54] C. Calude and I. Chiţescu, "Qualitative properties of P. Martin-Löf random sequences," Unione Matematica Italiana. Bollettino. B. Serie VII **3**, 229–240 (1989).

[55] J. von Neumann, "Various Techniques Used in Connection With Random Digits," National Bureau of Standards Applied Math Series **3**, 36–38 (1951), reprinted in *John von Neumann, Collected Works, (Vol. V)*, A. H. Traub, editor, MacMillan, New York, 1963, p. 768.

[56] P. Elias, "The Efficient Construction of an Unbiased Random Sequence," Ann. Math. Statist. **43**, 865–870 (1972).
http://dx.doi.org/10.1214/aoms/1177692552

[57] Y. Peres, "Iterating Von Neumann's procedure for extracting random bits," The Annals of Statistics **20**, 590–597 (1992).
http://www.jstor.org/stable/2242181

[58] M. Dichtl, "Bad and Good Ways of Post-processing Biased Physical Random Numbers," in *Fast Software Encryption. Lecture Notes in Computer Science Volume 4593/2007*, A. Biryukov, ed. (Springer, Berlin and Heidelberg, 2007), pp. 137–152, 14th International Workshop, FSE 2007, Luxembourg, Luxembourg, March 26-28, 2007, Revised Selected Papers.
http://dx.doi.org/10.1007/978-3-540-74619-5_9

[59] P. Lacharme, "Post-Processing Functions for a Biased Physical Random Number Generator," in *Fast Software Encryption. Lecture Notes in Computer Science Volume 5086/2008*, K. Nyberg, ed. (Springer, Berlin and Heidelberg, 2008), pp. 334–342, 15th International Workshop, FSE 2008, Lausanne, Switzerland, February 10-13, 2008, Revised Selected Papers.
http://dx.doi.org/10.1007/978-3-540-71039-4_21

[60] G. J. Chaitin, *Information, Randomness and Incompleteness* (World Scientific, Singapore, 1990), 2nd edn., this is a collection of G. Chaitin's early publications.

[61] E. Specker, *Selecta* (Birkhäuser Verlag, Basel, 1990).

[62] C. A. Hooker, *The Logico-Algebraic Approach to Quantum Mechanics. Volume I: Historical Evolution* (Reidel, Dordrecht, 1975).

[63] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987).

[64] K. Gödel, in *Collected Works. Publications 1929-1936. Volume I*, S. Feferman, J. W. Dawson, S. C. Kleene, G. H. Moore, R. M. Solovay, and J. van Heijenoort, eds. (Oxford University Press, Oxford, 1986).