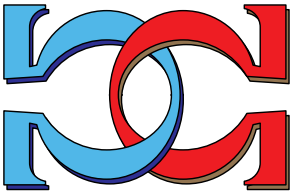

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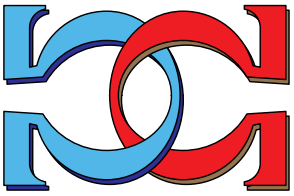
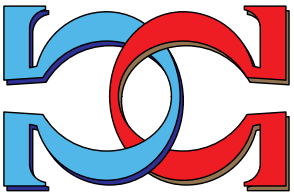
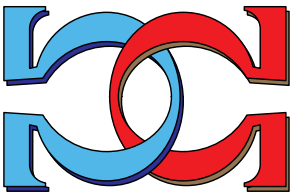




On the brightness of the Thomson lamp. A prolegomenon to quantum recursion theory



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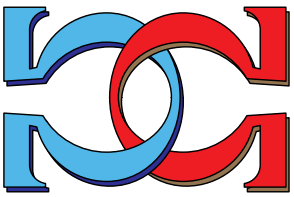

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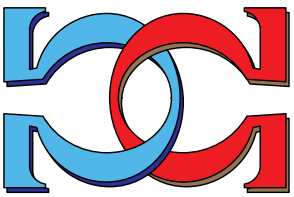

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On the brightness of the Thomson lamp. A prolegomenon to quantum recursion theory

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Abstract

Some physical aspects related to the limit operations of the Thomson lamp are discussed. Regardless of the formally unbounded and even infinite number of "steps" involved, the physical limit has an operational meaning in agreement with the Abel sums of infinite series. The formal analogies to accelerated (hyper)computers and the recursion theoretic diagonal methods are discussed. As quantum information is not bound by the mutually exclusive states of classical bits, it allows a consistent representation of fixed point states of the diagonal operator. In an effort to reconstruct the self-contradictory feature of diagonalization, a generalized diagonal method allowing no quantum fixed points is proposed.

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INTRODUCTION

Caveats at the very beginning of a manuscript may appear overly prudent and displaced, stimulating even more *caveats* or outright rejection. Yet, one should keep in mind that, to rephrase a *dictum* of John von Neumann [1], from an operational point of view [2], anyone who considers physical methods of producing infinity is, of course, in a state of sin. Such sinful physical pursuits qualify for Neils Henrik Abel's verdict that (Letter to Holmboe, January 16, 1826 [3, 4]), *"divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever."* This, of course, has prevented no-one, in particular not Abel himself, from considering these issues. Indeed, by reviving old eleatic ideas [5, 6], accelerated computations [7] have been the main paradigm of the fast growing field of hypercomputations (e.g., Refs. [8–10]). For the sake of hypercomputation, observers have been regarded on their path toward black holes [11–13], and automata have been densely embedded [14], to name just two "mind-boggling" proposals.

In what follows we shall discuss issues related to the quasi-physical aspects of hypercomputers, or more specifically, accelerated agents or processes approaching the limit of infinite computation. Thereby, we shall not be concerned with issues related to unbounded space or memory consumption discussed by Calude and Staiger [15], although we acknowledge their importance. In analogy to Benacerraf's discussion [16] of Thomson's proposal [17, 18] of a lamp which is switched on or off at geometrically decreasing time delays, Shagrir [19] suggested that the physical state has little or no relation to the states in the previous acceleration process.

This argument is not dissimilar to Bridgman's [2] argument against the use of Cantor-type diagonalization procedures on the basis that it is physically impossible to operationalize and construct some process "on top of" or after a nonterminating, infinite process. The method of diagonalization presents an important technique of recursion theory [20–22]. Some aspects of its physical realizability have already been discussed by the author [23–26]. In what follows we shall investigate some further physical issues related to "accelerated" agents and the operationalizability of the diagonalization operations in general.

CLASSICAL BRIGHTNESS OF THE THOMSON LAMP

The Thomson lamp is some light source which is switched off when it is on, and conversely switched on when it is off; whereby the switching intervals are geometrically (by a constant factor

smaller than unity) reduced or "squeezed." By construction, the switching process never "saturates," since the state always flips from "on" to "off" and then back to "on" again *ad infinitum;* yet, as this infinity of switching cycles is supposed to be reachable in finite time, one may wonder what happens at and after the accumulation point.

The Thomson process can formally be described at *intrinsic* (unsqueezed, unaccelerated) discrete time steps t; i.e., by the partial sum

$$s(t) = \sum_{n=0}^{t} (-1)^n = \begin{cases} 1 & \text{for even } t, \\ 0 & \text{for odd } t, \end{cases} \tag{1}$$

which can be interpreted as the result of all switching operations until time t. The intrinsic time scale is related to an *extrinsic* (squeezed, accelerated) time scale [27, p. 26]

$$\tau_0 = 0,\ \tau_{t>0} = \sum_{n=1}^{t} 2^{-n} = 2\left(1 - 2^{-t}\right) \quad . \tag{2}$$

In the limit of infinite intrinsic time $t \to \infty$, the proper time $\tau_\infty = 2$ remains finite.

If one encodes the physical states of the Thomson lamp by "0" and "1," associated with the lamp "on" and "off," respectively, and the switching process with the concatenation of "+1" and "-1" performed so far, then the divergent infinite series associated with the Tomson lamp is the Leibniz series [3, 28–30]

$$s = \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - \cdots \overset{\text{A}}{=} \frac{1}{2}. \tag{3}$$

Here, "A" indicates the Abel sum [3] obtained from a "continuation" of the geometric series, or alternatively, by $s = 1 - s$. As this shows, formal summations of the Leibnitz type (3) require specifications which could make them unique.

Moreover, divergent sums could be valuable *approximations* of solutions of differential equations which might be even *analytically solvable*. One of the best-known examples of this case is Euler's differential equation [4, 31, 32] $z^2 y' + y = z$, which has both (i) a formal solution as a power series $\hat{f}(z) = -\sum_{n=0}^{\infty} n!\,(-z)^n$, which *diverges* for all nonzero values of z; as well as (ii) an exact solution $\hat{f}(z) = e^{1/z}\mathrm{Ei}(-1/z)$ obtained by direct integration. Ei stands for the exponential integral $\mathrm{Ei}(z) = \int_z^\infty e^{-t}/t$. The difference between the exact and the series solution up to index k can be approximated [4] by $|f(z) - f_k(z)| \leq \sqrt{2\pi z} e^{-1/z}$, which is exponentially small for small z. To note another example, Laplace has successfully (with respect to the predictions made) used divergent power series for calculations of planetary motion. Thus it is not totally unreasonable to

suspect that the perturbation series solutions obtained via diagrammatical techniques [33–35] for the differential equations of quantum electrodynamics are of this "well behaved" type.

Therefore, at least in principle, the Thomson lamp could be perceived as a physical process governed by some associated differential equation, such as $y'(1-z)-1=0$, which has an exact solution $f(z)=\log(1/1-x)$; as well as a divergent series solution $\hat{f}(z)=\sum_{n=0}^{\infty}(-z)^n/n$, so that the first derivative of $\hat{f}(z)$, taken at $z=1$, yields the Leibniz series s.

If one is willing to follow the formal summation to its limit, then the Abel sum could be justified from the point of view of classical physics as follows. Suppose that the switching processes can be disregarded, an assumption not dissimilar to related concerns for Maxwell's demon [36]. Suppose further that all measurements are finite in the sense that the temporal resolution δ of the observation of the Thomson lamp cannot be made "infinitely small;" i.e., the observation time is finitely limited from below by some arbitrary but nonvanishing time interval. Then, the mean brightness of the Thomson lamp can be operationalized by measuring the *average* time of the lamp to be "on" as compared to being "off" in the time interval determined by the temporal resolution of the measuring device. This definition is equivalent to an idealized "photographic" plate or light-sensitive detector which additively collects light during the time the shutter is open. Then, such a device would record the cycles of Thomson's lamp up to temporal resolution δ, and would then integrate the remaining light emanating from it. Suppose that $\delta=1/2^t$, and that at the time $\sum_{n=1}^{t}2^{-n}=2\,(1-2^{-t})$, the Thomson lamp is in the "off" state. Then the "on" and "off" periods sum up to

$$\begin{aligned} s_0 &= \tfrac{1}{4}+\tfrac{1}{16}+\tfrac{1}{64}+\cdots=\tfrac{1}{3} \\ s_1 &= \tfrac{1}{2}+\tfrac{1}{8}+\tfrac{1}{32}+\cdots\cdots=\tfrac{2}{3}\,. \end{aligned} \tag{4}$$

By averaging over the initial time, there is a 50:50 chance that the Thomson lamp is "on" or "off" at highest resolution — which is equivalent to repeating the experiment with different offsets – resulting in an average brightness of $1/2$, which is identical to the Abel sum. In this way, the Abel sum can be justified from a physical point of view.

In a strict sense, this classical physics treatment of the brightness of the Thomson lamp is of little use for predictions of the state of the Thomson lamp *after* the limit of the switching cycles. One might still put forward that, in accordance with our operationalization of the average brightness, there is a 50:50 probability that it is in state "on" or "off." Just as for the Tomson lamp discussed above, one needs to be careful in defining the output of an accelerated Turing machine [19, 23–27, 37, 38].

QUANTUM STATE OF THE THOMSON LAMP

The quantum mechanical formalization of the Thomson lamp presents an intriguing possibility: as quantum information is not bound by two classically contradictory states, the quantum state of the Thomson lamp can be in a superposition thereof. At first glance, the fact that quantum information co-represents contradictory cases appears "mind-boggling" at best. As Hilbert pointedly stated (in Ref. [39, p. 163], he was not discussing quantum information theory): *"... the conception that facts and events could contradict themselves appears to me as an exemplar of thoughtlessness [56]."* Nevertheless, "quantum parallelism" appears to be an important property of quantum information and computation, which is formally expressed by a coherent summation of orthogonal pure states. This property can be used for quantum states which are formally invariant "fixed points" of diagonalization operators.

We note a particular aspect of the quantization of the electromagnetic field emanating from the Thomson lamp: the quanta recorded at the photographic emulsion or detector impinge on these light sensitive devices not continuously but in discrete "energy lumps" of $E_\nu = h\nu = hc/\lambda$, where ν and λ refer to the frequency and wavelength of the associated field mode. We shall also not be concerned with the photon statistics, which would require a detailed analysis of the light source [40].

In what follows, the notation of Mermin [41] will be used. Let $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$ be the representations of the "off" and "on" states of the Thomson lamp, respectively. Then, the switching process can be symbolized by the "not" operator $\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, transforming $|0\rangle$ into $|1\rangle$ and *vice versa*. Thus the quantum switching process of the Thomson lamp at time t is the partial product [42, Sect. 5.11]

$$\mathbf{S}(t) = \prod_{n=0}^{t} \mathbf{X}^t = \begin{cases} \mathbb{I}_2 & \text{for even } t, \\ \mathbf{X} & \text{for odd } t. \end{cases} \tag{5}$$

In the limit, one obtains an infinite product $\mathbf{S}$ of matrices with the two accumulation points mentioned above.

The eigensystem of $\mathbf{S}(t)$ is given by the two 50:50 mixtures of $|0\rangle$ and $|1\rangle$ with the two eigenvalues 1 and -1; i.e.,

$$\mathbf{S}(t)\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) = \pm\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) = \pm|\psi_\pm\rangle. \tag{6}$$

In particular, the state $|\psi_+\rangle$ associated with the eigenvalue $+1$ is a *fixed point* of the operator $\mathbf{S}(t)$. These are the two states which can be expected to emerge as the quantum state of the Thomson lamp in the limit of infinity switching processes. Note that, as for the classical case and for the formal Abel sum, they represent a fifty-fifty mixture of the "on" and "off" states.

QUANTUM FIXED POINT OF DIAGONALIZATION OPERATOR

In set theory, logic and in recursion theory [20–22, 43–45], the method of *proof by contradiction* (*reductio ad absurdum*) requires some "switch of the bit value," which is applied in a self-reflexive manner. Classically, total contradiction is achieved by supposing that some proposition is true, then proving that under this assumption it is false, and *vice versa*. The general proof method thus can be intuitively related to the application of a "not" operation.

The following considerations rely on the reader's informal idea of effective computability and algorithm, but they can be translated into a rigorous description involving universal computers such as Turing machines [44, Chapter C.1]. Assume that an algorithm A is restricted to classical bits of information. For the sake of contradiction, assume that there exists a (suspicious) halting algorithm h which outputs the code of a classical bit as follows:

$$h(B(X)) = \begin{cases} 0 & \text{whenever } B(X) \text{ does not halt,} \\ 1 & \text{whenever } B(X) \text{ halts.} \end{cases} \tag{7}$$

Then, suppose an algorithm A using h as a subprogram, which performs the diagonal argument by halting whenever (case 1) $h(B(X))$ indicates that $B(X)$ does not halt (diverges), and conversely (case 2) by not halting whenever $h(B(X))$ indicates that $B(X)$ halts (converges). Self-reflexivity reveals the antinomy of this construction: upon operating on its own code, A reaches at a total contradiction: whenever $A(A)$ halts, $h(A(A))$ outputs 1 and forces $A(A)$ not to halt. Conversely, whenever $A(A)$ does not halt, then $h(A(A))$ outputs 0 and steers $A(A)$ into the halting mode. In both cases one arrives at a complete contradiction. Classically, this contradiction can only be consistently avoided by assuming the nonexistence of A and, since the only nontrivial feature of A is the use of the peculiar halting algorithm h, the impossibility of any such halting algorithm.

As has already been argued [23–26], the quantum state $|\psi_+\rangle = (1/\sqrt{2})\,(|0\rangle + |1\rangle)$ is the quantum fixed point state of the "not" operator, which is essential for diagonal arguments, as $\mathbf{X}|\psi_+\rangle = |\psi_+\rangle$. Thus in quantum recursion theory, the diagonal argument consistently goes

through without leading to a contradiction, as $h(A(A))$ yielding $|\psi_+\rangle$ still allows a consistent response of A by a coherent superposition of its halting and nonhalting states. It should be noted, however, that the fixed point quantum "solution" of the halting problem cannot be utilized. In particular, if one is interested in the "classical" solution of the decision problem whether or not $A(A)$ halts, then one ultimately has to perform an irreversible measurement on the fixed point state. This causes a state reduction into the classical states corresponding to $|0\rangle$ and $|1\rangle$. Any single measurement yields an indeterministic result: According to the Born rule (cf. [46, p. 804], English translation in [47, p. 302] and [48]), when measured "along" the classical basis (observable) $\{|0\rangle, |1\rangle\}$, the probability that the fixed point state $|\psi_+\rangle$ returns at random one of the two classical states $|0\rangle$ (exclusive) or $|1\rangle$, is $1/2$. Thereby, classical undecidability is recovered. Thus, as far as problem solving is concerned, this method involving quantum information does not present an advantage over classical information. For the general case discussed, with regards to the question of whether or not a computer halts, the quantum "solution" fixed point state is equivalent to the throwing of a fair classical coin [49].

QUANTUM DIAGONALIZATION

The above argument used the continuity of quantum bit states as compared to the discrete classical spectrum of just two classical bit states for a construction of fixed points of the diagonalization operator modelled by $\mathbf{X}$. One could proceed a step further and allow *nonclassical diagonalization procedures*. Thereby, one could allow the entire range of two-dimensional unitary transformations [50]

$$\mathbf{U}_2(\omega,\alpha,\beta,\varphi) = e^{-i\beta}\begin{pmatrix} e^{i\alpha}\cos\omega & -e^{-i\varphi}\sin\omega \\ e^{i\varphi}\sin\omega & e^{-i\alpha}\cos\omega \end{pmatrix} \quad , \tag{8}$$

where $-\pi \leq \beta, \omega \leq \pi$, $-\frac{\pi}{2} \leq \alpha, \varphi \leq \frac{\pi}{2}$, to act on the quantum bit. A typical example of a nonclassical operation on a quantum bit is the "square root of not" gate operator ($\sqrt{\mathbf{X}} \cdot \sqrt{\mathbf{X}} = \sqrt{\mathbf{X}}$)

$$\sqrt{\mathbf{X}} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \quad , \tag{9}$$

which again has the fixed point state $|\psi_+\rangle$ associated with the eigenvalue $+1$. Yet, not all of these unitary transformations have eigenvectors associated with eigenvalues $+1$ and thus fixed points.

Indeed, only unitary transformations of the form

$$[\mathbf{U}_2(\omega,\alpha,\beta,\varphi)]^{-1}\,\mathrm{diag}(1,e^{i\lambda})\mathbf{U}_2(\omega,\alpha,\beta,\varphi) = \\ = \begin{pmatrix} \cos\omega^2 + e^{i\lambda}\sin\omega^2 & \frac{-1+e^{i\lambda}}{2}e^{-i(\alpha+\varphi)}\,\sin(2\,\omega) \\ \frac{-1+e^{i\lambda}}{2}e^{i(\alpha+\varphi)}\,\sin(2\,\omega) & e^{i\lambda}\cos\omega^2 + \sin\omega^2 \end{pmatrix} \tag{10}$$

for arbitrary λ have fixed points.

Applying nonclassical operations on quantum bits with no fixed points

$$[\mathbf{U}_2(\omega,\alpha,\beta,\varphi)]^{-1}\,\mathrm{diag}(e^{i\mu},e^{i\lambda})\mathbf{U}_2(\omega,\alpha,\beta,\varphi) = \\ = \begin{pmatrix} e^{i\mu}\cos(\omega)^2 + e^{i\lambda}\sin(\omega)^2 & \frac{e^{-i(\alpha+p)}}{2}\left(e^{i\lambda}-e^{i\mu}\right)\sin(2\,\omega) \\ \frac{e^{i(\alpha+p)}}{2}\left(e^{i\lambda}-e^{i\mu}\right)\sin(2\,\omega) & e^{i\lambda}\cos(\omega)^2 + e^{i\mu}\sin(\omega)^2 \end{pmatrix} \tag{11}$$

with $\mu,\lambda \neq n\pi$, $n \in \mathbb{N}_0$ gives rise to eigenvectors which are not fixed points, and which acquire nonvanishing phases μ,λ in the generalized diagonalization process. For a physical realization of theses diagonalization operators in terms of generalized beam splitters, the reader is referred to Refs. [51–54].

SUMMARY

In summary we have discussed some physical aspects related to the limit operations of the Thomson lamp. This physical limit, regardless of the formally unbounded and even infinite number of "steps" involved, has an operational meaning in agreement with the formal Abel sums of infinite series. We have also observed the formal analogies to accelerated (hyper)computers and have discussed the recursion theoretic diagonal methods. As quantum information is not bound by mutually exclusive states of classical bits, it allows a consistent representation of fixed point states of the diagonal operator. In an effort to reconstruct the self-contradictory feature of diagonalization and the resulting *reductio ad absurdum*, a generalized diagonal method allowing no quantum fixed points has been proposed.

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[56] *"... mir erscheint die Auffassung, als könnten Tatsachen und Ereignisse selbst miteinander in Widerspruch geraten, als das Musterbeispiel einer Gedankenlosigkeit."*