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On Oscillation-free ε-random Sequences

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Abstract

In this paper we discuss three notions of partial randomness or ε-randomness. ε-randomness should display all features of randomness in a scaled down manner. However, as Reimann and Stephan [15] proved, Tadaki [22] and Calude et al. [3] proposed at least three different concepts of partial randomness.

We show that all of them satisfy the natural requirement that any ε-non-null set contains an ε-random infinite word. This allows us to focus our investigations on the strongest one which is based on a priori complexity.

We investigate this concept of partial randomness and show that it allows—similar to the random infinite words—oscillation-free (w.r.t. to a priori complexity) ε-random infinite words if only ε is a computable number. The proof uses the dilution principle.

Alternatively, for certain sets of infinite words (ω-languages) we show that their most complex infinite words are oscillation-free ε-random. Here the parameter ε is also computable and depends on the set chosen.

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Contents

1 Introduction

Partial randomness was investigated in the papers by Tadaki [22] and Calude et al. [3]. It is a linear generalisation of Marin-Löf's concept of random sequences [13]. The concept of partial randomness tries to specify sequences as random to some degree ε, $0 < \varepsilon \leq 1$, where the case $\varepsilon = 1$ coincides with Martin-Löf randomness. In [22] and [3] several different generalisations of the concepts for random sequences were given. It turned out that some of them are equivalent, and there remained three approaches which were shown to be inequivalent recently by Reimann and Stephan [15].

To define random sequences (infinite words) Martin-Löf introduced the concept of sequential test and declared an infinite word as random if it withstands all sequential tests. It became clear soon that random infinite words are those which do not allow an unbounded increase of capital in a fair coin-tossing game when using semi-computable gambling strategies (see [16]). Schnorr [16] combined martingales (the capital functions in the game) with order functions to relativise the degree of randomness and in Section 17 of this book he considered martingales combined with exponential order functions, an idea which came up later in [19] and in a somewhat disguised form as s-gales in Lutz's papers [11, 12].

A different characterisation of Martin-Löf random sequences using the concept of Kolmogorov complexity was obtained by Levin, Schnorr and Chaitin. They used variants of Kolmogorov complexity. For a detailed description of the variants of Kolmogorov complexity see [24] and [10, Section 4.5.5]. Notably simple characterisations were obtained using prefix complexity (KP), monotone complexity (Km) and a priori complexity (KA).[1]

A simple idea what could be an example of a binary $\frac{1}{2}$-random infinite word is the following. Take $\xi = x_1 x_2 \cdots x_i \cdots$ to be a (1-)random infinite word and dilute it by inserting zeros at every other position to obtain $\xi' = x_1 0 x_2 0 \cdots x_i 0 \cdots$. This idea, of dilution was already used by Daley [5] to 'construct' infinite words having a Kolmogorov complexity function of a certain behaviour and appeared later in [18, 12, 3] to describe infinite words with large complexity oscillations.

As one observes easily the Kolmogorov complexity of the n-length prefix of a diluted word ξ' is about the complexity of the $\varepsilon \cdot n$-length

[1] We follow here, except for the monotone complexity, the notation of [24] who use KP, KM, and KA, whereas Li and Vitányi [10] use K, Km and KM, respectively.

prefix of the original word ξ where ε is the dilution coefficient (e.g. $\varepsilon = \frac{1}{2}$ in the above example). This was a motivation to consider the relative Kolmogorov complexity of an infinite word as the the limit of the quotient of the complexity of the n-length prefix and the length n (see [1, 17, 18]). Later it was discovered that the existence of Levin's universal semi-computable semi-measure [25] proves that this idea of relative Kolmogorov complexity coincides with Lutz's [12] constructive dimension (see [20] and the remark on p. 223 of [19]).

In our discussion on ε-randomness we will not pursue all lines indicated above but focus on Martin-Löf tests and prefix and a priori Kolmogorov complexities. Observe that universal semi-computable semi-measures and universal semi-computable martingales are in one-to-one correspondence and give rise to the definition of a priori complexity (see Section 2.1).

We first show that partial randomness based on on Martin-Löf tests and prefix and a priori Kolmogorov complexities all satisfy the natural requirement that non-null sets w.r.t. a related measure always contain partial random infinite words. Having shown that all these concepts are in some sense natural we focus on the strongest one, the one based on strong Martin-Löf tests or as shown in [3], equivalently, on a priory complexity.

For a priori complexity (1-)random infinite words show an oscillation-free behaviour (cf. [23]). This need not be true for ε-random infinite words (cf. [12, 3]). We investigate whether one can prove oscillation-freeness for partial random infinite words, too. We present proofs that, though non-1-random infinite words may display large complexity oscillations, ε-random infinite words having oscillation-free behaviour exist for all computable $\varepsilon > 0$.

We give two methods of 'construction' (or presentation) of such infinite words. The first is by dilution of 1-random infinite words, and the second by 'choosing' most complex infinite words in suitably defined sets of infinite words.

2 Notation and Preliminaries

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers and by $\mathbb{Q}$ the set of rational numbers. Let X be an alphabet of cardinality $|X| = r \geq 2$. By X^* we denote the set of finite words on X, including the *empty word* e, and X^ω is the set of infinite strings (ω-words) over X. Subsets of X^*

will be referred to as *languages* and subsets of X^ω as ω-*languages*.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. For a language W let $W^* := \bigcup_{i\in\mathbb{N}} W^i$, and by $W^\omega := \{w_1 \cdots w_i \cdots : w_i \in W \setminus \{e\}\}$ we denote the set of infinite strings formed by concatenating words in W. Furthermore $|w|$ is the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\eta)$ $(\eta \in X^* \cup X^\omega)$ by $w \sqsubseteq \eta$, and $\eta[0..n]$ is the n-length prefix of η provided $|\eta| \geq n$. A language $W \subseteq X^*$ is referred to as *prefix-free* provided $w \sqsubseteq v$ and $w, v \in W$ imply $w = v$.

We denote by $B/w := \{\eta : w \cdot \eta \in B\}$ the *left derivative* of the set $B \subseteq X^* \cup X^\omega$. A language $W \subseteq X^*$ is *regular* provided its set of left derivatives $\{W/w : w \in X^*\}$ is finite. In the sequel we assume the reader to be familiar with basic facts of language theory. As usual, the class of recursively enumerable languages is denoted by Σ_1, the class containing their complements by Π_1. Thus, $\Sigma_1 \cap \Pi_1$ is the class of recursive languages.

We consider the set X^ω as a metric space (Cantor space) (X^ω, ϱ) of all ω-words over the alphabet X where the metric ϱ is defined as follows.

$$\varrho(\xi, \eta) := \inf\{r^{-|w|} : w \sqsubset \xi \wedge w \sqsubset \eta\} .$$

This space is a compact, and $\mathcal{C}(F) := \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$ turns out to be the *closure* of the set F (smallest closed subset containing F) in (X^ω, ϱ).

2.1 Randomness and Kolmogorov complexity

A semi-measure on X^ω is defined by a function (called semi-measure on X^*) $\nu : X^* \to [0, \infty)$ having the following property:

$$\nu(e) > 0 \text{ and } \nu(w) \geq \textstyle\sum_{x\in X} \nu(wx) \text{ for all } w \in X^* . \tag{1}$$

This defines via $M_\nu(w \cdot X^\omega)$ a pre-measure M_ν on the balls $w \cdot X^\omega$ which can be extended to all Borel subsets of X^ω (cf. [6]). The usual Lebesgue measure on X^ω is obtained via the (semi-)measure $\mu(w) := r^{-|w|}$.

Recall further that a function $f : X^* \to (-\infty, \infty)$ is called *left computable* if the set of lower bounds $\{(w, q) : q \in \mathbb{Q} \wedge q < f(w)\} \in \Sigma_1$. Levin proved in [25] that there is a universal left computable semi-measure $\mathbf{M}$, that is, for every left computable semi-measure ν there is a constant c_ν such that $\forall w(w \in X^* \to \nu(w) \leq c_\nu \cdot \mathbf{M}(w))$.

We briefly recall the concept of Kolmogorov complexity of finite words. For a more comprehensive introduction see the textbooks [2] and [10]. To this end let $\varphi : X^* \to X^*$ be a partial-recursive function. The complexity of a word $w \in X^*$ with respect to φ is defined as

$$K_\varphi(w) := \inf\{|\pi| : \pi \in X^* \wedge \varphi(\pi) = w\}. \tag{2}$$

It is well known that there is an optimal partial-recursive function $\mathfrak{U} : X^* \to X^*$, that is, a function satisfying that for every partial-recursive function φ

$$\exists c_\varphi \forall w (w \in X^* \to K_{\mathfrak{U}}(w) \leq K_\varphi(w) + c_\varphi) \tag{3}$$

If one considers only partial-recursive functions φ with prefix-free domain $dom(\varphi) \subseteq X^*$ we obtain in the same way an optimal partial-recursive function $\mathfrak{U}_p$.

Proposition 2.1 *There is a partial recursive function $\mathfrak{U}_p : X^* \to X^*$ with prefix-free domain $dom(\mathfrak{U}_p)$ such that for every partial-recursive functions φ with prefix-free domain $dom(\varphi)$ there is a constant c_φ such that*

$$\forall w (w \in X^* \to K_{\mathfrak{U}_p}(w) \leq K_\varphi(w) + c_\varphi) .$$

Following [10] the complexity $\mathrm{KP} := K_{\mathfrak{U}_p}$ will be called *prefix complexity*.

From Levin's universal left computable semi-measure one derives the *a priori* complexity $\mathrm{KA}(w) := -\log_{|X|} \mathbf{M}(w)$ (cf. [10, 23, 24, 20]).

Finally, we recall the concept of Martin-Löf-tests.

Definition 2.2 A recursively enumerable set $\mathfrak{V} \subseteq X^* \times \mathbb{N}$ is referred to as a *sequential Martin-Löf-test* provided $V_{i+1} \cdot X^\omega \subseteq V_i \cdot X^\omega$, where $V_i := \{v : (v, i) \in \mathfrak{V}\}$, and $M_\mu(V_i \cdot X^\omega) < 2^{-i}$.

An ω-word is called *Martin-Löf-random* (*ML-random*) provided $\xi \notin \bigcap_{i \in \mathbb{N}} V_i \cdot X^\omega$ for all sequential Martin-Löf-tests.

Then the following equivalences are known (see e.g. [2, 10] and [23]).

Theorem 2.3 *Let $\xi \in X^\omega$. Then the following conditions are equivalent.*

1. *ξ is Martin-Löf-random.*
2. $\mathrm{KP}(\xi[0..n]) \geq_{\mathrm{ae}} n - O(1)$,
3. $\lim_{n\to\infty} \mathrm{KP}(\xi[0..n]) - n = \infty$, *and*
4. $\mathrm{KA}(\xi[0..n]) \geq_{\mathrm{ae}} n - O(1)$.

2.2 ε-randomness

In this part we briefly summarise the results of [22] and [3] on ε-randomness and Reimann's and Stephan's hierarchy result [15]. First we relativise the concept of Martin-Löf test in two ways.

Definition 2.4 A recursively enumerable set $\mathfrak{V} \subseteq X^* \times \mathbb{N}$ is referred to as an *Martin-Löf-ε-test* provided

1. $V_{i+1} \cdot X^\omega \subseteq V_i \cdot X^\omega$, and
2. $\forall i \big(\mu^{(\varepsilon)}(V_i) := \sum_{v \in V_i} r^{-\varepsilon \cdot |v|} < r^{-i}\big)$.

A set $\mathfrak{V} \subseteq X^* \times \mathbb{N}$ is called a *strong Martin-Löf-ε-test* if it satisfies 1. and

2'. $\forall i \, \forall C \big(C \subseteq V_i \wedge C \text{ is prefix-free } \rightarrow \mu^{(\varepsilon)}(C) < r^{-i}\big)$.

We call $\xi \in X^\omega$ (*strongly*) *Martin-Löf-ε-random* if and only if $\xi \notin \bigcap_{i \in \mathbb{N}} V_i \cdot X^\omega$ for all (strong) Martin-Löf-ε-tests.

In fact, every Martin-Löf-ε-test is a strong Martin-Löf-ε-test, the attribute strong refers to the fact (supported by Theorem 2.8 below) that not every ML-ε-random ω-word is also strongly ML-ε-random. The following equivalences between Martin-Löf-ε-tests and Kolmogorov complexity are known.

Lemma 2.5 ([22]) *Let $0 < \varepsilon \leq 1$ be computable. Then an ω-word $\xi \in X^\omega$ is ML-ε-random if and only if* $\mathrm{KP}(\xi[0..n]) \geq_{\mathrm{ae}} \varepsilon \cdot n - O(1)$.

Lemma 2.6 ([3]) *Let $0 < \varepsilon \leq 1$ be computable. Then an ω-word $\xi \in X^\omega$ is strongly ML-ε-random if and only if* $\mathrm{KA}(\xi[0..n]) \geq_{\mathrm{ae}} \varepsilon \cdot n - O(1)$.

Another possibility is to generalise condition 3 of Theorem 2.3.

Definition 2.7 ([22]) An ω-word $\xi \in X^\omega$ is referred to as *strongly Chaitin-ε-random* provided $\lim_{n \to \infty} \mathrm{KP}(\xi[0..n]) - \varepsilon \cdot n = \infty$.

The hierarchy of these notions was finally established in the paper by Reimann and Stephan..

Theorem 2.8 ([15]) *Let $0 < \varepsilon < 1$ be a rational number. Then every ML-ε-random ω-word is strongly Chaitin-ε-random, and every strongly Chaitin-ε-random ω-word is strongly ML-ε-random, and none of these implications can be reversed.*

Here the question arises which one of the concepts of ε-randomness is a natural generalisation of (1-)randomness. For 1-randomness it is known that every Lebesgue non-null set $F \subseteq X^\omega$ contains a random ω-word. A similar condition for $\varepsilon < 1$ can be formulated using Hausdorff dimension and measure.

We recall the definition of the Hausdorff measure and Hausdorff dimension of a subset of (X^ω, ϱ) (see e.g. [6, 7]). In the setting of languages this can be read as follows (see e.g. [18]). For $F \subseteq X^\omega$ and $0 \leq \gamma \leq 1$ the equation

$$\mathbb{L}_\gamma(F) := \lim_{l\to\infty} \inf\Big\{ \sum_{w\in W} r^{-\gamma\cdot|w|} : F \subseteq W \cdot X^\omega \wedge \forall w(w \in W \rightarrow |w| \geq l) \Big\} \quad (4)$$

defines the γ-*dimensional metric outer measure* on X^ω. The measure $\mathbb{L}_\gamma$ satisfies the following.

Corollary 2.9 *If* $\mathbb{L}_\gamma(F) < \infty$ *then* $\mathbb{L}_{\gamma+\delta}(F) = 0$ *for all* $\delta > 0$.

Then the *Hausdorff dimension* of F is defined as

$$\dim F := \sup\{\gamma : \gamma = 0 \vee \mathbb{L}_\gamma(F) = \infty\} = \inf\{\gamma : \mathbb{L}_\gamma(F) = 0\}.$$

Theorem 2.10 ([14]) *Let* $F \subseteq X^\omega$ *and* $\mathbb{L}_\varepsilon(F) > 0$. *Then for every constant* $c > -\log_{|X|} \mathbb{L}_\varepsilon(F)$ *there is a* $\xi \in F$ *such that* $\mathrm{KA}(\xi[0..n]) \geq \varepsilon \cdot n - c$.

This theorem proves that, for computable ε, every $\mathbb{L}_\varepsilon$-non-null set contains a strongly ML-ε-random ω-word.

A similar theorem proving that, for computable ε, every $\mathbb{L}_\varepsilon$-non-null set contains a strongly Chaitin-ε-random ω-word can be found in [3, Corollary 5.6].

3 Oscillation-free ε-random ω-words

Random ω-words ξ satisfy, except for the lower bounds mentioned in Theorem 2.3 also the upper bounds $\mathrm{KP}(\xi[0..n]) \leq n + \mathrm{KP}(\iota(n)) + O(1)$, where $\iota(n)$ is the nth word in a recursive enumeration of X^*, and $\mathrm{KA}(\xi[0..n]) \leq n + O(1)$ (see [2, 10, 23]). For ω-words of lower complexity low upper bounds on the complexity need not be true. As it was mentioned in the introduction, there are ω-words having large complexity oscillations.

Tadaki showed that there are strongly Chaitin-ε-random ω-words ξ having $\kappa(\xi) := \limsup_{n\to\infty} \frac{\mathrm{KP}(\xi[0..n])}{n} \leq \varepsilon$. In this section we want to show that it is possible, for computable reals $0 < \varepsilon < 1$, to 'construct' ε-random ω-words satisfying $\mathrm{KA}(\xi[0..n]) \leq \varepsilon \cdot n + O(1)$, that is, ε-random ω-words having no oscillation w.r.t. the a priori complexity KA.

We derive two methods. The first one is a generalisation of the dilution principle and uses prefix-monotone recursive mappings $\varphi : X^* \to X^*$. The second one selects maximal complex ω-words in suitably chosen constructively given subsets of X^ω.

3.1 A generalised dilution principle

In this section we consider *prefix-monotone* mappings, that is, mappings $\varphi : X^* \to X^*$ satisfying $\varphi(w) \sqsubseteq \varphi(v)$ whenever $w \sqsubseteq v$. We call a function $g : \mathbb{N} \to \mathbb{N}$ a *modulus function* for φ provided $|\varphi(w)| = g(|w|)$ for all $w \in X^*$. This, in particular, implies that $|\varphi(w)| = |\varphi(v)|$ for $|w| = |v|$ when φ has a modulus function.

Every prefix-monotone mapping $\varphi : X^* \to X^*$ defines as a limit a partial mapping $\overline{\varphi} :\subseteq X^\omega \to X^\omega$ in the following way: $\mathbf{pref}(\overline{\varphi}(\xi)) = \mathbf{pref}(\varphi(\mathbf{pref}(\xi)))$ whenever $\varphi(\mathbf{pref}(\xi))$ is an infinite set, and $\overline{\varphi}(\xi)$ is undefined when $\varphi(\mathbf{pref}(\xi))$ is finite.

We obtain our first result.

Theorem 3.1 *Let $\varphi : X^* \to X^*$ be a one-to-one prefix-monotone recursive function with strictly increasing modulus function $g : \mathbb{N} \to \mathbb{N}$. Then $\overline{\varphi} : X^\omega \to X^\omega$ is also one-to-one and*

$$\big|\mathrm{KA}\big(\overline{\varphi}(\xi)[0..g(n)]\big) - \mathrm{KA}\big(\xi[0..n]\big)\big| \leq O(1) \textit{ for all } \xi \in X^\omega \textit{ and all } n \in \mathbb{N}\,.$$

Proof. The mapping $\overline{\varphi}$ is one-to-one because $w \in X^*$, $x, y \in X$ and $x \neq y$ imply that $\varphi(wx)$ and $\varphi(wy)$ are incomparable w.r.t. $\sqsubseteq$.

In order to prove $\mathrm{KA}\big(\overline{\varphi}(\xi)[0..g(n)]\big) \geq \mathrm{KA}(\xi[0..n]) - c$ we consider the semi-measure $\nu : X^* \to [0, \infty)$ defined by $\nu(w) := \mathbf{M}(\varphi(w))$. It is immediate that ν is left computable. Since $\mathbf{M}(\varphi(w)) \geq \sum_{v \in X^{g'(w)}} \mathbf{M}(\varphi(w) \cdot v) \geq \sum_{x \in X} \mathbf{M}(\varphi(wx))$ where $g'(w) := g(|w|) - g(|w| - 1)$ the function ν is indeed a semi-measure[2]. Thus $\mathbf{M}(\varphi(w)) = \nu(w) \leq c \cdot \mathbf{M}(w)$ yields the assertion.

To prove the converse, we define

$$\begin{array}{rcll} \nu'(e) & := & \mathbf{M}(e) & \text{and} \\ \nu'(v) & := & \sum_{\varphi(w) \sqsupseteq v, |v| > g(|w|-1)} \mathbf{M}(w) & \text{for } v \neq e\,. \end{array}$$

In particular, $\nu'(\varphi(w)) = \mathbf{M}(w)$. The sum is finite, thus ν' is left computable.

[2] We set $g(-1) := 0$.

Moreover we have

$$\sum_{y\in X}\nu'(vy) = \sum_{y\in X}\sum_{\substack{\varphi(w)\sqsupseteq vy\\ |vy|>g(|w|-1)}}\mathbf{M}(w)$$

$$= \begin{cases} \sum\limits_{\substack{\varphi(w)\sqsupseteq v\\ |v|>g(|w|-1)}}\mathbf{M}(w) & \text{, if } |v|>g(|w|-1) \text{ and}\\ \sum\limits_{y\in X}\sum\limits_{x\in X}\sum\limits_{\varphi(w'x)\sqsupseteq vy}\mathbf{M}(w'x) & \text{, } \begin{array}{l}\text{where } \varphi(w')=v\,,\\ \text{if } |v|=g(|w|-1)\,.\end{array}\end{cases}$$

The sum in the former case is $\nu'(v)$. In the latter case, $\varphi(w'x)\sqsupseteq\varphi(w')y$ implies $\varphi(w'x)\not\sqsupseteq\varphi(w')y'$ whenever $y,y'\in X$, $y'\neq y$. Consequently, y is uniquely determined by x and the sum simplifies to $\sum\limits_{x\in X}\sum\limits_{\varphi(w'x)\sqsupseteq vy}\mathbf{M}(w'x)\leq\sum\limits_{x\in X}\mathbf{M}(w'x)\leq\mathbf{M}(w')=\nu'(v)$.

Now, $\mathbf{M}(w)=\nu'(\varphi(w))\leq c'\cdot\mathbf{M}(\varphi(w))$ yields $\mathrm{KA}\big(\overline{\varphi}(\xi)[0..g(n)]\big)\leq\mathrm{KA}(\xi[0..n])-\log_r c'$. ❏

We need still the following technical result on computable reals ε, $0<\varepsilon<1$.

Lemma 3.2 *Let ε, $0<\varepsilon<1$, be computable. Then there are $c_1,c_2>0$ and an increasing recursive function $g:\mathbb{N}\to\mathbb{N}$ such that $0<g(n+1)-g(n)\leq c_1$ and $|n-\varepsilon\cdot g(n)|\leq c_2$, for all $n\in\mathbb{N}$.*

For the sake of completeness we give a proof.

Proof. The function $g:\mathbb{N}\to\mathbb{N}$ can be defined as follows.

$$\begin{aligned} g(0) &:= 0 \text{ and}\\ g(n+1) &:= g(n)+\min\{k : k\in\mathbb{N}\wedge k\geq 1\wedge \tfrac{n+1}{g(n)+k}<\varepsilon\}\end{aligned}$$

Thus $0\leq\varepsilon\cdot g(n)-n\leq\varepsilon$ and $0<g(n+1)-g(n)\leq 1+\frac{1}{\varepsilon}$, for all $n\in\mathbb{N}$. Moreover, g is a computable function provided $\varepsilon, 0<\varepsilon<1$, is a computable real number. ❏

For the particular g constructed in the proof of Lemma 3.2 we have $g(n)\leq\ell<g(n+1)$ if and only if $\lfloor\varepsilon\cdot\ell\rfloor=n$. This yields the following consequence of Theorem 3.1.

Theorem 3.3 *Let ε, $0<\varepsilon<1$ be a computable number. Then there is a a one-to-one prefix-monotone recursive function $\varphi:X^*\to X^*$ with strictly increasing modulus function $g:\mathbb{N}\to\mathbb{N}$ such that*

$$\big|\mathrm{KA}\big(\overline{\varphi}(\xi)[0..\ell]\big)-\mathrm{KA}\big(\xi[0..\lfloor\varepsilon\cdot\ell\rfloor]\big)\big|\leq O(1)$$

for all $\ell\in\mathbb{N}$ and all $\xi\in X^\omega$ satisfying $\mathrm{KA}(\xi[0..n+1])\leq\mathrm{KA}(\xi[0..n])+O(1)$.

In particular, if we choose $\xi \in X^\omega$ to be random then $\overline{\varphi}(\xi)$ is non-oscillating strongly ML-ε-random.

We conclude this section by an example which shows that not requiring these strong assumptions on the mappings φ in Theorem 3.1 and on the modulus in Lemma 3.2 may lead to large complexity oscillations in $\overline{\varphi}(X^\omega)$.

Example 3.4 Let $m_i := \sum_{j=0}^{2i} j!$ a sequence of rapidly growing natural numbers and define the prefix-monotone mapping $\varphi : \{0,1\}^* \to \{0,1\}^*$ as follows.

$$\begin{aligned} \varphi(e) &:= e \\ \varphi(wa) &:= \begin{cases} \varphi(w)a & \text{, if } |\varphi(w)a| \notin \{m_i : i \in \mathbb{N}\} \text{ and} \\ \varphi(w)a0^{(2i+1)!} & \text{, if } |\varphi(w)a| = m_i. \end{cases} \quad , \end{aligned}$$

that is, φ dilutes the input by rarely inserting very long blocks of zeros. Then $\overline{\varphi}(\{0,1\}^\omega) = \prod_{i=0}^{\infty}\{0,1\}^{(2i)!} \cdot 0^{(2i+1)!}$.

Now, one easily observes that the ω-word $\mathbf{x}$ 'constructed' in [3, Example 5.2] belongs to $\overline{\varphi}(\{0,1\}^\omega)$. For every $\delta > 0$ this ω-word has infinitely many prefixes $w \sqsubset \mathbf{x}$ with $\mathrm{KP}(w) \le \delta \cdot |w|$ and infinitely many prefixes $v \sqsubset \mathbf{x}$ with $\mathrm{KP}(v) \ge (1-\delta) \cdot |v|$.

3.2 Maximally complex ω-words

In [18] it was shown that for regular ω-languages $F \subseteq X^\omega$ and (simple) Kolmogorov complexity most complex ω-words in F show the same (scaled down by a factor $\dim F$) behaviour of their complexity function $\mathrm{K}(\xi[0..n])$ as (1-)random ω-words. In this section we transfer this result to the a priori complexity KA.

As usual, we call an ω-language $F \subseteq X^\omega$ *regular* provided there are an $n \in \mathbb{N}$ and regular languages W_i, V_i, $i = 1, \ldots, n$, such that $F = \bigcup_{i=1}^{n} W_i \cdot V_i^\omega$. As mentioned in [18, Theorem 1.8] the languages V_i can be chosen to be prefix-free.

The lower bound can be derived via Theorem 2.10 from [18, Theorem 4.7].

Lemma 3.5 ([18]) *If $F \subseteq X^\omega$ is a non-empty regular ω-language then* $\mathbb{L}_{\dim F}(F) > 0$.

Corollary 3.6 *Let $F \subseteq X^\omega$ be regular and* $\dim F > 0$. *Then F contains a strongly ML-*$\dim F$*-random ω-word.*

For the proof of the upper bound we need some more known facts on regular ω-languages and their Hausdorff dimension.

Lemma 3.7 *If $V \subseteq X^*$ is a non-empty prefix-free regular language then there is a unique value $1 \geq \gamma \geq 0$ such that $\sum_{v\in V} r^{-\gamma\cdot|v|} = 1$ and this value satisfies $\gamma = \dim V^\omega$.*

The following identity is useful to estimate the Hausdorff dimension of a regular ω-language.

$$\dim \bigcup_{i=1}^{n} W_i \cdot V_i^\omega = \max\{\dim V_i^\omega : i = 1, \ldots, n\} \tag{5}$$

Before proceeding to our upper bound we mention still that also the a priori complexity of an ω-word does not increase much by pre-multiplication with a finite word.

$$\forall w \exists c_w (\mathrm{KA}((w\cdot\xi)[0..n]) \leq \mathrm{KA}(\xi[0..n]) + c_w) \tag{6}$$

Now we can prove our results. Similar results hold for simple Kolmogorov complexity (see [18, Theorem 4.8]).

Theorem 3.8 *Let $F \subseteq X^\omega$ be a regular ω-language and let $\dim F > 0$. Then for every $\xi \in F$ there is a c_ξ such that $\mathrm{KA}(\xi[0..n]) \leq \dim F \cdot n + c_\xi$.*

In view of Lemma 3.7 and Eqs. (5) and (6) the proof of our theorem follows from the subsequent lemma which shows that for regular ω-languages of a special shape we can do better.

Lemma 3.9 *Let $V \subseteq X^*$ be a prefix-free regular language having at least two elements. Then for the unique value $\gamma > 0$ such that $\sum_{v\in V} r^{-\gamma\cdot|v|} = 1$ there is a $c > 0$ such that $\mathrm{KA}(\xi[0..n]) \leq \gamma\cdot n + c$ for every ξ in the closure of V^ω, $\mathcal{C}(V^\omega)$, that is, for all ξ with $\mathbf{pref}(\xi) \subseteq \mathbf{pref}(V^\omega)$.*

Proof. Let $\gamma > 0$ be the unique solution of $\sum_{v\in V} r^{-\gamma\cdot|v|} = 1$. We define a computable measure ν on X^* as follows.

1. $\nu(w) := r^{-\gamma\cdot|w|}$ if $w \in V^*$,
2. $\nu(w) := \sum_{wv\in V} r^{-\gamma\cdot|w\cdot v|}$ if $w \in \mathbf{pref}(V)$,
3. $\nu(w) := 0$ if $w \notin \mathbf{pref}(V^*) = V^* \cdot \mathbf{pref}(V)$, and
4. $\nu(w) := \nu(u)\cdot\nu(v)$ if $w = u\cdot v$ with $u \in V^*$ and $v \in \mathbf{pref}(V) \setminus V$.

Observe that, for $w \in V^* \cap \mathbf{pref}(V) = V \cup \{e\}$, (i) and (ii) coincide and that the decomposition in (iv) is unique because V is prefix-free.

In view of (ii) the identity $\sum_{x\in X} \nu(wx) = \nu(w)$ is obvious for $w \in \mathbf{pref}(V) \setminus V$. Then the general identity $\sum_{x\in X} \nu(wx) = \nu(w)$ follows from the inductive definition of ν.

Now, consider $\nu(w) := \sum_{wv\in V} r^{-\gamma\cdot|w\cdot v|} = r^{-\gamma\cdot|w|} \cdot \sum_{v\in V/w} r^{-\gamma\cdot|v|}$ for $w \in \mathbf{pref}(V) \setminus V$. Since V is regular, the set $\{V/v : v \in X^*\}$ is finite. Thus, the minimum $c' := \min\{\sum_{v\in V/w} r^{-\gamma\cdot|v|} : w \in \mathbf{pref}(V) \setminus V\}$ exists and is positive. Then in view of (iv), we obtain $\nu(w) \geq c' \cdot r^{-\gamma\cdot|w|}$ for $w \in \mathbf{pref}(V^*)$.

On the other hand $c \cdot \mathbf{M}(w) \geq \nu(w)$ for a suitable $c > 0$ and all $w \in X^*$. This yields $\mathrm{KA}(w) \leq \gamma \cdot |w| - \log_r \frac{c}{c'}$ for all $w \in \mathbf{pref}(V^*) = \mathbf{pref}(V^\omega)$. ❏

It would be desirable to extend this construction to a broader class of ω-languages, but already for very simple classes of non-regular ω-languages a general construction fails. Here the corresponding examples can be taken from [18, 21] and the estimates for the asymptotic complexities $\kappa(\xi) := \limsup_{n\to\infty} \frac{\mathrm{KP}(\xi[0..n])}{n}$ and $\underline{\kappa}(\xi) := \liminf_{n\to\infty} \frac{\mathrm{KP}(\xi[0..n])}{n}$ in Section 6 of [18] yield ω-words with large complexity oscillations, that is, having $\kappa(\xi) - \underline{\kappa}(\xi) > 0$:

Example 3.10 Let $X := \{0,1,2,3\}$ and set $W := \bigcup_{n=1}^{\infty} \{0,1\}^n \cdot 2^{3n}$. For this prefix-free linear language of simple structure we have $\dim W^\omega = \frac{1}{4}$ ([21]) and thus $\underline{\kappa}(\xi) \leq \frac{1}{4}$ for all $\xi \in W^\omega$. On the other hand, Corollary 6.11 and Eq. (6.13) of [18] show that there are $\xi \in W^\omega$ with $\kappa(\xi) \geq \frac{1}{2}$.

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