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Indifferent Sets

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Abstract

We define the notion of *indifferent set* with respect to a given class of $\{0,1\}$-sequences. Roughly, for a set A in the class, a set of natural numbers I is *indifferent for* A with respect to the class if it does not matter how we change A at the positions in I: the new sequence continues to be in the given class. We are especially interested in studying those sets that are indifferent with respect to classes containing different types of stochastic sequences.

For the class of Martin-Löf random sequences, we show that every random sequence has an infinite indifferent set and that there is no universal indifferent set. We show that indifferent sets must be sparse, in fact sparse enough to decide the halting problem. We prove the existence of co-c.e. indifferent sets, including a co-c.e. set that is indifferent for every 2-random sequence with respect to the class of random sequences.

For the class of absolutely normal numbers, we show that there are computable indifferent sets with respect to that class and we conclude that there is an absolutely normal real number in every non-trivial many-one degree.

1 Introduction

Intuitively a random sequence A of 0s and 1s should be indistinguishable from one produced by tossing a coin infinitely many times and writing 0 if it comes up heads and 1 if it comes up tails. Now, we can transform A by flipping any single bit, but the transformed sequence continues to be random, since the notion of randomness does no depend on the value of a single bit. Moreover, even with an intuitive and informal notion of randomness, it is reasonable to think that replacing finitely many bits of A by arbitrary fixed bits would not turn A into a non-random sequence. However, changing infinitely many bits does not necessarily preserves randomness. Even if one keeps infinitely many bits from the original A, the transformed sequence may fail to be random. Indeed, suppose we transform $A = h_0h_1h_2h_3\dots$ into

$\tilde{A} = 0h_1 0h_3 0h_5 \ldots$ It is not reasonable to think that $\tilde{A}$ is random, since all even positions are clearly predictable. This means that the set $I = 2\mathbb{N}$ is not *indifferent* for A, in the sense that it is not true that every B that agrees with A on the positions of $\mathbb{N} \setminus I$ is also random. Here, the problem seems to be that the set I of even numbers is not sparse enough. On the other hand, any finite set D is indifferent for A, because every B that agrees with A on all the positions of $\mathbb{N} \setminus D$ is also random.

Let us give a formal definition of the notion of indifferent set. For any $A, B \in 2^\omega$ and a further $X \subseteq \mathbb{N}$, we say that A *agrees with* B *on* X if $A(x) = B(x)$ for every $x \in X$. Then the sequence A is the same as B except, perhaps, at the positions not in X.

Definition 1. For a class $\mathcal{C} \subseteq 2^\omega$, a set $A \in \mathcal{C}$ and a further set $I \subseteq \mathbb{N}$, we say that I is *indifferent for* A *with respect to* $\mathcal{C}$ if each set B that agrees with A on $\mathbb{N} \setminus I$ is also in $\mathcal{C}$. In this case, we also say that I is $\mathcal{C}$*-indifferent for* A. We just say that I is $\mathcal{C}$*-indifferent* if it is indifferent for some $A \in \mathcal{C}$, and when the class $\mathcal{C}$ is clear from the context, we simply say that I is *indifferent*.

In this paper we tackle some questions related to the existence of indifferent sets and we investigate some of their computability theoretic properties. We are especially interested in studying those sets that are indifferent with respect to the class of Martin-Löf random sequences, but we also analyze indifferent sets with respect to a much larger class: the absolutely normal numbers.

In Section 3 we answer the first fundamental question of whether there are infinite indifferent sets with respect to the class of Martin-Löf random sequences. More specifically, in Corollary 6 we prove that every random A has an A'-computable infinite indifferent set. Taking A Martin-Löf random and low, this implies that there is an infinite Δ^0_2 set I that is indifferent for a A. In Theorem 7 we show that I may be chosen in such a way that every set B that agrees with A on $\mathbb{N} \setminus I$ is GL_1 and in Theorem 9 we prove that, roughly speaking, the set I may also contain blocks of every length. This last result may seem contrary to our intuition that an indifferent set I should be sparse, since it is not necessary for every two single elements of I to be very far apart one from the other. We also prove that there is no single universal indifferent set: Theorem 10 shows that for every indifferent I there is a random for which I is not indifferent.

As we mentioned above, one suspects that indifferent sets must be sparse in some way. Section 4 makes this precise. Let I be an infinite set and view it as the range of a strictly increasing function $p\colon \mathbb{N} \to \mathbb{N}$. Theorem 12 states that if I is a co-c.e. indifferent set (the existence of these sets is shown in Section 5), then p dominates every partial recursive function, so I is quite sparse. Dropping the condition that I is co-c.e., we can still show that function $n \mapsto p(n^2)$ dominates every partial recursive function. From this, we conclude in Corollary 14 that p dominates every total recursive function (so I is *dominant*) and that $I \geq_T \emptyset'$.

Once we know that there are infinite indifferent sets in Δ^0_2, a natural question is whether there are, for instance, c.e. or co-c.e. indifferent sets with respect to the class of Martin-Löf random sequences. No c.e. set can be dominant, ruling out one possibility. Section 5 explores co-c.e. indifferent sets. In Theorem 15 it is shown that every low Martin-Löf random sequence has a co-c.e. indifferent set and in Corollary 19 we show that there is a co-c.e. set indifferent for every 2-random sequence (still with respect to the class of Martin-Löf random sequences), hence for almost every sequence.

In Section 6 we turn our attention to the class of absolutely normal reals, a notion of *randomness* much weaker than Martin-Löf randomness. Absolutely normal reals are those that satisfy the law of large numbers, in a generalized sense (see page 17 for the formal definition). Using a proof technique similar to the one used for the proof of Theorems 7 and 17, we conclude in Corollary 21 that there is a computable set I that is indifferent for a computable A with respect to the class of absolutely normal reals. This implies that there are absolutely normal reals in every non-trivial many-one degree (Corollary 22), a situation that is clearly false for Martin-Löf randomness.

We have mainly studied indifference for Martin-Löf randomness. The notion can also be investigated for other randomness notions, such as Schnorr randomness, and even for classes such as 1-genericity and variants of immunity.

2 Basic definitions

In general, we use the notation and terminology adopted by Robert I. Soare in [14]. If A is a set of natural numbers then $A(x) = 1$ if $x \in A$; otherwise $A(x) = 0$. For $A \subseteq \mathbb{N}$, $\overline{A}$ denotes $\mathbb{N} \setminus A$. We denote by $A \upharpoonright n$ the string of length n that consists of the bits $A(0) \dots A(n-1)$. For $n_0, \dots, n_k$ different numbers in $\{0, \dots, |\sigma| - 1\}$ and $h_0, \dots, h_k \in \{0, 1\}$ we denote by $\sigma[n_0 \leftarrow h_0, \dots, n_k \leftarrow h_k]$ the string τ of length $|\sigma|$ such that $\tau(n) = \sigma(n)$ if $n \notin \{n_0, \dots, n_k\}$ and $\tau(n_i) = h_i$ for all $i \in \{0, \dots, k\}$.

For $\sigma \in 2^{<\omega}$ and $X \in 2^\omega$ we write $\sigma \prec X$ to mean that σ is a prefix of X. For $\sigma \in 2^{<\omega}$, let $[\sigma]^{\preceq} = \{X \in 2^\omega : \sigma \prec X\}$ be the basic open set generated by σ.

Let $V_{e,s} = \{\sigma : |\sigma| \leq s \wedge (\exists \rho \in W_{e,s})\ \rho \preceq \sigma\}$ and $P_{e,s} = \{\sigma : |\sigma| = s \wedge \sigma \notin V_{e,s}\}$, where W_e is the e-th c.e. set. Then the e-th Σ^0_1-class $\mathcal{V}_e$ and the e-th Π^0_1-class $\mathcal{P}_e$ will be represented by $(V_{e,s})$ and $(P_{e,s})$ respectively, so that $\mathcal{V}_e = \bigcup_s [V_{e,s}]^{\preceq}$ and $\mathcal{P}_e = \bigcap_s [P_{e,s}]^{\preceq}$.

Let $\mathcal{C} \subseteq 2^\omega$ and $\sigma \in 2^\omega$. We define $\mathcal{C}|\sigma = \{X \in 2^\omega : \sigma X \in \mathcal{C}\}$. We denote with $\mu : 2^\omega \to \mathbb{R}$ the usual Lebesgue measure in the Cantor space. Notice that $\mu(\mathcal{C}|\sigma) = 2^{|\sigma|}\mu\left(\mathcal{C} \cap [\sigma]^{\preceq}\right)$.

We will repeatedly use the following result (a proof can be found in [12]):

Lemma 2 (Lebesgue density theorem)**.** *Let $\mathcal{C}$ be a measurable subset of 2^ω with $\mu(\mathcal{C}) > 0$, and let $\delta < 1$. Then there exists $\sigma \in 2^{<\omega}$ such that $\mu(\mathcal{C}|\sigma) \geq \delta$.*

A measurable set $\mathcal{C}$ has *density* d at X if $\lim_n \mu(\mathcal{C}|(X \upharpoonright n)) = d$. Define

$$\phi(\mathcal{C}) = \{X \in 2^\omega : \mathcal{C} \text{ has density 1 at } X\}.$$

For Theorem 18 we will also use the following result (see also [12] for a proof):

Lemma 3 (Full Lebesgue density theorem)**.** *If $\mathcal{C}$ is measurable then so is $\phi(\mathcal{C})$ and*

$$\mu\left((\mathcal{C} \setminus \phi(\mathcal{C})) \cup (\phi(\mathcal{C}) \setminus \mathcal{C})\right) = 0.$$

A machine M is *prefix-free* if the domain of M is an antichain under the prefix relation of strings, that is, if σ is in the domain of M then no proper extension may also be in it. Let $(M_d)_{d \in \mathbb{N}}$ be an effective listing of all prefix-free machines. The universal prefix-free machine U is given by $U^A(0^d 1 \sigma) = M_d^A(\sigma)$. Let $K : 2^{<\omega} \to \mathbb{N}$ be the prefix Kolmogorov complexity, that is,

$$K(\sigma) = \min\{|\rho| : U(\rho) = \sigma\}.$$

Martin-Löf [10] introduced a notion of randomness that has been widely accepted in the field. A ML-test is a uniformly c.e. sequence $(G_i)_{i\in\mathbb{N}}$ of sets $G_i \subseteq 2^{<\omega}$ such that $\mu\left([G_i]^{\preceq}\right) \leq 2^{-i}$. A set $A \in 2^\omega$ *fails* the test if $A \in \bigcap_i [G_i]^{\preceq}$, otherwise A *passes* the test. A is *Martin-Löf random* if A passes each ML-test. Let MLR denote the class of Martin-Löf random sequences. In this paper we will just call them *random* sequences.

Schnorr [13] found a characterization of the random sequences in terms of the prefix Kolmogorov complexity. This characterization is here used in place of the original definition: A is random if and only if

$$(\exists c)(\forall n)\ K(A \upharpoonright n) > n - c.$$

Hence random sets have highly incompressible prefixes. For each $b \in \mathbb{N}$, define

$$R_b = \{\sigma \in 2^{<\omega} \colon K(\sigma) \leq |\sigma| - b\}.$$

Then it can be shown that $\bigcap_b [R_b]^{\preceq} = 2^\omega \setminus \mathsf{MLR}$ and $\mu\left([R_b]^{\preceq}\right) \leq 2^{-b}$. Therefore, $(R_b)_{b\in\mathbb{N}}$ is a *universal* Martin-Löf test, and for every $A \in \mathsf{MLR}$ there is a large enough b such that A is in the Π^0_1-class $2^\omega \setminus [R_b]^{\preceq}$.

Observe that any finite I is trivially indifferent for any random A. Indeed, there are $2^{\|I\|}$ many $B \in 2^\omega$ that agree with A on $\overline{I}$ and for any such B we can compute $A \upharpoonright n$ from $B \upharpoonright n$ using only the values of A on I. Hence for any such B there is $d \in \mathbb{N}$ such that $(\forall n)\ K(A \upharpoonright n) \leq K(B \upharpoonright n) + d$, and so B is also random.

3 Indifferent sets and autoreducibility

In this section we introduce the notion of *autoreducibility* and in Theorem 5 we show that every non autoreducible set belonging to a Π^0_1-class $\mathcal{P}$ has an infinite $\mathcal{P}$-indifferent set. In Proposition 4 we show that no random is autoreducible [15]. This implies that every random has an indifferent set.

Trahtenbrot introduced in [15] the notion of autoreducibility. A set A is *autoreducible* if A is redundant in the sense that for each x, one can determine $A(x)$ via queries to A other than x. More precisely, there is a Turing functional Φ such that

$$(\forall x)\ A(x) = \Phi^{A\setminus\{x\}}(x).$$

For example, the set $B \oplus B$ is autoreducible, for each set B. Thus, each many-one degree contains an autoreducible set, and each c.e. many-one degree contains a c.e. autoreducible set. However, not all c.e. sets are autoreducible, as Ladner [8] showed that there is a non autoreducible c.e. set of degree $\mathbf{0'}$.

Our intuition is that being random is incompatible with having redundancy. Random sets live up to our expectations here. Trahtenbrot [15] showed that no Kolmogorov–Loveland stochastic sequence can be autoreducible, hence no random sequence can be. We prove this for completeness.

Proposition 4. *Suppose there is an infinite computable set B and a functional Φ such that $A(x) = \Phi^{A\setminus\{x\}}(x)$ for all $x \in B$. Then A is not random.*

Proof. For simplicity, assume that $0 \in B$. Let u_i be the use of $\Phi^{A \setminus \{i\}}(i)$. Let $b_0 = 0$ and $b_{i+1} = \min\{b \in B : b > \max(u_i, b_i)\}$ and let $\sigma_i = A(b_i+1) \dots A(b_{i+1}-1)$. There is a prefix-free machine M that on input $\tau_k = 0^{|k|}1k\sigma_0\sigma_1 \dots \sigma_{k-1}$ computes

$$A(b_0)\sigma_0 A(b_1)\sigma_1 \dots A(b_{k-1})\sigma_{k-1} = A \upharpoonright b_k.$$

(Note that we identify k with its binary code and write $|k|$ or $\log k$ to denote the length of this code.) The idea is that we first obtain k from the input τ_k and then at each step $i = 0, \dots, k-1$ we calculate $\gamma_i = A \upharpoonright b_i$ leaving unread a portion $\rho_i = \sigma_i \dots \sigma_{k-1}$ of the input. Start with $\rho_0 = \sigma_0\sigma_1 \dots \sigma_{k-1}$ and $\gamma_0 = \emptyset$. To compute $A(b_i)$, we find the least s such that $\Phi_s^{\gamma_i 0 \rho_i}(i) \downarrow$. Once we find it, we know that $\Phi_s^{\gamma_i 0 \rho_i}(b_i) = A(b_i)$ and we only have read a prefix α_i of $\gamma_i 0 \rho_i$. Since $|\alpha_i| = u_{b_i}$, we can compute b_{i+1} and then obtain σ_i from ρ_i (just take the initial $b_{i+1} - b_i - 1$ bits from ρ_i). At this point we know $\gamma_i = A \upharpoonright b_i$, we calculated $A(b_i)$ and we read from the input $\sigma_i = A(b_i+1) \dots A(b_{i+1}-1)$. Then we define $\gamma_{i+1} = \gamma_i A(b_i)\sigma_i = A \upharpoonright b_{i+1}$ and leave an unread input $\rho_{i+1} = \sigma_{i+1} \dots \sigma_{k-1}$. We finally output γ_k having read all the input τ_k. Hence M is prefix-free and there is c such that for all k,

$$K(A \upharpoonright b_k) \le 2\log k + b_k - k + c,$$

and so we conclude that A is not random. $\square$

The next result shows the existence of infinite indifferent sets in a general setting.

Theorem 5. *Let $\mathcal{P}$ be a Π^0_1-class and suppose $A \in \mathcal{P}$ is not autoreducible. Then there is an infinite set $I \le_T A'$ such that I is indifferent for A with respect to $\mathcal{P}$.*

Proof. Let $(P_s)_{s \in \mathbb{N}}$ be a recursive approximation of the given Π^0_1-class $\mathcal{P} = \bigcap_s [P_s]^{\preceq}$. First let us show that there is a number n such that the singleton set $\{n\}$ is $\mathcal{P}$-indifferent for A. Assume not, then, for each x, one of A and $A[x \leftarrow 1 - A(x)]$ (the set where the bit in position x has flipped) is not on $\mathcal{P}$. This allows us to compute $A(x)$ from $A \setminus \{x\}$, as follows: Search for $s > x$ such that $A[x \leftarrow 1] \upharpoonright s \notin P_s$ or $A[x \leftarrow 0] \upharpoonright s \notin P_s$. If the first case applies output 0, otherwise output 1.

An infinite set $I = \{n_0 < n_1 < \dots\}$ that is $\mathcal{P}$-indifferent for A can now be computed inductively. Suppose we already have an indifferent set $\{n_0 < \dots < n_k\}$. Then A is a member of the Π^0_1-class

$$\begin{aligned} \mathcal{Q}_k = \{Y :\ Y \upharpoonright n_k + 1 = A \upharpoonright n_k + 1 \quad \wedge \quad & (\forall h_0, \dots, h_k \in \{0,1\}) \\ & Y[n_0 \leftarrow h_0, \dots, n_k \leftarrow h_k] \in \mathcal{P}\}. \end{aligned}$$

Now, by the argument above let n_{k+1} be an $\mathcal{Q}_k$-indifferent point for A. Then $n_{k+1} > n_k$ since all $Y \in \mathcal{Q}_k$ extend $A \upharpoonright n_k + 1$.

To see that the whole set I is $\mathcal{P}$-indifferent for A, we use that $\mathcal{P}$ is closed: suppose that Y is obtained from A by replacing the bit $A(n_i)$ by h_i. For each k, the set $Y_k = A[n_0 \leftarrow h_0, \dots, n_k \leftarrow h_k]$ is in $\mathcal{P}$, and the distance $d(Y_k, Y)$ is at most $2^{-n_{k+1}}$. Here the distance is defined in the following way: for $X, Y \in 2^\omega$, if $X = Y$ then $d(X,Y) = 0$, otherwise $d(X,Y) = 2^{-n}$, where n is minimal such that $X(n) \neq Y(n)$ (it is known that $(2^\omega, d)$ is a metric space). Thus $Y \in \mathcal{P}$.

Finally, we verify that $I \le_T A'$: let $\mathcal{Q}_{-1} = \mathcal{P}$. To compute $n_0, n_1, \dots$ inductively, note that for $k \ge -1$, n_{k+1} may be defined as the least n such that

$$(\forall s)\ \ (A[n \leftarrow 0] \upharpoonright s \in Q_{k,s} \ \wedge\ A[n \leftarrow 1] \upharpoonright s \in Q_{k,s}).$$

where $(Q_{k,s})_{s\in\mathbb{N}}$ is a computable approximation of $\mathcal{Q}_k = \bigcap_s [Q_{k,s}]^{\preceq}$.

Hence n_{k+1} can be computed from an index for the Π^0_1-class $\mathcal{Q}_k$ using A' as an oracle. Next we may find an index for $\mathcal{Q}_{k+1}$ using A. □

Corollary 6. *Every random set A has an A'-computable infinite* MLR*-indifferent set.*

Proof. For any random A, choose b large enough such that $A \in 2^\omega \setminus [R_b]^{\preceq}$. By Proposition 4, A is not autoreducible, so by Theorem 5, A has an A'-computable infinite MLR-indifferent set. □

By the Low Basis Theorem [6], there is a random set that is low. The above corollary implies that every low random A has an infinite Δ^0_2 MLR-indifferent set I. In fact, this las assertion will be improved in Theorem 15.

The following theorem proves in a different way the existence of such I but it also guarantees that any set B agreeing with A on $\overline{I}$ is GL_1 (that is, $B' \equiv_T B \oplus \emptyset'$).

Theorem 7. *There is an infinite Δ^0_2 set I and a low $A \in$ MLR such that I is MLR-indifferent for A. Furthermore, if $B \in 2^\omega$ agrees with A on $\overline{I}$, then B is GL_1.*

Proof. Let $\mathcal{P} = 2^\omega \setminus [R_b]^{\preceq}$ for some b. Clearly, $\mathcal{P}$ is a Π^0_1-class such that $\emptyset \neq \mathcal{P} \subseteq$ MLR. The idea is to start with $\mathcal{Q}_0 = \mathcal{P}$ and find a string σ such that $\mu\left((\mathcal{Q}_0|\sigma 0) \cap (\mathcal{Q}_0|\sigma 1)\right) > 0$. This means that there are two random sets of the form $\sigma 0X$ and $\sigma 1X$, so we define A starting with $\sigma 0$ and we let $|\sigma|$ be the first indifferent point (corresponding to the position of the last 0). We can go on in the same way with the Π^0_1-class $\mathcal{Q}_1 = (\mathcal{Q}_0|\sigma 0) \cap (\mathcal{Q}_0|\sigma 1)$. To guarantee that any $B \in 2^\omega$ that agrees with A on $\overline{I}$ is also GL_1, instead of considering always $\mathcal{Q}_{s+1} = (\mathcal{Q}_s|\sigma 0) \cap (\mathcal{Q}_s|\sigma 1)$ (for the last σ chosen), sometimes we consider $\mathcal{Q}_{s+1}$ as a subtree of $(\mathcal{Q}_s|\sigma 0) \cap (\mathcal{Q}_s|\sigma 1)$ such that either $(\forall X \in \mathcal{Q}_{s+1})\ J^{\tau X}(e)\downarrow$ or $(\forall X \in \mathcal{Q}_{s+1})\ J^{\tau X}(e)\uparrow$, for some $\tau \prec B$ that depends on e. Here $J^A(x)$ is the jump of A, that is $J^A(x) = \{x\}^A(x)$. Since $\emptyset'$ can decide which of the two cases holds, we have that B is GL_1.

Construction. We construct $A = \sigma_0 0 \sigma_1 0 \sigma_2 0 \ldots$ and $I = \{n_0, n_1, n_2 \ldots\}$ by stages.

- *Step* 0. Let $\mathcal{Q}_0 = \mathcal{P}$.
- *Step* $2e{+}1$. Define σ_e as the least σ such that $\mu\left(\mathcal{Q}_{2e}|\sigma\right) > 1/2$. Define the e-th indifferent point as $n_e = e + \sum_{j\leq e} |\sigma_j|$. Also define the new Π^0_1-class $\mathcal{Q}_{2e+1} = (\mathcal{Q}_{2e}|\sigma 0) \cap (\mathcal{Q}_{2e}|\sigma 1)$.
- *Step* $2e+2$. Let $\mathcal{T}^e_0 = \mathcal{Q}_{2e+1}$ and for $i \in \{0,\ldots,2^{e+1}-1\}$ let ρ^e_i be the i-th string of length $e+1$. For all $i = 0,\ldots,2^{e+1}-1$, do the following:
  1. Let $\beta^e_i = (A \upharpoonright n_e)0[n_0 \leftarrow \rho^e_i(0),\ldots,n_e \leftarrow \rho^e_i(e)]$, that is, β^e_i coincides with $(A \upharpoonright n_e)0$ except at the indifferent points defined so far, $n_0,\ldots,n_e$, where it has the bits $\rho^e_i(0),\ldots,\rho^e_i(e)$.
  2. If $\mathcal{T}^e_i \cap \{X\colon J^{\beta^e_i X}(e)\uparrow\} \neq \emptyset$ then $\mathcal{T}^e_{i+1} = \mathcal{T}^e_i \cap \{X\colon J^{\beta^e_i X}(e)\uparrow\}$. Otherwise $\mathcal{T}^e_{i+1} = \mathcal{T}^e_i$.

  Finally, let $\mathcal{Q}_{2e+2} = \mathcal{T}^e_{2^{e+1}}$.

Verification. Notice that

$$\begin{aligned} 1/2 &< \mu(\mathcal{Q}_{2e}|\sigma) \\ &= (\mu(\mathcal{Q}_{2e}|\sigma 0) + \mu(\mathcal{Q}_{2e}|\sigma 1))/2 \end{aligned}$$

and so $\mu(\mathcal{Q}_{2e}|\sigma 0) + \mu(\mathcal{Q}_{2e}|\sigma 1) > 1$, which implies $\mu(\mathcal{Q}_{2e+1}) > 0$. Observe also that $\mathcal{Q}_{2e+2} \subseteq \mathcal{Q}_{2e+1}$ and $(A \upharpoonright n_e)0\mathcal{Q}_{2e+1} \subseteq \mathcal{P}$. In the odd steps we guarantee that for all $h_0, \ldots, h_e \in \{0,1\}$,

$$(A \upharpoonright n_e)0[n_0 \leftarrow h_0, \ldots, n_e \leftarrow h_e]\mathcal{Q}_{2e+1} \subseteq \mathcal{P}.$$

Since $(A \upharpoonright n_e)0\mathcal{Q}_{2e+2}$ is a nonempty Π^0_1-class included in $\mathcal{P}$, it does not have measure zero. Indeed, suppose by contradiction that it has measure zero. Then $(A \upharpoonright n_e)0\mathcal{Q}_{2e+2}$ would induce a ML-test and all elements in $(A \upharpoonright n_e)0\mathcal{Q}_{2e+2}$ would be non-random, contradicting the fact that $(A \upharpoonright n_e)0\mathcal{Q}_{2e+2} \subseteq \mathcal{P} \subseteq \mathsf{MLR}$. So $\mu((A \upharpoonright n_e)0\mathcal{Q}_{2e+2}) > 0$ and Lemma 2 may be safely applied in the odd steps. Clearly $A \in \mathcal{P}$ and hence it is random. By construction, if $B \in 2^\omega$ agrees with A on $\overline{I}$ then $B \in \mathcal{P}$, and hence it is random.

At step $2e+1$, we have a computable approximation of $\mathcal{Q}_{2e} = \bigcap_i \mathcal{Q}_{2e,i}$. Observe that $\mu(\mathcal{Q}_{2e}|\sigma) > 1/2$ is equivalent to $(\forall i)\ \mu(\mathcal{Q}_{2e,i}|\sigma) > 1/2$ and therefore $\emptyset'$ can find σ in the odd steps. At step $2e+2$, $\emptyset'$ can also construct Q_{2e+2}, since $\mathcal{T}^e_i \cap \{X \colon J^{\beta^e_i X}(e)\uparrow\}$ is a Π^0_1-class. Hence, A is Δ^0_2.

Suppose $B \in 2^\omega$ agrees with A on $\overline{I}$. To determine if $e \in B'$, we consider $\mathcal{Q}_{2e+2}$, and i such that $\rho_i(j) = B(n_j)$ for $0 \le j \le e$ and β^e_i as in the construction. Notice that $\beta^e_i \prec B$ by hypothesis. At that stage of the construction, there are two possibilities:

- If $\mathcal{T}_i \cap \{X \colon J^{\beta^e_i X}(e)\uparrow\} \neq \emptyset$ then $\mathcal{T}_{i+1} = \mathcal{T}_i \cap \{X \colon J^{\beta^e_i X}(e)\uparrow\}$. Hence $(\forall X \in \mathcal{T}_{i+1})\ J^{\beta^e_i X}(e)\uparrow$ and, since $\mathcal{Q}_{2e+2} \subseteq \mathcal{T}_{i+1}$, we have $(\forall X \in \mathcal{Q}_{2e+2})\ J^{\beta^e_i X}(e)\uparrow$. Since $B \in \beta^e_i\mathcal{Q}_{2e+2}$, we conclude $J^B(e)\uparrow$.
- Else, $\mathcal{T}_i \cap \{X \colon J^{\beta^e_i X}(e)\uparrow\} = \emptyset$ and $\mathcal{T}_{i+1} = \mathcal{T}_i$. Thus $(\forall X \in \mathcal{Q}_{2e+2})\ J^{\beta^e_i X}(e)\downarrow$ and hence $J^B(e)\downarrow$.

Since $\mathcal{T}_i \cap \{X \colon J^{\tau X}(e)\uparrow\}$ is a Π^0_1-class, we can decide if it is empty or not, using $\emptyset'$. So $B' \le_T B \oplus \emptyset'$. For $B = A$ we obtain that A is low, since $A \le_T \emptyset'$. □

We mentioned in the introduction that it is reasonable to think that the elements of an infinite MLR-indifferent set should be sparse. In the next section we make this intuition precise. For now, we prove that there are infinite MLR-indifferent sets consisting of blocks of bits of arbitrary length. This means that the indifferent points need not be dispersed; one can have large groups of consecutive indifferent points (of course, these groups will be dispersed). To prove this result, we apply the same reasoning used in the proof of Theorem 7. We first need an auxiliary lemma that follows easily from Lemma 2.

Lemma 8. *Let $\mathcal{C}$ be a measurable set of 2^ω with $\mu(\mathcal{C}) > 0$. For any $k > 0$ there exists a string σ such that $\mu\left(\bigcap_{|\tau|=k} \mathcal{C}|\sigma\tau\right) > 0$.*

Proof. We know that for any σ we have

$$\mu(\mathcal{C}|\sigma) = \sum_{|\tau|=k} \mu\left((\mathcal{C}|\sigma) \cap [\tau]^{\preceq}\right) = 2^{-k} \sum_{|\tau|=k} \mu(\mathcal{C}|\sigma\tau)$$

and

$$\sum_{|\tau|=k} \mu(\mathcal{C}|\sigma\tau) = 2^k - \sum_{|\tau|=k} \mu(2^\omega \setminus (\mathcal{C}|\sigma\tau)) \leq 2^k - 1 + \mu\left(\bigcap_{|\tau|=k} \mathcal{C}|\sigma\tau\right).$$

Therefore, for any σ we have $2^k\mu(\mathcal{C}|\sigma) \leq 2^k - 1 + \mu\left(\bigcap_{|\tau|=k} \mathcal{C}|\sigma\tau\right)$. By Lemma 2, there is a string σ such that $\mu(\mathcal{C}|\sigma) > 1 - 2^{-k}$. For such σ we have

$$2^k - 1 + \mu\left(\bigcap_{|\tau|=k} \mathcal{C}|\sigma\tau\right) > 2^k(1 - 2^{-k})$$

and so $\mu\left(\bigcap_{|\tau|=k} \mathcal{C}|\sigma\tau\right) > 0$. □

Theorem 9. *Let $(k_i)_{i\in\mathbb{N}}$ be a Δ^0_2 sequence of natural numbers greater than 0. There is a set $I \subseteq \mathbb{N}$ such that:*

- *I has disjoint blocks of consecutive numbers of length k_i, i.e.*

$$I = \bigcup_i \{n_i, \ldots, n_i + k_i - 1\}$$

  for a sequence $(n_i)_{i\in\mathbb{N}}$ such that $n_i + k_i - 1 < n_{i+1}$ for all i.

- *I is as in Theorem 7.*

Proof. Follow the proof of Theorem 7 to define $A = \sigma_0 0^{k_0} \sigma_1 0^{k_1} \sigma_2 0^{k_2} \ldots$ At step $2e + 1$ find σ least such that $\mu\left(\bigcap_{|\tau|=k_e} \mathcal{Q}_{2e}|\sigma\tau\right) > 0$. The existence of this σ is guaranteed by Lemma 8. At stage $2e + 1$ define $\mathcal{Q}_{2e+1} = \bigcap_{|w|=k_e} \mathcal{Q}_{2e}|\sigma w$. At stage $2e + 2$ consider each string β^e_i of length $\sum_{j\leq e} k_j$ and proceed in the same way. □

An interesting question is whether there is a universal indifferent set I with respect to the class of random sequences, in the sense that I is MLR-indifferent for every random A. We close this section by showing that there is no such universal indifferent set. In Section 5 we will produce an infinite set I that is MLR-indifferent for all 2-random sequences, hence for almost all random sequences.

Theorem 10. *For every infinite set I, there is a random for which I is not* MLR*-indifferent.*

Proof. On the one hand, van Lambalgen [16, 17] showed that if A is random then B is A-random if and only if $B \oplus A$ is random. On the other hand, the Kučera-Gács Theorem [7, 5] states that every set is weak truth-table reducible to a random set.

Let $J = I \cap 2\mathbb{N}$ and assume $\|J\| = \infty$ (the argument is similar if $\|I \cap (2\mathbb{N} + 1)\| = \infty$). By the Kučera-Gács Theorem we take a random $A \geq_{wtt} J$. We also take a set B that is A-random. By the result of van Lambalgen, $B \oplus A$ is random.

Now, let $\tilde{B} \in 2^\omega$ be such that $\tilde{B}(i) = 0$ for all $i \in J/2$ and $\tilde{B}(i) = B(i)$ for all $i \notin J/2$. Since $A \geq_{wtt} J$, it is clear that $\tilde{B}$ cannot be A-random. Again by van Lambalgen's result, $\tilde{B} \oplus A$ is not random.

Observe that $B \oplus A$ and $\tilde{B} \oplus A$ differ at most at the positions of J. So J (and hence I) is not MLR-indifferent for $A \oplus B$. □

4 The sparseness of indifferent sets

We now prove that indifferent sets are sparse. Let $I \subseteq \mathbb{N}$ be infinite and let $p\colon \mathbb{N} \to \mathbb{N}$ be strictly increasing such that $\operatorname{range} p = I$. Recall that I is *hyperimmune* if it is not dominated by a total recursive function. It is *dominant* if it dominates every total recursive function. We say that $p\colon \mathbb{N} \to \mathbb{N}$ is *partial dominant* if for any partial recursive function ψ,

$$(\forall^\infty b)\,[\psi(b)\downarrow \Rightarrow\ \psi(b) \le p(b)]\,.$$

Note that if p is partial dominant, then $p \ge_T \emptyset'$. This is immediate because $b \in \emptyset'$ iff $b \in \emptyset'_{p(b)}$, except for finitely many b.

We show that any infinite MLR-indifferent set I is dominant and complete (i.e., computes $\emptyset'$), and that if I is also assumed to be co-c.e., then it must be partial dominant. To warm up, we prove that indifferent sets are hyperimmune.

Theorem 11. *Any infinite* MLR*-indifferent set is hyperimmune.*

Proof. Suppose I is MLR-indifferent for some random A and assume for a contradiction that I is not hyperimmune. Then there exists a strictly increasing computable function $f\colon \mathbb{N} \to \mathbb{N}$ such that $I \cap \{f(j), \dots, f(j+1)-1\} \neq \emptyset$ for all j (this follows from [14, Theorem 2.3]).

Let $m_j = \min I \cap \{f(j), \dots, f(j+1)-1\}$ and let $B \in 2^\omega$ be defined in the following way:

$$B(i) = \begin{cases} A(i) & \text{if } i \notin \{m_0, m_1, m_2, \dots\}; \\ A(i) & \text{if } i = m_j \text{ and } \|A \cap \{f(j), \dots, f(j+1)-1\}\| \text{ is odd;} \\ 1 - A(i) & \text{if } i = m_j \text{ and } \|A \cap \{f(j), \dots, f(j+1)-1\}\| \text{ is even.} \end{cases}$$

That is, we define B like A but we flip at most one bit in every block starting at position $f(j)$ and ending at position $f(j+1)-1$ so that B has always an odd number of 1s in every such block. By hypothesis, B is also random.

We claim that B is autoreducible. To compute $B(x)$ from $B \setminus \{x\}$, find j such that $f(j) \le x < f(j+1)$. If $\|(B \setminus \{x\}) \cap \{f(j), \dots, f(j+1)-1\}\|$ is odd, then $B(x) = 0$. Otherwise, $B(x) = 1$. But Proposition 4 states that B cannot be both random and autoreducible, so we have a contradiction. □

To some extent, this result confirms our intuition that MLR-indifferent sets must be sparse. In the special case where I is co-c.e.—which we will show to be possible in the next section—we can prove a much stronger sparseness condition.

Theorem 12. *If I be an infinite co-c.e.* MLR*-indifferent set, then it is partial dominant.*

Proof. We may assume, without loss of generality, that $A(i) = 0$ for $i \in I$. Let $p\colon \mathbb{N} \to \mathbb{N}$ be strictly increasing such that $\operatorname{range} p = I$. Suppose ψ is a partial recursive function. Fix b such that $\psi(b)\downarrow$ and $\psi(b) > p(b)$, and let $\tilde{b} = \|I \cap \{0, \dots, \psi(b)-1\}\|$. Notice that $\tilde{b} \ge b$.

To describe $A \restriction \psi(b)$ we need to code b, $\tilde{b}$ and the $\psi(b) - \tilde{b}$ bits $A(j)$, for $0 \le j < \psi(b)$ and $j \notin I$ in a prefix way. The procedure for computing $A \restriction \psi(b)$ from those parameters is the following:

1. Read b and $\tilde{b}$, and calculate $\psi(b)$.

2. Enumerate the complement of I until we see $\psi(b) - \tilde{b}$ elements in $\{0, \dots, \psi(b) - 1\}$, i.e. find the least stage s such that $\|\overline{I}_s \cap \{0, \dots, \psi(b) - 1\}\| = \psi(b) - \tilde{b}$. Once we reach this stage, no more elements will be enumerated into $\overline{I} \cap \{0, \dots, \psi(b) - 1\}$, so

$$(\forall t \geq s)\ \|\overline{I}_t \cap \{0, \dots, \psi(b) - 1\}\| = \psi(b) - \tilde{b}.$$

3. Copy the rest of the $\psi(b) - \tilde{b}$ bits from the input and interleave 0 in each position of $I_s \cap \{0, \dots, \psi(b) - 1\}$.

We use $2|b| + 1$ bits to describe $\psi(b)$, we use $2|\tilde{b}| + 1$ bits to describe $\tilde{b}$, and we use $\psi(b) - \tilde{b}$ bits to describe the needed bits of A. Hence there is a constant c such that

$$\begin{aligned} K(A \upharpoonright \psi(b)) &\leq 2 \log b + 2 \log \tilde{b} + \psi(b) - \tilde{b} + c \\ &\leq 4 \log \tilde{b} + \psi(b) - \tilde{b} + c. \end{aligned}$$

Since A is random, $\tilde{b} - 4 \log \tilde{b} \leq d$ for some constant d. This is possible for only finitely many $\tilde{b}$s, and therefore for only finitely many bs. □

It is open whether every infinite MLR-indifferent set is partial dominant. We come close in the next theorem. Our coding method is not very sophisticated; we use our control over an unknown subset of size $n^2/2$ to code $2 \log n - 1$ bits of information. A cleverer coding method might be able to code more with control over fewer bits, but the present result is sufficient to prove that indifferent sets are quite sparse and that they decide the halting problem.

Theorem 13. *Let I be an infinite* MLR*-indifferent set. Let $p\colon \mathbb{N} \to \mathbb{N}$ be strictly increasing such that* $\operatorname{range} p = I$. *Then for any partial recursive function ψ,*

$$(\forall^\infty b)\ \left[\psi(b) \downarrow \Rightarrow\ \psi(b) \leq p(b^2)\right].$$

Proof. Choose a Marin-Löf random sequence A for which I is indifferent. Assume, for a contradiction, that $(\exists^\infty b)\ \psi(b) \downarrow > p(b^2)$. We inductively define a sequence $\{n_0, n_1, \dots\}$ as follows. Choose n_0 such that $\psi(n_0) \downarrow > p(n_0^2)$. Once n_i has been defined, choose n_{i+1} such that $n_{i+1}^2 \geq 2(\psi(n_i) + 2\lfloor \log n_i \rfloor - 1)$ and $\psi(n_{i+1}) \downarrow > p(n_{i+1}^2)$.

Now we will define a sequence B that agrees with A on $\overline{I}$, but which will turn out not to be random. We define B in stages. At the end of stage i, we will have determined $B \upharpoonright (\psi(n_i) + 2\lfloor \log n_i \rfloor - 1)$. Since $\psi(n_{i+1}) > p(n_{i+1}^2)$, when we define $B \upharpoonright \psi(n_{i+1})$ at stage $i + 1$, we have at least

$$\begin{aligned} n_{i+1}^2 - (\psi(n_i) + 2\lfloor \log n_i \rfloor - 1) &\geq n_{i+1}^2 - n_{i+1}^2/2 \\ &= n_{i+1}^2/2 \end{aligned}$$

positions of I to work with. This is enough to control the value of

$$\|B \upharpoonright \psi(n_{i+1})\| \pmod{\lfloor n_{i+1}^2/2 \rfloor},$$

which in turn is enough to code $2\lfloor \log n_{i+1} \rfloor - 1$ bits, so we can let B agree with A on $\{\psi(n_{i+1}), \dots, \psi(n_{i+1}) + 2\lfloor \log n_{i+1} \rfloor - 2\}$ and define $B \upharpoonright \psi(n_{i+1})$ so that $\|B \upharpoonright \psi(n_{i+1})\| \pmod{\lfloor n_{i+1}^2/2 \rfloor}$ codes these bits.

Now let us estimate the complexity of $B \upharpoonright (\psi(n_i) + 2\lfloor \log n_i \rfloor - 1)$. We can describe n_i in $\log n_i + 2 \log \log n_i + O(1)$ bits. Then we can calculate $\psi(n_i)$ and read in the bits of $B \upharpoonright \psi(n_i)$. From $\|B \upharpoonright \psi(n_i)\| \pmod{\lfloor n_i^2/2 \rfloor}$, we can determine the remaining bits of $B \upharpoonright (\psi(n_i) + 2\lfloor \log n_i \rfloor - 1)$. Therefore, there is a c such that

$$K(B \upharpoonright (\psi(n_i) + 2\lfloor \log n_i \rfloor - 1)) \leq \psi(n_i) + \log n_i + 2 \log \log n_i + c.$$

If B were random, then $\log n_i - 2 \log \log n_i$ would be bounded above by a constant. This is possible for only finitely many n_i, hence B is not random and I is not an indifferent sequence for A. □

Corollary 14. *If I is an infinite* MLR*-indifferent set, then I is dominant and $I \geq_T \emptyset'$.*

Proof. Let $p \colon \mathbb{N} \to \mathbb{N}$ be strictly increasing such that $\operatorname{range} p = I$. Let f be a total recursive function. Define $g(b) = \max\{f(0), \ldots, f((b+1)^2 - 1)\}$. By Theorem 13, $g(b) \leq p(b^2)$, except for finitely many b. For any $a \in \mathbb{N}$, let b be the least integer such that $a < (b+1)^2$. So $a \geq b^2$. Then $f(a) \leq g(b) \leq p(b^2) \leq p(a)$, except for finitely many a. Thus, I is dominant.

To see that $I \geq_T \emptyset'$, note that $b \in \emptyset'$ iff $b \in \emptyset'_{p(b^2)}$, except for finitely many b. □

5 Co-c.e. indifferent sets

We mentioned in Section 1 that every finite set is trivially MLR-indifferent. By the results of the previous section, we know that there are Δ^0_2 infinite MLR-indifferent sets. We wonder if there are, for example, infinite c.e. indifferent sets for the class of Martin-Löf random sequences. Theorem 11 answers this question negatively because no c.e. set can be hyperimmune. On the other hand, there are infinite co-c.e. MLR-indifferent sets.

Theorem 15. *Every low random set A has an infinite co-c.e.* MLR*-indifferent set.*

Proof. The set

$$L = \{\langle k, n \rangle : (\exists m \geq k)\ K(A \upharpoonright m) \leq m + n\}$$

is c.e. relative to A and hence Δ^0_2. It is known that A is Martin-Löf random if and only if $\lim_n K(A \upharpoonright n) - n = \infty$ (this follows, for example from the result of Miller and Yu [11] stating that $\sum_n 2^{n - K(Z \upharpoonright n)} < \infty$ for each Martin-Löf random Z). Then for each n there is k such that $\langle k, 2n \rangle \notin L$. Furthermore, there is a function $f \leq \emptyset'$ such that $\langle f(n), 2n \rangle \notin L$, i.e.

$$(\forall m \geq f(n))\ K(A \upharpoonright m) - m > 2n.$$

Having f, there is a co-c.e. set $I = \bigcap_s I_s$ such that $p(n)$, the n-th element of I, satisfies $p(n) \geq f(n)$. Given m, let $\sigma_m = A(p(0))A(p(1)) \ldots A(p(n_m - 1))$ where $n_m = \max\{i : p(i) \leq m\}$. On the one hand, since $|\sigma_m| = n_m$ and $m \geq f(n_m)$, there is a constant c such that for all m,

$$\begin{aligned} K(\sigma_m) &\leq 2n_m + c \\ &< K(A \upharpoonright m) - m + c \end{aligned}$$

On the other hand, from a program for computing $B \upharpoonright m$ and a program for computing σ_m, one can compute $A \upharpoonright m$ in the following way: first obtain $B \upharpoonright m$ and m. Then obtain σ_m and $n_m = |\sigma_m|$. Find s such that $\|I_s \cap \{0, \ldots, m\}\| = n_m$. The n_m elements of $I_s \cap \{0, \ldots, m\}$ are $p(0), p(1), \ldots, p(n_m - 1)$, and $A(p(i)) = \sigma(i)$, for $i \in \{0, \ldots, n_m - 1\}$. Since $B \upharpoonright m$ differs from $A \upharpoonright m$ at most in the positions $p(0), p(1), \ldots, p(n_m - 1)$, we can compute $A \upharpoonright m$ from all the data already computed. Therefore, there is a constant d such that for all m,

$$\begin{aligned} K(A \upharpoonright m) &\leq K(B \upharpoonright m) + K(\sigma_m) + d \\ &< K(B \upharpoonright m) + K(A \upharpoonright m) - m + c + d. \end{aligned}$$

This implies that $K(B \upharpoonright m) > m - (c + d)$ for all m and hence B is also random. $\square$

An interesting open question is whether Chaitin's Ω has an infinite co-c.e. indifferent set.

For the case of a general Π^0_1-class $\mathcal{P}$ of positive measure, one can also prove that there are infinite co-c.e. $\mathcal{P}$-indifferent sets. We begin with an easy lemma.

Lemma 16. *Let $\mathcal{C}$ be a measurable subset of 2^ω. If $\mu(\mathcal{C}|\sigma 0) + \mu(\mathcal{C}|\sigma 1) > 1$ then for all n there exists τ of length n such that $\mu(\mathcal{C}|\sigma\tau 0) + \mu(\mathcal{C}|\sigma\tau 1) > 1$.*

Proof. For $n = 1$, suppose $\mu(\mathcal{C}|\sigma 00) + \mu(\mathcal{C}|\sigma 01) \leq 1$ and $\mu(\mathcal{C}|\sigma 10) + \mu(\mathcal{C}|\sigma 11) \leq 1$. Then

$$\begin{aligned} 2 &\geq \mu(\mathcal{C}|\sigma 00) + \mu(\mathcal{C}|\sigma 01) + \mu(\mathcal{C}|\sigma 10) + \mu(\mathcal{C}|\sigma 11) \\ &= 2(\mu(\mathcal{C}|\sigma 0) + \mu(\mathcal{C}|\sigma 1)) \\ &> 2, \end{aligned}$$

and this is a contradiction, so $\mu(\mathcal{C}|\sigma 00) + \mu(\mathcal{C}|\sigma 01) > 1$ or $\mu(\mathcal{C}|\sigma 10) + \mu(\mathcal{C}|\sigma 11) > 1$. Hence, by a simple induction we can prove that for any n, there is τ of length n such that $\mu(\mathcal{C}|\sigma\tau 0) + \mu(\mathcal{C}|\sigma\tau 1) > 1$. $\square$

Theorem 17. *Let $\mathcal{P}$ be a Π^0_1-class of positive measure. There is an infinite co-c.e. set that is $\mathcal{P}$-indifferent for a Δ^0_2 set A.*

Proof. We use the same idea as in the proof of Theorem 7, but instead of using $\emptyset'$ to find (in the odd stages) some σ such that $\mu(\mathcal{Q}_i|\sigma 0) + \mu(\mathcal{Q}_i|\sigma 1) > 1$, for some Π^0_1-class $\mathcal{Q}_i$, we find the first σ in the lexicographic order such that $\mu(\mathcal{Q}_{i,s}|\sigma 0) + \mu(\mathcal{Q}_{i,s}|\sigma 1) > 1$, where $(\mathcal{Q}_{i,s})_{s \in \mathbb{N}}$ is a recursive approximation of $\mathcal{Q}_i$. Hence, we do not need $\emptyset'$ anymore and we enumerate $\overline{I}$, restraining ourselves from putting into $\overline{I}$ those positions that are candidates for indifferent points. We use a marker n_i to indicate the candidate for the i-th indifferent point. Each time some marker n_j has to grow, we ensure that all n_k for $k > j$ are properly shifted. By Lemma 16, each marker is moved finitely often, so $\overline{I}$ is well defined and I is infinite. **Construction.** Let $(P_s)_{s \in \mathbb{N}}$ be a recursive approximation of the given Π^0_1-class $\mathcal{P} = \bigcap_s [P_s]^{\preceq}$.

We computably enumerate $\overline{I} = \bigcup_s \overline{I}_s$ and we define a Δ^0_2-approximation of A.

- *Step* 0. Let $\overline{I}_0 = \{0\}$ and $A_0 = \emptyset$.
- *Step* $s + 1$.
  1. Let $\mathcal{Q}_{0,s} = [P_s]^{\preceq}$ and $n_{-1,s} = 0$.
  2. Define $A_s = \sigma_{0,s} 0 \sigma_{1,s} 0 \ldots \sigma_{s,s} 0$ in the following way: for $i = 0, \ldots, s$:
     (a) Let $\sigma_{i,s}$ be the least string in $\mathcal{Q}_{i,s}$ such that
        i. $\mu(\mathcal{Q}_{i,s}|\sigma_{i,s} 0) + \mu(\mathcal{Q}_{i,s}|\sigma_{i,s} 1) > 1$ and

ii. $i + \sum_{j \le i} |\sigma_{j,s}| \notin \overline{I}_s$

(b) Set $\mathcal{Q}_{i+1,s} = (\mathcal{Q}_{i,s}|\sigma_{i,s}0) \cap (\mathcal{Q}_{i,s}|\sigma_{i,s}1)$.

3. Define the first $s+1$ candidates for indifferent points at stage $s+1$ as

$$n_{i,s} = i + \sum_{j \le i} |\sigma_{j,s}|$$

(for $i = 0, \ldots, s$).

4. Define

$$\overline{I}_{s+1} = \overline{I}_s \cup \bigcup_{0 \le i \le s} \{n_{i-1,s} + 1, \ldots, n_{i,s} - 1\}.$$

Verification. Observe that conditions of steps 2(a)i and 2(a)ii are computable because $\mathcal{Q}_{i,s}$ is clopen. This is the main difference with respect to the construction of Theorem 7; we are forced to consider candidates for the indifferent points, which may change in further stages. Let us analyze the marker $n_{0,s}$ for successive stages $s = 0, 1, 2, \ldots$ By Lemma 2 there is a τ, such that $\mu(\mathcal{P}|\tau) > 1/2$ and hence $\mu(\mathcal{P}|\tau 0) + \mu(\mathcal{P}|\tau 1) > 1$. We also know by Lemma 16 that there are extensions of τ of every length with the same property. The construction will eventually find some such extension. That is, there is a stage s_0 such that for all $t \ge s_0$, $\mu(\mathcal{Q}_{0,t}|\sigma_{0,t}0) + \mu(\mathcal{Q}_{0,t}|\sigma_{0,t}1) > 1$ and the marker for the first indifferent point is stable from stage s_0 on, i.e. $\sigma_{0,t} = \sigma_{0,s_0}$ and $n_{0,t} = n_{0,s_0} \notin \overline{I}_t$. Therefore n_{0,s_0} is the first indifferent point. By construction we guarantee that $\mu(\mathcal{Q}_{1,t}) > 0$ for all $t \ge s_0$ and then we can repeat the argument for the candidate to the second indifferent point. By induction it can be shown that every marker will be changed finitely often, that is for each $i \ge 0$ there is a stage s_i such that for all $t \ge s_i$, $\sigma_{i,t} = \sigma_{i,s_i}$ and $n_{i,t} = n_{0,s_i} \notin \overline{I}_t$. Since each time we detect two consecutive candidates for indifferent points $n_{i,s}$, $n_{i+1,s}$ we enumerate into $\overline{I}$ all n such that $n_{i,s} < n < n_{i+1,s}$, we finally have $I = \{n_{0,s_0}, n_{1,s_1}, n_{2,s_2}, \ldots\}$. By construction, I is an infinite co-c.e. set indifferent for the set $A = \sigma_{0,s_0}0\sigma_{1,s_1}0\sigma_{2,s_2}0\cdots = \lim_s A_s \in \mathcal{P}$. □

Theorem 17 constructs a co-c.e. that is $\mathcal{P}$-indifferent for a single Δ^0_2 sequence in $\mathcal{P}$. One can modify the proof to obtain a co-c.e. set that is $\mathcal{P}$-indifferent for most sequences in $\mathcal{P}$.

Theorem 18. *For any $\varepsilon > 0$ and any Π^0_1-class $\mathcal{P}$, there is an infinite co-c.e. set I such that*

$$\mu(\{A \in \mathcal{P} \colon I \text{ is not } \mathcal{P}\text{-indifferent for } A\}) < \varepsilon.$$

Proof. The construction is similar to the one from Theorem 17. Let $(P_s)_{s\in\mathbb{N}}$ be a recursive approximation of the given Π^0_1-class $\mathcal{P} = \bigcap_s \mathcal{P}_s$, where $\mathcal{P}_s = [P_s]^{\preceq}$. We computably enumerate $\overline{I} = \bigcup_s \overline{I}_s$.

- *Step* 0. Let $\overline{I}_0 = \{0\}$.
- *Step* $s+1$. Let $n_{-1,s} = 0$. For $i = 0, \ldots, s$ define $n_{i,s}$, the new marker for the i-th indifferent point, as the least number $n > n_{i-1,s}$ such that

  1. $n \notin \overline{I}_s$ and
  2. $\mu(\{A \in \mathcal{P}_s \colon \mu(\mathcal{P}_s|(A \upharpoonright n)) < 1 - 2^{-2i-3}\}) < 2^{-2i-3}$.

Define

$$\overline{I}_{s+1} = \overline{I}_s \cup \bigcup_{0\le i\le s} \{n_{i-1,s}+1,\ldots,n_{i,s}-1\}.$$

Verification. Let us see that $n_{i,s}$, the candidate for the i-th indifferent point, eventually stabilizes. By Lemma 3, there is a k_i such that for all $k \ge k_i$,

$$\mu(\{A \in \mathcal{P}\colon \mu(\mathcal{P}|(A\upharpoonright k)) < 1-2^{-2i-3}\}) < 2^{-2i-3}/2.$$

Taking s_i large enough that $\mu\left(\mathcal{P}_{s_i}\setminus\mathcal{P}\right) < 2^{-2i-3}/2$, we have that for any $s \ge s_i$,

$$\mu(\{A \in \mathcal{P}_s\colon \mu(\mathcal{P}_s|(A\upharpoonright k)) < 1-2^{-2i-3}\}) < 2^{-2i-3}.$$

Now assume, by induction, that $n_{i-1,s}$ has stabilized. If $s \ge s_i$ and $n_{i,s} \ge k_i$, then $n_{i,s}$ has also stabilized. Hence, its value changes only finitely often. Taking $n_i = \lim_s n_{i,s}$, we have shown that $I = \{n_0, n_1, n_2, \ldots\}$ is infinite. By construction, it is a co-c.e. set.

Now we ask, for how many $A \in \mathcal{P}$ is $A[n_i \leftarrow 1-A(n_i)]$ not in $\mathcal{P}$? If $\mu(\mathcal{P}|(A\upharpoonright n_i)) \ge 1-2^{-2i-3}$, then the probability that $A[n_i \leftarrow 1-A(n_i)] \notin \mathcal{P}$ is at most 2^{-2i-3}. Thus by the choice of n_i,

$$\mu(\{A \in \mathcal{P}\colon A[n_i \leftarrow 1-A(n_i)] \notin \mathcal{P}\}) \le 2^{-2i-3}\mu(\mathcal{P}) + 2^{-2i-3} \le 2^{-2i-2}.$$

In other words, $\{n_i\}$ is $\mathcal{P}$-indifferent for all $A \in \mathcal{P}$ except a set of measure at most 2^{-2i-2}. We prove, by induction, that $\{n_0, n_1, \ldots, n_i\}$ is $\mathcal{P}$-indifferent for all $A \in \mathcal{P}$ except a set of measure $\sum_{k=0}^{i} 2^{-k-2}$. For $i = 0$, it is immediate from the previous calculation. Assume that it is true for $i-1$. Take $A \in \mathcal{P}$ for which $\{n_0, n_1, \ldots, n_{i-1}\}$ is indifferent. If $\{n_0, n_1, \ldots, n_i\}$ is not indifferent for A, then there is a $B \in \mathcal{P}$ such that A and B agree except on $\{n_0, n_1, \ldots, n_{i-1}\}$, but $B[n_i \leftarrow 1-B(n_i)] \notin \mathcal{P}$. The measure of sequences $B \in \mathcal{P}$ with the latter property is at most 2^{-2i-2} and each agrees with 2^i sequences A except on $\{n_0, n_1, \ldots, n_{i-1}\}$. Therefore, there are at most $2^i 2^{-2i-2} = 2^{-i-2}$ sequences A for which $\{n_0, n_1, \ldots, n_{i-1}\}$ is indifferent but $\{n_0, n_1, \ldots, n_i\}$ is not. This proves the claim.

Take $A \in \mathcal{P}$. If $\{n_0, n_1, \ldots, n_i\}$ is $\mathcal{P}$-indifferent for A, for all i, then I is $\mathcal{P}$-indifferent for A. This is because $\mathcal{P}$ is a closed set: if B agrees with A on $\overline{I}$, then it is the limit of elements of $\mathcal{P}$, hence also in $\mathcal{P}$. Thus,

$$\mu(\{A \in \mathcal{P}\colon I \text{ is not } \mathcal{P}\text{-indifferent for } A\}) \le \sum_{k=0}^{\infty} 2^{-k-2} = 1/2.$$

Finally, let $I_i = \{n_i, n_{i+1}, n_{i+2}, \ldots\}$, which is again an infinite co-c.e. set. By the same reasoning as above,

$$\mu(\{A \in \mathcal{P}\colon I_i \text{ is not } \mathcal{P}\text{-indifferent for } A\}) \le 2^{-2i-1}.$$

For large enough i, we have $2^{-2i-1} < \varepsilon$. □

Recall that a sequence is 2-random if it is Martin-Löf random relative to $\emptyset'$. By analyzing the previous proof, we will show that there is an infinite co-c.e. set I that is indifferent for every 2-random sequence with respect to the class of Martin-Löf random sequences. It is natural to ask if there is an infinite $I \subseteq \mathbb{N}$ indifferent for every 2-random sequence with respect to the class of 2-*random* sequences; by relativizing Theorem 10, we can show that this is impossible.

Corollary 19. *There is an infinite co-c.e. set I that is* MLR*-indifferent for every 2-random sequence.*

Proof. Let I be the infinite co-c.e. set given by the proof above applied to $\mathcal{P} = 2^\omega \setminus [R_1]^{\preceq}$. We claim that I is MLR-indifferent for every 2-random sequence. Fix i. Using $\emptyset'$, we can find an s such that $\mu(\mathcal{P}_s \setminus \mathcal{P}) \leq 2^{-2i-1}$. Let

$$\mathcal{G}_i = \{A \in \mathcal{P}_s \colon I_i \text{ is not } \mathcal{P}\text{-indifferent for } A\}.$$

If I_i is not $\mathcal{P}$-indifferent for A, then some finite subset of I_i is not $\mathcal{P}$-indifferent for A; this follows from the closure of $\mathcal{P}$. So $\mathcal{G}_i$ is a $\Sigma^0_1[\emptyset']$-class uniformly in i. Furthermore,

$$\begin{aligned} \mu(\mathcal{G}_i) &\leq \mu(\mathcal{P}_s \setminus \mathcal{P}) + \mu(\{A \in \mathcal{P} \colon I_i \text{ is not } \mathcal{P}\text{-indifferent for } A\}) \\ &\leq 2^{-2i-1} + 2^{-2i-1} = 2^{-2i} \leq 2^{-i}. \end{aligned}$$

So $(\mathcal{G}_i)_{i\in\mathbb{N}}$ is a Martin-Löf test relative to $\emptyset'$. Hence, if $A \in \mathcal{P}$ is 2-random, then I_i is $\mathcal{P}$-indifferent for A, for some i. Now assume that B agrees with A on $\overline{I}$. Then there is a B' that agrees with A on $\overline{I_i}$ and differs from B on a finite set. Because I_i is $\mathcal{P}$-indifferent for A, we have $B' \in \mathcal{P}$, so B' is random. Thus B is also random. Therefore, I is MLR-indifferent for any 2-random $A \in \mathcal{P}$.

We still must handle the case of a 2-random $A \notin \mathcal{P}$. Consider the Σ^0_1 classes

$$\mathcal{S}_i = \left\{X \colon \text{no } A \in \mathcal{P} \text{ agrees with } X \text{ on } \overline{\{0, \dots, i\}}\right\}.$$

It follows from Lemma 2 that $\lim_i \mu(\mathcal{S}_i) = 0$. Using $\emptyset'$, we can pick out a subsequence $(\mathcal{S}_{i_m})_{m\in\mathbb{N}}$ such that $\mu(\mathcal{S}_{i_m}) \leq 2^{-m}$, making it a Martin-Löf test relative to $\emptyset'$. So, if A is 2-random (in fact, it is enough for A to be *weakly* 2-random), then there is an $A' \in \mathcal{P}$ that differs from A on a finite set. But this means that $A' \in \mathcal{P}$ is 2-random, so I is MLR-indifferent for A'. Therefore, I is MLR-indifferent for A. □

6 Indifference for being absolutely normal

In Theorem 17 we showed that for any Π^0_1-class of positive measure $\mathcal{P}$, there is a co-c.e. $\mathcal{P}$-indifferent set. The next results uses the same technique but starting from a rather simpler class of reals $\mathcal{C}$ and constructing a computable $\mathcal{C}$-indifferent set. The key point in this construction is that the class $\mathcal{C}$ may not only be computably approximated, but the error at each step of the approximation may be computably bounded.

Theorem 20. *Let $\mathcal{C}$ be a Π^0_1-class with positive measure and let $(\mathcal{C}_i)_{i\in\mathbb{N}}$ be a computable approximation of clopen sets such that $\mathcal{C} = \bigcap_i \mathcal{C}_i$. Let $r \colon \mathbb{N} \to \mathbb{Q}$ be computable such that $\mu(\mathcal{C}_i \setminus \mathcal{C}) \leq r(i)$ and $\lim_i r(i) = 0$. Then there is an infinite computable set that is $\mathcal{C}$-indifferent for a computable $A \in \mathcal{C}$.*

Proof. Uniformly in s we define $(\mathcal{C}_{s,i})_{i\in\mathbb{N}}$, a c.e. sequence of finite clopen sets, a string $\sigma_s \in 2^{<\omega}$ and a function $r_s \colon \mathbb{N} \to \mathbb{Q}$ such that:

1. $\mu(\mathcal{C}_s) > 0$;
2. for all $h_1, \dots, h_s \in \{0, 1\}$, $\sigma_1 h_1 \dots \sigma_s h_s \mathcal{C}_s \subseteq \mathcal{C}$;

3. for any i, $\mu(\mathcal{C}_{s,i} \setminus \mathcal{C}_s) \leq r_s(i)$ and $\lim_i r_s(i) = 0$;

where $\mathcal{C}_s = \bigcap_i \mathcal{C}_{s,i}$.

Construction. We define the required objects by stages:

- *Step* 0. Let $\mathcal{C}_0 = \mathcal{C}$ and $r_0 = r$.
- *Step s+1.* Suppose $\sigma_1, \dots, \sigma_s$, $\mathcal{C}_0, \dots, \mathcal{C}_s$ and $r_0, \dots, r_s$ satisfying conditions 1–3 have already been defined. Do the following search for $n = 1, 2, 3 \dots$ At stage n:
  - Let σ be the n-th string in the length-lexicographic order.
  - Let m be the least number such that $r_s(m) \leq 2^{-|\sigma|-2}$.
  - If $\mu(\mathcal{C}_{s,m}|\sigma) \leq 3/4$ then go to stage $n+1$; else terminate the search.

  Define

$$\begin{aligned} \sigma_{s+1} &= \sigma; \\ \mathcal{C}_{s+1,i} &= \mathcal{C}_{s,i}|\sigma_{s+1}0 \cap \mathcal{C}_{s,i}|\sigma_{s+1}1; \\ r_{s+1} &= 2^{|\sigma_{s+1}|+2} r_s. \end{aligned}$$

Verification. The search of step $s+1$ must eventually terminate because by Lemma 2 there is a string σ with $\mu(\mathcal{C}_s|\sigma) > 3/4$ and hence $\mu(\mathcal{C}_{s,m}|\sigma) > 3/4$ for all m. Suppose at step $s+1$ we find string $\sigma = \sigma_{s+1}$ such that $r_s(m) \leq 2^{-|\sigma|-2}$ and $\mu(\mathcal{C}_{s,m}|\sigma) > 3/4$. Then

$$\begin{aligned} \mu\left(\mathcal{C}_s \cap [\sigma]^{\preceq}\right) &\geq \mu\left(\mathcal{C}_{s,m} \cap [\sigma]^{\preceq}\right) - r_s(m) \\ &> 3 \cdot 2^{-|\sigma|-2} - 2^{-|\sigma|-2} \\ &= 2^{-|\sigma|-1} \end{aligned}$$

and therefore $\mu(\mathcal{C}_s|\sigma) > 1/2$. Then

$$\begin{aligned} \mu(\mathcal{C}_{s+1}) &= \mu(\mathcal{C}_s|\sigma 0 \cap \mathcal{C}_s|\sigma 1) \\ &\geq \mu(\mathcal{C}_s|\sigma 0) + \mu(\mathcal{C}_s|\sigma 1) - 1 \\ &= 2\mu(\mathcal{C}_s|\sigma) - 1 \\ &> 0. \end{aligned}$$

This shows that condition 1 is true. Since both $\sigma 0 \mathcal{C}_{s+1}$ and $\sigma 1 \mathcal{C}_{s+1}$ are included in $\mathcal{C}_s$, condition 2 is also verified. To verify condition 3, let $\mathcal{A}_i = \mathcal{C}_{s,i}|\sigma 0$, $\mathcal{A} = \bigcap_i \mathcal{A}_i$, $\mathcal{B}_i = \mathcal{C}_{s,i}|\sigma 1$ and $\mathcal{B} = \bigcap_i \mathcal{B}_i$. Since $(\mathcal{A}_i \cap \mathcal{B}_i) \setminus (\mathcal{A} \cap \mathcal{B}) \subseteq (\mathcal{A}_i \setminus \mathcal{A}) \cup (\mathcal{B}_i \setminus \mathcal{B})$ then

$$\begin{aligned} \mu(\mathcal{C}_{s+1,i} \setminus \mathcal{C}_{s+1}) &= \mu((\mathcal{A}_i \cap \mathcal{B}_i) \setminus (\mathcal{A} \cap \mathcal{B})) \\ &\leq \mu(\mathcal{A}_i \setminus \mathcal{A}) + \mu(\mathcal{B}_i \setminus \mathcal{B}) \\ &= \mu((\mathcal{C}_{s,i} \setminus \mathcal{C}_s)|\sigma 0) + \mu((\mathcal{C}_{s,i} \setminus \mathcal{C}_s)|\sigma 1) \\ &\leq 2^{|\sigma|+2} \mu(\mathcal{C}_{s,i} \setminus \mathcal{C}_s) \\ &\leq 2^{|\sigma|+2} r_s(i) \\ &= r_{s+1}(i). \end{aligned}$$

Finally, we define $A = \sigma_1 0 \sigma_2 0 \sigma_3 0 \dots$ and we have that $\sigma_1 h_1 \sigma_2 h_2 \sigma_3 h_3 \dots \in \mathcal{C}$ for any $h_i \in \{0, 1\}$. Hence, for $i \geq 1$ we define $n_i = i - 1 + \sum_{1 \leq s \leq i} |\sigma_s|$ and we define the computable $\mathcal{C}$-indifferent set as $\{n_1, n_2, n_3 \dots\}$. $\square$

The idea of *normality* for reals is that every digit and block of digits appears equally frequent in its expansion for base q. Of course, this definition depends on the base. *Absolutely normal* reals are normal in every base. More precisely, a real A is *normal* in base q if for every word $\gamma \in \{0, \dots, q-1\}$

$$\lim_{n\to\infty} Q(A, q, \gamma, n)/n = q^{-|\gamma|},$$

where $Q(A, q, \gamma, n)$ denotes the number of occurrences of the word γ in the first n digits after the fractional point in the expansion of A in the base of q. A is *absolutely normal* if it is normal to every base $q \geq 2$. Let AN be the class of all absolutely normal reals. Each random real is absolutely normal, but the reverse is not true.

An exponential complexity bound for computing an absolutely normal number follows from the work of Lutz [9], Ambos-Spies, Terwjin and Zheng [2] and Ambos-Spies and Mayordomo [1] on reals that are random with respect to polynomial-time martingales (i.e., no polynomial-time computable martingale succeeds on such a real). On the one hand, one can formulate a quadratic-time computable martingale which succeeds on all reals in $[0, 1]$ that are not absolutely normal. Therefore, being n^2-computably random already implies being absolutely normal. On the other hand, they show that there exist n^2-computably random sequences computable in exponential time.

A direct construction of computable absolutely normal reals was shown in [3, 4]. The construction is based on the existence of a sequence $(\mathcal{D}_i)_{i\in\mathbb{N}}$ such that $\mathcal{D}_{i+1} \subseteq \mathcal{D}_i \subseteq [0, 1]$, and such that $\mathcal{D} = \bigcap_i \mathcal{D}_i$ has positive measure and only contains absolutely normal reals. Each $\mathcal{D}_i$ is the union of finitely many intervals with rational endpoints. Furthermore, the whole sequence is computable, in the sense that we can bound the error at each step, i.e. there is a computable function $r\colon \mathbb{N} \to \mathbb{Q}$ such that $\mu(\mathcal{D}_i \setminus \mathcal{D}) \leq r(i)$ and $\lim_i r(i) = 0$. Moreover, each $(\mathcal{D}_i)_{i\in\mathbb{N}}$ is uniformly computably, that is, there are computable functions $f\colon \mathbb{N}\times\mathbb{N} \to \mathbb{Q}\cap[0, 1]$ and $g\colon \mathbb{N} \to \mathbb{N}$ such that

$$\mathcal{D}_i = \bigcup_{j=1}^{g(i)} (f(i, j), f(i, j+1))\,.$$

Now, there is nothing special in Theorem 20 requiring $\mathcal{C}_i$ to be clopen instead of finite sets of intervals with rational endpoints. One could replace $\mathcal{C}$ and $(\mathcal{C}_i)_{i\in\mathbb{N}}$ with the sets $\mathcal{D}$ and $(\mathcal{D}_i)_{i\in\mathbb{N}}$ described above and the same argument goes through. Then, we obtain the following result:

Corollary 21. *There is an infinite computable* AN*-indifferent set for a computable set.*

Proof. Immediate from Theorem 20 and the discussion above. □

This also shows that there are absolutely normal reals not only in the degree $\mathbf{0}$, but in *each* Turing degree, in fact, in each non-trivial many-one degree. This, of course, is false for random reals.

Corollary 22. *For every $A \notin \{\emptyset, \mathbb{N}\}$, there is an absolutely normal real B such that $A \equiv_m B$.*

Proof. By Corollary 21, let $I = \{n_0, n_1, n_2, \dots\}$ be an infinite computable AN-indifferent set for a computable $\tilde{B}$. Let $A \in 2^\omega$ and define B in the following way: $B(n_i) = A(i)$ for all $i \in \mathbb{N}$ and $B(n) = \tilde{B}(n)$ if $n \notin I$. Since $\tilde{B}$ is absolutely normal, B also is. By construction it is clear that $A \leq_m B$, and if $A \notin \{\emptyset, \mathbb{N}\}$, $B \leq_m A$. □

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References

[1] Klaus Ambos-Spies and Elvira Mayordomo. Resource bounded measure and randomness. In A. Sorbi, editor, *Complexity Logic and Recursion Theory*, pages 1–47. Marcel Dekker, New York NY, 1997.

[2] Klaus Ambos-Spies, Sebastiaan Terwijn, and Xizhong Zheng. Resource bounded randomness and weakly complete problems. *Theoretical Computer Science*, 172:195–207, 1997.

[3] Verónica Becher and Santiago Figueira. An example of a computable absolutely normal number. *Theoretical Computer Science*, 270:947–958, 2002.

[4] Verónica Becher, Santiago Figueira, and Rafael Picchi. Turing's unpublished algorithm for normal numbers. *Theoretical Computer Science*, 70:126–138, 2007.

[5] Péter Gács. Every set is reducible to a random one. *Information and Control*, 70:186–192, 1986.

[6] Carl Jockusch Jr. and Robert I. Soare. Π^0_1 classes and degrees of theories. *Transactions of the American Mathematical Society*, 173:33–56, 1972.

[7] Antonin Kučera. Measure, Π^0_1 classes, and complete extensions of PA. *Lecture Notes in Mathematics*, 1141:245–259, 1985.

[8] Richard E. Ladner. Mitotic recursively enumerable sets. *The Journal of Symbolic Logic*, 38(2):199–211, 1973.

[9] Jack H. Lutz. Category and measure in complexity classes. *SIAM Journal of Computing*, 19:1100–1131, 1990.

[10] Per Martin-Löf. The definition of random sequences. *Information and Control*, 9:602–619, 1966.

[11] Joseph S. Miller and Liang Yu. On initial segment complexity and degrees of randomness. *Transactions of the American Mathematical Society*, 360(6):3193–3210, 2008.

[12] John C. Oxtoby. *Measure and Category.* Springer, New York, 2nd edition, 1980.

[13] Claus-Peter Schnorr. Zufälligkeit und Wahrscheinlichkeit. *Lecture Notes in Mathematics*, 218, 1971.

[14] Robert I. Soare. *Recursively enumerable sets and degrees.* Springer, Heidelberg, 1987.

[15] B. A. Trahtenbrot. On autoreducibility. *Dokl. Akad. Nauk SSSR*, 192:1224–1227, 1970.

[16] Michael van Lambalgen. *Random sequences.* PhD thesis, University of Amserdam, 1987.

[17] Michael van Lambalgen. The axiomatization of randomness. *Journal of Symbolic Logic*, 55:1143–1167, 1990.