CDMTCS
Research
Report
Series

Incompleteness: A Personal Perspective

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CDMTCS-324
June 2008

Centre for Discrete Mathematics and
Theoretical Computer Science

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Abstract

After proving the completeness of the system of predicate logic in his doctoral dissertation (1929), Gödel has continued the investigation of the completeness problem for more comprehensive formal systems, especially for systems encompassing all known methods of mathematical proof. In 1931 (see [28, 25]) Gödel proved his famous *(first) incompleteness theorem*, which in modern terms reads:

> Every computably enumerable, consistent axiomatic system containing elementary arithmetic is incomplete, that is, there exist true sentences unprovable by the system.

Our aim is to present a personal view of the incompleteness phenomenon. We will focus on interesting/natural concrete independent sentences, on the source of incompleteness, and on how common the incompleteness phenomenon is. Some open questions will be briefly stated.

1 The incompleteness theorem

In modern language the incompleteness theorem [28] can be expressed in the following form:

Theorem 1 *Every axiomatic system $\mathcal{F}$ which is*
(1) finitely specified,
(2) rich enough to include the arithmetic, and
(3) arithmetically sound,
is incomplete; that is, there exists (and can be effectively constructed) a sentence of arithmetic which
(A) can be expressed in $\mathcal{F}$,

(B) is true, and
(C) is unprovable by $\mathcal{F}$.

Our main example of an axiomatic theory is the Zermelo–Fraenkel set theory with choice, ZFC. We fix an interpretation of Peano Arithmetic (PA) in ZFC. Each sentence of the language of PA has a translation into a sentence of the language of ZFC, determined by the interpretation of PA in ZFC. A "sentence of arithmetic" indicates a sentence of the language of ZFC that is the translation of some sentence of PA.

Condition (1) says that there is a computable function listing all axioms and inference rules (which could be infinite): the axioms and inference rules form a c.e. set. We cannot take as axioms all true arithmetical sentences because this set is not c.e.

Condition (2) says that $\mathcal{F}$ has all the symbols and axioms used in arithmetic, the symbols for 0 (zero), S (successor), $+$ (plus), $\times$ (times), $=$ (equality) and the axioms making them work (as, for example, $x + S(y) = S(x + y)$). Condition (2) says that $\mathcal{F}$ is "infinite"; it cannot be satisfied if we do not have individual terms for the natural numbers. For example, Tarski proved that the Euclidean geometry, which refers to points, circles and lines, is complete.

Finally, (3) says that every sentence of arithmetic proved by $\mathcal{F}$ is true; in ZFC a true sentence of arithmetic is a sentence whose translation in PA is true in the standard model of PA.

Incompleteness shows that there are more true sentences than provable sentences (in a given axiomatic theory with the required properties). A closer analogy is the relation between what is true and what can be proved in court.[1]

One could replace conditions (B) and (C) in Theorem 1 by the following condition

(B') is neither provable or disprovable by $\mathcal{F}$,

which makes no reference to truth.

In what follows we will fix

> *an axiomatic theory $\mathcal{F}$ satisfying the properties (1), (2), (3) in Theorem 1*

and let X be the underlying alphabet of $\mathcal{F}$. A sentence which $\mathcal{F}$ cannot prove nor disprove is called *independent* (of $\mathcal{F}$). A set $A \subseteq X^*$ is said to be *expressible* in $\mathcal{F}$ (or *definable* in the language of $\mathcal{F}$) if there is a predicate H of $\mathcal{F}$ such that for all $x \in X^*$, $H(x)$ is true iff $x \in A$.

Let *Arith* be the set of true sentences of arithmetic expressible in PA.

[1]The Scottish judicial system which admits three forms of verdict, guilty, not-guilty and not-proven, comes closer to the spirit of incompleteness.

Theorem 2 *No axiomatic theory $\mathcal{F}$ can prove all sentences of Arith.*

2 Incompleteness as a theorem in computability theory

We assume that the reader is familiar with the notion of computable function and set (on naturals or strings) and with the basic theorem stating the existence of a computable function f such that the set it enumerates $S = \{f(0), f(1), f(2), \ldots, f(n), \ldots\}$ is not computable. Sets enumerated by computable functions are called computably enumerable (c.e.). For details see any textbook in computability theory, for example [35].

Inspired by Davis [20], we consider all propositions s_n of the form "$n \notin S$", where S is the above set and n is a natural number. Let $\mathcal{F}$ be an axiomatic theory containing (among other sentences) all propositions s_n. We assume that: a) $\mathcal{F}$ is sound for all s_n, i.e. whenever $\mathcal{F}$ proves s_n, then $n \notin S$, b) there is a computable function t which enumerates all propositions s_n that $\mathcal{F}$ can prove, i.e. $\{t(0), t(1), \ldots, t(m), \ldots\} = \{s_i \mid \mathcal{F} \text{ proves } s_i, i \geq 0\}$.

In this setting Gödel's incompleteness theorem can be stated as follows:

Theorem 3 *If $\mathcal{F}$ is an axiomatic system satisfying a) and b) above, then there is a natural number N such that $N \notin S$, but $\mathcal{F}$ cannot prove s_N ($\mathcal{F}$ cannot prove the true proposition s_N).*

Proof. Assume by contradiction that there is no such N, i.e. for all n, $\mathcal{F}$ proves s_n iff $n \notin S$. We consider the function $g(n)$ defined by the following algorithm:

> "in parallel generate the pairs $(f(0), t(0)), (f(1), t(1)), \ldots$ till at some stage we get $i \geq 0$ such that $n = f(i)$ (in which case we put $g(n) = 1$), or $j \geq 0$ with $s_n = t(j)$(when we put $g(n) = 0$),"

where f enumerates S and t comes from b).

We claim that g is computable. Indeed, if $n \in S$, then $n = f(i)$, for some $i \geq 0$, so $g(n) = 1$; otherwise, $n \notin S$, so $s_n = t(j)$, for some $j \geq 0$, hence $g(n) = 0$. Because $\mathcal{F}$ is sound it is impossible to have a pair i, j such that $n = f(i)$ and $s_n = t(j)$ and because $n \in S$ or $n \notin S$ it is impossible for the procedure computing $g(n)$ to continue indefinitely. We have reached a contradiction because for all $n \geq 0$, $g(n) = 1$ iff $n \in S$, showing that S is computable. $\Box$

3 Interpreting incompleteness

Gödel wrote his paper very carefully. Speculating on his extreme caution, Feferman [24] stated that Gödel "could have been more centrally involved in the development of the fundamental concepts of modern logic—*truth* and *computability*—than he was." Gödel took pain to convince various people about the validity of his assertions and results, but he avoided any public debate and considered his results to have been accepted by those whose opinion mattered to him.

There is a variety of reactions to incompleteness, ranging from pessimism to optimism or simple dismissal. For pessimists, this result can be interpreted as the final, definite failure of any attempt to formalise the whole of mathematics. For example, H. Weyl acknowledged that the incompleteness theorem has exercised a "constant drain on the enthusiasm" with which he engaged himself in mathematics. In contrast, scientists like Dyson acknowledge the limit placed by incompleteness on our ability to discover the truth in mathematics, but interpret this in an optimistic way, as a guarantee that mathematics will go on forever (see Barrow [1], pp. 218–221). According to Smoryński [36], "students who hear of Gödel Theorem either recover from it or else go on to become experts in mathematical logic". A lucid analysis of the impact of the incompleteness theorem in physics is presented in Barrow [2] (see also [16, 38, 12]).

Here are two opinions expressed in 2006 with the occasion of Gödel's centenary and 75 years since Gödel's incompleteness theorem was published. For Davis [20], "Gödel's theorem had made it clear that no single formal system could be devised that would enable all mathematical truths, even those expressible in terms of basic operations on the natural numbers, to be provided with a formal proof." Feferman [23] wrote: "my view of Gödel's incompleteness theorems is that their relevance to mathematical logic (and its offspring in the theory of computation) is paramount; further, their philosophical relevance is significant, but in just what way is far from settled; and finally, their mathematical relevance outside of logic is very much unsubstantiated but is the object of ongoing, tantalizing efforts."

4 Three questions

In what follows we will discuss the following three questions on incompleteness:

- Are there interesting/natural concrete independent sentences?
- What is the source of incompleteness?
- How common is the incompleteness phenomenon?

An analogy suggested in [23] can be used to clarify these questions. Cantor's diagonal proof shows the existence of transcendental reals but doesn't provide any natural/interesting concrete examples. Liouville constructed an interesting class of examples of transcendental reals, but his method was not directly useful for showing that a natural example of real (like π, e) is transcendental; however, Liouville's method shows a source of transcendence (Liouville numbers can be approximated "quite closely" by rationals). Ferdinand von Lindemann's proof showed that π, the most interesting real number, is transcendental. Finally, are there "many" transcendental reals? The answer is yes in both measure and category [34].

5 Are there interesting independent sentences?

Gödel's proof shows the existence of independent sentences: even more, one can effectively construct infinitely many such sentences, but[2] it gives no interesting concrete examples of independent sentences.

An axiomatic system is *consistent* if it does not prove the assertion "0=1". The first interesting example appears in *Gödel's second incompleteness theorem*:

Theorem 4 *Every axiomatic theory $\mathcal{F}$ cannot prove its own consistency.*

Gödel [29] found the first combinatorial $\forall\exists$–sentence[3] which is independent in PA. Referring to normalisation for a typed extension of lambda-calculus—the system T, Gödel's independent sentence is

> *Each term has a normal form T.*

Gödel's proof was extended and improved by Girard (see [26] and the discussion in Longo [31]). Other combinatorial $\forall\exists$–sentences true but unprovable in PA include Paris and Harrington modified form of the finite Ramsey theorem [33] and Kruskal-Friedman theorem [31]; they generated many other results (see [4] for a list). Diophantine examples are discussed by Matiyasevich in [32]; for a uniform way to generate true and unprovable sentences see [11].

Interesting $\forall\exists$–sentences appear in algorithmic information theory (see also the discussion in [5]). A prefix-free (Turing) machine is a Turing machine from bit strings to bit strings whose domain is a prefix-free set. A prefix-free machine is universal if it can simulate every prefix-free machine. The prefix complexity of the string x (induced by U) is defined by $H(x)(= H_U(x)) = \min\{|w| \mid U(w) = x\}$ (see more in [7]).

As first example consider all sentences of the form

$$H(x) > m,$$

[2] As Cantor's proof of the existence of transcendental reals.

[3] A sentence of the form $\forall x \exists y R(x, y)$, where R is a computable predicate is a $\forall\exists$–sentence.

where x is a string and m is a natural number. Clearly, $H(x) > m$ iff $\forall y \exists t$ ($U(y)$ stops in time t and $U(y) = x \Longrightarrow |y| > m$), is a $\forall\exists$–sentence. Chaitin [14] (see also the presentation in [19]) proved the following:[4]

Theorem 5 *Consider an axiomatic theory $\mathcal{F}$. Then, there exists a constant c (depending on $\mathcal{F}$) such that if $\mathcal{F}$ proves a sentence of the form "$H(x) > m$", then $m < c$.*

As H is unbounded, there are infinitely many true sentences of the form "$H(x) > m$" that $\mathcal{F}$ cannot prove. The axiomatic theory can be coded itself by a string, so it makes sense to talk about $H(\mathcal{F})$.[5]

Interesting examples of independent $\forall\exists$–sentences appear in connection with the bits of Chaitin's Omega number. The halting probability Ω_U of a prefix-free universal machine U (see [15]) is defined by $\Omega_U = \sum_{U(x) \text{ is defined}} 2^{-|x|}$. In [15] Chaitin proved

Theorem 6 *Assume that ZFC is arithmetically sound. Then, for every prefix-free universal machine U, ZFC can determine the value of only finitely many bits of Ω_U, and one can give a bound on the number of bits of Ω_U which ZFC can determine.*

The real Ω_U depends on U, and so by constructing a special U Solovay [37] proved the following:

Theorem 7 *There effectively exists a prefix-free universal machine U such that ZFC (if arithmetically sound) cannot determine any bit of Ω_U.*

This result was generalised in [6] as follows:

Theorem 8 *Assume that ZFC is arithmetically sound. Consider a prefix-free machine U which PA proves universal and assume that Ω_U is written in binary as follows:*

$$\Omega_U = 0.\omega_0\omega_1 \ldots \omega_{i-1}\omega_i\omega_{i+1} \ldots, \text{ where } \omega_0 = \omega_1 = \ldots = \omega_{i-1} = 1, \omega_i = 0.$$

Then, we can effectively construct a prefix-free universal machine U' (depending upon ZFC and U) such that PA proves universal, $\Omega_U = \Omega'_U$, and ZFC can determine at most i initial bits of Ω'_U.

[4] In [13] the state complexity of 3-tape-symbol Turing machines was used to prove a similar result.

[5] A false interpretation of Theorem 5 might say that the complexity of theorems proven by $\mathcal{F}$ is bounded by $H(\mathcal{F}) + c$. Indeed, if the set of theorems proven by $\mathcal{F}$ is infinite, then their program-size complexities will be arbitrarily large.

Is it possible to find simpler concrete independent sentences? A natural idea is to look at $\forall$–sentences[6]. Goldbach's conjecture or Riemann hypothesis are $\forall$–sentences. Are they independent of ZFC? Of course, *this is not known.* However, as they are $\forall$–sentences one can associate to each of them a program which never halts iff the conjecture is true. In [8] such programs have been effectively constructed, Π_G (for Goldbach's conjecture) has 3,484 bits and Π_R (for Riemann hypothesis) has 7,780 bits. Solving the Halting Problem for relatively small-size programs would solve these questions.

Two similar conjectures seem to be different. Define the function $T(x) = x/2$, if x is even, and $T(x) = 3x + 1$, if x is odd. The famous conjecture by Collatz is:

Collatz' conjecture. *For every $a > 0$, there is an iteration N such that $T^N(a) = 1$.*

The reverse (mirror) of a number is the number formed with the same decimal digits but written in the opposite order. For example, the mirror of 12 is 21, the mirror of 131072 is 270131, etc. Start with the decimal representation of a natural a, reverse the digits and add the constructed number to a; iterate this process till the result is a palindrome. Following [21] we have:

The palindrome conjecture. *For every a, a palindrome number will be obtained after finitely many iterations of the above procedure.*

In [8] it was proved that Collatz conjecture is a $\forall$–sentence, but the proof—based on the fact that the set of natural numbers a satisfying the conjecture is c.e.—*is not constructive.* The same argument applies also to the palindrome conjecture. *We don't know whether there is no constructive proof for the fact that each conjecture is a $\forall$–sentence.* This suggests that *the Collatz and palindrome conjectures are more likely to be unprovable than Goldbach or Riemann conjectures.*

Another possibility to get a simpler independent statement is suggested by the following two conjectures stated by Dyson [22].

Dyson's first conjecture. *The reverse (in decimal) of a power of two is never a power of five.*

Dysons plausibility argument is based on the following heuristics:

[6] A sentence of the form $\forall x R(x)$, where R is a computable predicate is a $\forall$–sentence.

> The digits in a big power of two seem to occur in a random[7] way without any regular pattern. If it ever happened that the reverse of a power of two was a power of five, this would be an unlikely accident, and the chance of it happening grows rapidly smaller as the numbers grow bigger. If we assume that the digits occur at random, then the chance of the accident happening for any power of two greater than a billion is less than one in a billion. It is easy to check that it does not happen for powers of two smaller than a billion. So the chance that it ever happens at all is less than one in a billion. That is why I believe the statement is true.

Dyson's second conjecture. *Dyson's first conjecture is unprovable in ZFC.*

Dyson's argument in favour of the second conjecture is:

> But the assumption that digits in a big power of two occur at random also implies that the statement is unprovable. Any proof of the statement would have to be based on some non-random property of the digits. The assumption of randomness means that the statement is true just because the odds are in its favour. It cannot be proved because there is no deep mathematical reason why it has to be true.

6 What is the source of incompleteness?

In Theorem 5 the high H–complexity of sentences "$H(x) > m$" with $m > c$ is a source of their unprovability. Chaitin has formulated the following "information-preservation principle":

> *The theorems of a finitely specified theory cannot be significantly more complex than the theory itself.*

Is every true sentence s with $H(s) > c$ unprovable by the theory? Unfortunately, the answer is *negative* because only finitely many sentences s have complexity $H(s) < c$ in contrast with the fact that the set of all theorems of the theory is infinite; see also [17] (reprinted in [18] pp. 55–81) and [38], pp. 123–125.

Chaitin's "information-preservation principle" was proved in [9] for δ, a computable variation of the prefix-free complexity H:

[7] These binary strings are not random in algorithmic information theory sense [7] because their H–complexity is about the logarithm of their length.

$$\delta(x) = H(x) - |x|.$$

To exclude the fact that the "information-preservation principle" is a consequence of some particular way of writing/coding the theorems in the given axiomatic theory we need to show that it is invariant with respect to every "acceptable" codification of sentences of the theory.

Let X be an alphabet with Q elements for the axiomatic theory $\mathcal{F}$. Consider a computable, one-to-one binary coding g of the set of sentences of $\mathcal{F}$. The δ-complexity of a sentence $u \in \mathcal{F}$ induced by g is defined by:

$$\delta_g(u) = H_2(g(u)) - \lceil \log_2 Q \rceil \cdot |u|_Q. \tag{1}$$

In [9] one proves the following result:

Theorem 9 *For every axiomatic theory $\mathcal{F}$ and for any computable, one-to-one function g, we can compute a bound N such that no sentence x with complexity $\delta_g(x) > N$ can be proved in the theory.*

Question 1. *Find other natural measures of complexity for which Chaitin's "heuristic principle" holds true.*

Sentences expressed by strings with large δ–complexity are unprovable. Theorem 9 does not hold true for an arbitrary finitely-specified theory as there are c.e. sets containing strings of arbitrary large δ-complexity. It is possible to have incomplete theories without high δ-complexity sentences; for example, an incomplete theory for propositional tautologies.

Question 2. *In the context of Theorem 9, are there independent sentences x with low δ_g-complexity?*

7 How common is the incompleteness phenomenon?

Is incompleteness a mere linguistic trick, an accidental phenomenon? To answer this question we need to measure the "size of the set of independent sentences" of an axiomatic theory $\mathcal{F}$. There are two possibilities and an important restriction: we can use either topological or probabilistic methods, but we have to work with constructive notions as the space of sentences is countable. We shall see that in both topological and probabilistic terms incompleteness is ubiquitous.

For every non c.e. set $A \subseteq X^*$ expressible in $\mathcal{F}$, the set $I(A)$ of all independent sentences of the form "$s \in A$" is non-empty and, indeed, infinite. *How large is $I(A)$?*

We start with a few elementary facts in topology, see [27]. The equivalence class of x induced by the relation $\equiv$ is denoted by $[x]_\equiv$. A set is saturated with respect to an equivalence relation if it is a union of equivalence classes. Let τ be a topology on X^* and let $\mathbf{C}_\tau$ be its closure operator. A set $A \subseteq X^*$ is said to be rare with respect to τ if for every $x \in X^*$ and every open neighbourhood N_x of x, one has $N_x \not\subseteq \mathbf{C}_\tau(A)$. A subset of X^* is dense if its closure is equal to X^* and it is co-rare if its complement is rare. A dense set is "larger" than a rare one, and a co-rare set is "larger" than a dense set.

The topologies considered as examples in the sequel are generated by partial orders on X^* in the standard way: the closure operator $\mathbf{C}_{\tau_\leq}$ is given by $\mathbf{C}_{\tau_\leq}(A) = \{u \in X^* \mid \exists v \in A, u \leq v\}$, for $A \subseteq X^*$ (see [27], p. 57–58). For any $u \in X^*$, let $N_u^{\leq} = \{v \mid v \in X^*,\ u \leq v\}$ be the open neighbourhood of u.

To exclude situations in which equivalence classes tend to form "clusters" we will consider only topologies with the following property:

> (F): *There is a computable equivalence relation $\equiv$ on X^* such that for every $x \in X^*$ and every open neighbourhood N_x of x, the set $\{y \mid y \in X^*,\ N_x \cap [y]_\equiv = \emptyset\}$ is finite.*

In [10] one proves the following:

Theorem 10 *Suppose that the topology τ is generated by a computable and length preserving partial order and satisfies (F) with respect to a computable equivalence relation $\equiv$. For every non c.e. set $A \subseteq X^*$ expressible in an axiomatic theory $\mathcal{F}$ saturated by $\equiv$, the set $I(A)$ is co-rare in τ.*

There are many natural topologies satisfying the above condition in Theorem 10, the prefix–topology among them (see the examples and Lemma 5.1 in [10]).

The topological results have been reinforced in [9] in probabilistic terms. Let g be a computable, one-to-one binary coding for the sentences of $\mathcal{F}$, and consider a) the probability $p_g^{\mathrm{prov}}(n)$ that a sentence of length n is provable in $\mathcal{F}$ and, b) the probability $p_g^{\mathrm{true}}(n)$ that a sentence of length n is true.[8]

[8]These probabilities depend on g in the same way as the complexity δ_g depends on g (see (1).

Theorem 11 *In every axiomatic theory $\mathcal{F}$, for all g, we have $\lim_{n\to\infty} p_g^{\rm prov}(n) = 0$, but $\lim_{n\to\infty} p_g^{\rm true}(n) > 0$.*

Even if there exist independent sentences with low δ_g-complexity sentences (see Question 2), in view of Theorem 11, the probability that a true sentence of length n with δ_g-complexity less than or equal to N is unprovable in the theory tends to zero when n tends to infinity.

The complexity measure ρ defined by $\rho(x) = H(x)/|x|$ is bounded, so Theorem 9 is trivially valid for ρ. However, Grenet [30] proved that Theorem 11 fails to be true for ρ.

Question 3. **[P. Cholak]** *Is there is a sequence of computable, one-to-one binary codings g_i of sentences such that $\lim_{n,i\to\infty} p_{i,n}^{\rm true} = 0$?*

Acknowledgement

I thank Elena Calude, Bruno Grenet and Marius Zimand for useful comments which improved the paper.

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