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Physical Unknowables

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Physical unknowables*

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Abstract

Different types of physical unknowables are discussed. Provable unknowables are derived from reduction to problems which are known to be recursively unsolvable. Recent series solutions to the *n*-body problem and related to it, chaotic systems, may have no computable radius of convergence. Quantum unknowables include the random occurrence of single events, complementarity and value indefiniteness.

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> … as we know, there are known knowns;
> there are things we know we know.
> We also know there are known unknowns;
> that is to say we know there are some things we do not know.
> But there are also unknown unknowns –
> the ones we don't know we don't know.
> *United States Secretary of Defense Donald H. Rumsfeld*
> *at a Department of Defense news briefing on February 12, 2002*

I. ISLANDS OF PRELIMINARY INSIGHTS IN AN OCEAN OF IGNORANCE

Throughout history, the demand to form the physical world according to people's wishes has been counterbalanced with the inability to predict and manipulate large portions of the human habitat. As time passed, humankind was able to figure out ways to tune ever increasing fragments of the world according to its needs. From a purely behavioral perspective, this is brought about in the way of opportunistic quasi-causal rules of the following kind, "if we do this, we obtain that." A typical example of such a rule is, "if we move a particular kind of electric on/off switch, the lights go on, and the room turns from dark to bright."

How do we arrive at those kinds of rules? Guided by our suspicions, thoughts, formalisms and by pure chance, we "fiddle" and "roam around," inspecting portions of our world and examining their behavior. We observe repeating patterns of behavior and pin them down by reproducing them. A physical behavior is anything that can be observed and thus operationally obtained and measured; e.g., the rise and fall of the sun, the ignition of fire, the formation and the melting of ice. Note that, due to the finiteness of the resolution, all kinds of physical behaviors, even the ones that appear continuous, can be discretized. Ultimately, all physical experiences can be broken down into yes-no propositions representable by zeroes and ones, by sequences of single clicks in detectors.

As we observe physical behaviors, we attempt to "understand" them by trying to figure out the "cause" (1) or "reason" for the behavioral patterns. That is, we invent virtual parallel worlds of thoughts and intellectual concepts such as "electric field" or "mechanical force" to "explain" the behavioral patterns. We call these creations of our minds "physical theories." Contemporary physical theories are heavily formalized, utilizing almost every branch of mathematics and formal logic which could have been imagined so far. A "good theory" provides us with the feeling of a key properly fitting into the lock of a treasure chest, a key unlocking new ways of world comprehension and manipulation.

The methods we employ are pretty reliable. Reliability yields a feeling of security and consolation. It strengthens the belief in the applicability and the overall validity of the method.

Ideally, an "explanation" should be as compact as possible and should apply to as many behavioral patterns as possible. We also have the feeling that, as we are able to manipulate more and more fragments of our habitat, we are converging to some final truth. Ultimately, we seek theories of everything (2) predicting and manipulating the phenomena at large.

In the extreme form, we envision ourselves as becoming empowered with omniscience and omni-influence: we presume that our ability to manipulate and tune the world is limited by our own phantasies alone, and any constraints whatsoever can be bypassed or overcome one way or another. Indeed, some of what in the past has been called magic, mystery and the beyond has been realized in everyday life. Many wonders of witchcraft have been transferred into the realm of the physical sciences. Take, for example, our abilities to fly, the capability to transmute lead into gold, to listen and speak to far away friends, or to cure bacterial diseases with a few pills of antibiotics.

Thereby, we not only trust the rules merely syntactically in the operational sense, but we take for granted the semantic significance of the physical theories that "let us understand" the behavioral patterns and even lead us to novel predictions of behaviors. Pointedly stated, we not only accept physical theories as pure abstractions and constructions of our own mind, but we associate meaning and truth to them. We grant absolute status to our own constructions of mind, purporting that they somehow are metaphysically real and eternal; so much so that only very reluctantly we admit their preliminary character.

Alas, the possibility to formulate theories per se; and in particular the applicability of formal, mathematical models, comes as a surprise. There appears to be an unreasonable effectiveness of mathematics in the natural sciences (3) which seems difficult to explain within science proper. It is not too unreasonable to speculate that any such reasoning might be metaphysical.

Sometimes we have the strength to face suspicions that, to put it in analogy to Shakespeare's poetry, our own constructions and the baseless fabric of our vision, just like the great globe itself, shall dissolve and leave not a rack behind. Our physical theories are such stuff as dreams are made on, and our little islands of transient insights are rounded with an ocean of ignorance.

II. PROVABLE PHYSICAL UNKNOWABLES

In the past century, unknowability has been formally derived along the notion of unprovability, accompanied by a precise meaning of provability (4–6). In formal logic (7) and the foundations of mathematics (8; 9) as well as theoretical computer sciences (10; 11), unprovability has been established as a concept proper. Those theoretical frameworks proved strong enough to derive some of their own limitations; among them their incompleteness and overall consistency.

This is a remarkable departure from informal suspicions and observations regarding the limitations of our worldview. No longer is one reduced to informal, heuristic contemplations and comparisons about what one knows or can do versus one's unpredictability and incapability. Formal unknowability is about formal proofs of unpredictability and impossibility.

Almost since its discovery, attempts (12; 13) have been made to translate formal incompleteness into the physical science, mostly by reduction to the halting problem (14–16). Here reduction means that physical undecidability is linked or reduced to logical undecidability. A typical example for such a reduction is the embedding of a Turing machine or any type of computer capable of universal computation into a physical system. As a consequence, the physical system inherits any type of unsolvability derivable for universal computers, such as the unsolvability of the halting problem: since the computer is part of the physical system, so are its behavioral patterns.

A clear distinction should be made from the onset regarding two different types of unknowables in the natural sciences: unknowables about physical systems and their phenomena and behaviors on one hand, and unknowables of the formal theoretical descriptions and models on the other hand. This section will mostly concern the first type of physical unknowability; the one which is associated with deterministic physical systems.

A. Intrinsic self-referential observers

Every physical observation is essentially (i) discrete, (ii) finite and (iii) self-referential. Whereas finiteness and discreteness has been briefly mentioned earlier, self-referentiality is a seldom recognized system science aspect of physical world perception.

Let us start with the assumption that there exist observers measuring objects, and that observer and object are distinct from one another. That is, there exists a "cut" between observer and object. Through that cut, information is exchanged.

If we insist on idealized one-way observation, information is transferred from the object to the observer via the cut. In this scenario, the object is a transmitter, and the observer is the receiver.

Symbolically, we may regard the object as an agent contained in a "black box," whose only relevant emanations are representable by finite strings of zeroes and ones appearing on the cut, which can be modeled by any kind of screen or display. In this purely syntactic point of view, a physical theory should be able to render identical symbols like the ones appearing through the cut. That is, a physical theory should be able to mimic or emulate the black box it purports to apply to. This view is often adapted in quantum mechanics, where it is difficult to find any meaning (17) for the theory.

A sharp distinction between a physical object and an extrinsic, outside observer is a rarely affordable abstraction. Examples are astronomy, blackbody radiation and classical physical configurations allowing an infinitely small (relative to the entire system) subsystem to convey the information transfer.

We are mostly interested in another scenario, in which the observer is part of the system to be observed. There, the measurement process is modeled symmetrically, and information is exchanged between observer and object bidirectionally.

The symmetrical configuration makes a distinction between observer and object purely conventional. The cut is constituted by the information exchanged. We tend to associate with the "measurement apparatus" one of the two subsystems which in comparison is "larger" and "more classical" and up-linked with some conscious observer. The rest of the system we call the "measured object."

Intrinsic observers face all kinds of self-referential situations. Among the most interesting are paradoxical self-referential statements. These have been known both informally as puzzling amusement and artistic perplexion, as well as a formalized scientifically valuable resource. There is an English phrase stating that one should not bite the hand that feeds oneself. In German, the saying amounts to the advice not to cut the very tree branch one is sitting on. The liar paradox is already mentioned in the Bible's Epistle to Titus, 1:12 stating that, "one of Crete's own prophets has said it: 'Cretans are always liars, evil brutes, lazy gluttons.' He has surely told the truth."

In what follows, paradoxical self-referentiality will be used to argue against the solvability of the general induction problem, as well as for a pandemonium of undecidabilities related to physical systems and their behaviors. All of them are based on intrinsic observers embedded in the system they observe.

It is not totally unreasonable to speculate that the limits of "intrinsic self-expression" seems to be what Gödel himself considered the gist of his incompleteness theorems. In a reply to a letter by Burks (reprinted in Ref. (18, p. 55); see also Ref. (19, p.554)), Gödel states,

> *"... that a complete epistemological description of a language A cannot be given in the same language A, because the concept of truth of sentences of A cannot be defined in A. It is this theorem which is the true reason for the existence of undecidable propositions in the formal systems containing arithmetic."*

B. What is an acceptable form of proof?

There exist ancient and informal notions of proof. An example (20) is the Babylonian notion to "prove" arithmetical statements by considering "large number" cases of algebraic formulae such as (21, Chapter V), $\sum_i^n i^2 = (1/3)(1+2n)\sum_i^n i$. Another example is knowledge acquired by revelation or by authority. Oracles occur in modern computer science, but only as idealized concepts whose physical realization is highly questionable if not forbidden.

The contemporary notion of proof is formalized and algorithmic. Around 1930 mathematicians could still hope for a "mathematical theory of everything" which consists of a finite number of axioms and algorithmic derivation rules by which all true mathematical statements could formally be derived. In particular, as expressed in Hilbert's 2nd problem, it should be possible to prove the consistency of the axioms of arithmetic.

Shortly afterwards, Gödel (7), Tarski (8), and Turing (10) put an end to this formalist program. They first formalized the concepts of proof and computation in general, equating them with algorithmic content. Then, they translated self-referential statements of the kind mentioned above into the formalism.

From a purely syntactic point of view, every formal system of mathematics can be identified with a computation and *vice versa*. Indeed, as stated by K. Gödel in a *Postscript*, dated from June 3rd, 1964 (22, pp. 369-370),

> *... due to A. M. Turing's work, a precise and unquestionably adequate definition of the general concept of formal system can now be given, the existence of undecidable arithmetical propositions and the non-demonstrability of the consistency of a system in the same system can now be proved rigorously for* every *consistent formal system containing a certain amount of finitary number theory.*
>
> *Turing's work gives an analysis of the concept of "mechanical procedure" (alias "algorithm" or "computation procedure" or "finite combinatorial procedure"). This concept is shown to be equivalent with that of a "Turing machine." A formal system can simply be defined to be any mechanical procedure for producing formulas, called provable formulas.*

What is an algorithm? In Turing's own words (10),

> *"whatever can (in principle) be calculated on a sheet of paper by the usual rules is*

computable."

These concretions limit the expressiveness of any formalism, for either it is too restricted to allow the representation of rich patterns of behavior, or it is bounded by self-referentiality. They, however, do not exclude revelations and knowledge of truth transcending the algorithmic formalism.

C. Undecidability of the general forecasting problem

Logical and undecidabilities are based on intrinsic paradoxical self-reference. Can we make use of paradoxical self-reference in physics? Is it possible to find physical expressions corresponding to, for instance, the liar paradox? Can we apply the "Gödelian program" to physics?

Indeed, we can argue that for any deterministic system strong enough to support universal computation, the general forecast or prediction problem is provable unsolvable. This will be shown by reduction to the halting problem.

Gödel had doubts about the relevance of formal incompleteness to physics, in particular to quantum mechanics. The author was told by professor Wheeler that this resentment (also mentioned in Ref. (23, pp. 140-141)) may have been due to Einstein's negative opinion of quantum theory; to the extend that Einstein may have "brainwashed" Gödel into believing that all efforts in this direction were in vain.

One of the first researchers getting interested in the application of paradoxical self-reference to physics was the philosopher Popper, who published two almost forgotten papers (12; 13) discussing, among other issues, Russell's Paradox of Tristram Shandy (24): In Volume 1, Chapter XIV, Shandy finds that he could publish two volumes of his life every year, covering a time span far smaller than the time it took him to write these volumes. This de-synchronization, Shandy concedes, will rather increase than diminish as he advances; and one may thus have serious doubts whether he will ever complete his autobiography.

More recently, there have been attempts to bring together researchers interested in the relevance of Gödelian incompleteness in physics. One of those meetings took place in Santa Fe (15), another one in Abisko (16).

A straightforward embedding of a universal computer into a physical system results in the fact that, due to the reduction to the recursive undecidability of the halting problem, certain future events cannot be forecasted and are thus provable indeterministic. Here reduction again means that physical undecidability is linked or reduced to logical undecidability.

For the sake of getting an (algorithmic) taste of what paradoxical self-reference is like, we present the sketch of an algorithmic proof (by contradiction) of the unsolvability of the halting problem. Consider a universal computer U and an arbitrary algorithm $B(X)$ whose input is a string of symbols X. Assume that there exists a "halting algorithm" $\texttt{HALT}$ which is able to decide whether B terminates on X or not. The domain of $\texttt{HALT}$ is the set of legal programs. The range of $\texttt{HALT}$ are classical bits.

Using $\texttt{HALT}(B(X))$ we shall construct another deterministic computing agent A, which has as input any effective program B and which proceeds as follows: Upon reading the program B as input, A makes a copy of it. This can be readily achieved, since the program B is presented to A in some encoded form $\ulcorner B \urcorner$, i.e., as a string of symbols. In the next step, the agent uses the code $\ulcorner B \urcorner$ as input string for B itself; i.e., A forms $B(\ulcorner B \urcorner)$, henceforth denoted by $B(B)$. The agent now hands $B(B)$ over to its subroutine $\texttt{HALT}$. Then, A proceeds as follows: if $\texttt{HALT}(B(B))$ decides that $B(B)$ halts, then the agent A does not halt; this can for instance be realized by an infinite $\texttt{DO}$-loop; if $\texttt{HALT}(B(B))$ decides that $B(B)$ does *not* halt, then A halts.

The agent A will now be confronted with the following paradoxical task: take the own code as input and proceed to determine whether or not it halts. Then, whenever $A(A)$ halts, $\texttt{HALT}(A(A))$, by the definition of A, would force $A(A)$ not to halt. Conversely, whenever $A(A)$ does not halt, then $\texttt{HALT}(A(A))$ would steer $A(A)$ into the halting mode. In both cases one arrives at a complete contradiction. Classically, this contradiction can only be consistently avoided by assuming the nonexistence of A and, since the only nontrivial feature of A is the use of the peculiar halting algorithm $\texttt{HALT}$, the impossibility of any such halting algorithm.

A universal computer can in principle be embedded into or realized by physical systems (14). An example for such a physical system is the computer on which I am currently typing this manuscript. It follows by reduction that there exist physical observables, in particular forecasts about whether or not such computer will ever halt in the sense sketched above, which are provable undecidable.

D. The busy beaver function as the maximal recurrence time

The busy beaver function (25–28) addresses the following question: given a finite system; i.e., a system whose algorithmic description is of finite length. What is the biggest number producible by such a system before halting?

Let $\Sigma(n)$ denote the busy beaver function of n. Originally, T. Rado (25) asked how many 1's a Turing machine with n possible states and an empty input tape could print on that tape before halting. The first values of the Turing busy beaver function $\Sigma_T(x)$ are finite and are known (27; 28): $\Sigma_T(1) = 1$, $\Sigma_T(2) = 4$, $\Sigma_T(3) = 6$, $\Sigma_T(4) = 13$, $\Sigma_T(5) \geq 1915$, $\Sigma_T(7) \geq 22961$, $\Sigma_T(8) \geq 3 \cdot (7 \cdot 3^{92} - 1)/2$.

Consider a related question: what is the upper bound of running time — or, alternatively, recurrence time — of a program of length n bits before terminating? An answer to that question confers a feeling of how long we have to wait for the most time-consuming program of length n bits to hold. That, of course, is a worst-case scenario. Many programs of length n bits will have halted long before the maximal halting time.

We mention without proof (26; 29) that this bound can be represented by the busy beaver function: $\texttt{TMAX}(n) = \Sigma(n + O(1))$ is the minimum time at which all programs of size smaller than or equal to n bits which halt have done so.

Knowledge of TMAX would "solve" the halting problem quantitatively. Because if the maximal halting time would be known and bounded by any computable function of the program size of n bits, one would have to wait just a little bit longer than $\texttt{TMAX}(n)$ to make sure that every program of length n — also this particular program — would have terminated. Otherwise, the program would run forever. In this sense, knowledge of TMAX is equivalent to a perfect predictor. Since the latter one does not exist, we may expect that TMAX cannot be a computable function. Indeed, for large values of n, $\Sigma(n)$ grows faster than any computable function of n.

By reduction we obtain upper bounds for the recurrence of any kind of physical behavior: for deterministic systems representable by n bits, the recurrence time grows faster than any computable number of n. This bound from below for possible behaviors may be interpreted as a qualitative measure of the impossibility to predict and forecast such behaviors by algorithmic means.

E. Undecidability of the induction problem

Induction in physics is the inference of general rules dominating and generating physical behaviors from these behaviors. For any deterministic system strong enough to support universal computation, the general induction problem is provable unsolvable. Induction is thereby reduced to the unsolvability of the rule inference problem (30–34),

Informally, the algorithmic idea of the proof is to take any sufficiently powerful rule or method of induction and, in using it, define some functional behavior which is not identified by it. This amounts to constructing an algorithm which (passively!) "fakes" the "guesser" by simulating some particular function φ until the guesser pretends to guess this function correctly. In a second, diagonalization step, the "faking" algorithm then switches to a different function $\varphi^* \neq \varphi$, such that the guesser's guesses become incorrect.

More formally, assume two (universal) computers U and U'. Suppose that the second computer U' executes an arbitrary algorithm p unknown to computer U, the "guesser." The task of U, which is called the rule inference problem, is to conjecture the "law" or algorithm p by analysing the behavior of $U'(p)$. The recursive unsolvability of the rule inference problem (30) states that this task cannot be performed by any effective computation.

For the sake of contradiction, assume (34) that there exists a "perfect guesser" U which can identify all total recursive functions (wrong). Then it is possible to construct a function $\varphi^* : \mathbb{N} \rightarrow \{0,1\}$, such that the guesses of U are wrong infinitely often, thereby contradicting the above assumption.

Define $\varphi^*(0) = 0$. One may construct φ^* by simulating U. Suppose the values $\varphi^*(0)$, $\varphi^*(1)$, $\varphi^*(2)$, $\cdots$, $\varphi^*(n-1)$ have already been constructed. Then, on input n, simulate U, based on the previous series $\{0,\varphi^*(0)\},\{1,\varphi^*(1)\},\{2,\varphi^*(2)\},\cdots,\{n-1,\varphi^*(n-1)\}$, and define $\varphi^*(n)$ equal to 1 plus the guess of U of $\varphi^*(n)$ mod 2. In this way, U can never guess φ^* correctly; thereby making an infinite number of mistakes.

One can also interpret this result in terms of the recursive unsolvability of the halting problem, which in turn is related to the busy beaver function: there is no recursive bound on the time the guesser has to wait in order to make sure that his guess is correct.

F. Results in classical recursion theory with implications for theoretical physics

The following theorems of recursive (i.e., computable) analysis have some implications to theoretical physics (35). (i) There exist recursive monotone bounded sequences of rational numbers whose limit is no computable number (36). A concrete example of such a number is Chaitin's Omega number (11; 37), the "halting probability" for a computer (using prefix-free code), which can be defined by a sequence of rational numbers with no computable radius of convergence.

(ii) There exist a recursive real function which has its maximum in the unit interval at no recursive real number (38). This has implication for the principle of least action.

(iii) There exists a real number r such that $G(r) = 0$ is recursively undecidable for $G(x)$ in a class of functions which involves polynomials and the sine function (39). This again has some bearing on the principle of least action.

(iv) There exist uncomputable solutions of the wave equations for computable initial values (40; 41).

III. BEHAVIOR OF THREE OR MORE CLASSICAL BODIES

An extreme deterministic position was formulated by Laplace, stating that (42, Chapter II)

> *Present events are connected with preceding ones by a tie based upon the evident principle that a thing cannot occur without a cause which produces it. This axiom, known by the name of the principle of sufficient reason, extends even to actions which are considered indifferent ...*
>
> *We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it an intelligence sufficiently vast to submit these data to analysis it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes.*

In the late 18th hundred, the issue seemed worthy and pressing enough to establish a prize by King Oscar II of Sweden, advised by Martin Leffler, who published the following question formulated by Weierstrass:

Given a system of arbitrarily many mass points that attract each according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.

Poincaré's original prize–winning contribution contained errors. The necessary corrections led the author to the conclusion that sometimes small variations in the initial values could lead to huge variations in the evolution of a physical system in later times. In Poincaré's own words (43, Chapter 4, Sect. II, pp.56-57)[1]:

If we would know the laws of Nature and the state of the Universe precisely for a certain time, we would be able to predict with certainty the state of the Universe for any later time. But [[. . .]] it can be the case that small differences in the initial values produce great differences in the later phenomena; a small error in the former may result in a large error in the latter. The prediction becomes impossible and we have a "random phenomenon."

A. Deterministic chaos

Poincaré's recognition of possible instabilities in n-body problems was the first indication of what today is called "deterministic chaos." In chaotic systems it is practically impossible to specify the initial value precise enough to allow long-term predictions.

A stronger assumption supposes that the initial values are elements of a continuum and thus are not representable by any algorithmically compressible number; in short, that they are random (44). Classical, deterministic chaos results from "unfolding" such a random initial value drawn from the "continuum urn" by a recursive, deterministic function.

A weaker form of deterministic chaos just expresses the fact that linear deviations of initial values which lie within the measurement precision result in exponential divergences in the future evolution of the system. For further discussions, the interested reader is referred to Refs. (45–49)

[1] Würden wir die Gesetze der Natur und den Zustand des Universums für einen gewissen Zeitpunkt genau kennen, so könnten wir den Zustand dieses Universums für irgendeinen späteren Zeitpunkt genau voraussagen. Aber [[. . .]] es kann der Fall eintreten, daß kleine Unterschiede in den Anfangsbedingungen große Unterschiede in den späteren Erscheinungen bedingen; ein kleiner Irrtum in den ersteren kann einen außerordentlich großen Irrtum für den letzteren nach sich ziehen. Die Vorhersage wird unmöglich und wir haben eine "zufällige Erscheinung".

B. Convergence of the general solution

More than one hundred years after its formulation as quoted above, the n–body problem has been solved by Wang (50–52). The 3–body problem was already solved in 1912 (53). The solutions are given in terms of power series.

Yet, in order to be practically applicable, the radius of convergence of the series must be known. We might already expect from deterministic chaos that these series solution have a "very slow" convergence. Even the prediction of behaviors in insignificantly short times may require the summation of a huge number of terms, making these series unusable for any practical work (51).

Alas, the complications regarding convergence of the series solutions are far more serious. Suppose we are able to construct a universal computer based on the n–body problem. This can, for instance, be achieved by ballistic computation, such as the "Billiard Ball" model of computation (54; 55) which effectively "embeds" a universal computer into a system of n–bodies. Then, by reduction, it follows that certain predictions are impossible.

What are the consequences of this reduction for the convergence of the series solutions? It can be expected that not only do the series converge "very slowly," like in deterministic chaotic systems, but that in general there does not exist any computable radius of convergence for the series solutions. This is very similar to Chaitin's Omega number (11; 37) representing the halting probability of a universal computer, or the busy beaver function. The Omega number can be "enumerated" by series solutions from "pseudo-algorithms" computing its very first digits. Yet, due to the uncomputable growth of the time required to determine whether or not terms possibly contribute, the series lack any computable radius of convergence.

IV. QUANTUM UNKNOWABLES

A third group of physical unknowables arise in the quantum context. Throughout its development, although a highly successful theory, quantum mechanics, in particular its interpretation and meaning, has been received controversially within the community. Some of its founding fathers, such as Schrödinger, De Broglie and Einstein had a very critical view on its validity and considered quantum mechanics a preliminary theory which should give way to a more complete one. Others, among them Bohr and Heisenberg, claimed that quantum unknowables will stay with us forever. Over the years, the latter view seems to have prevailed (56), although it was not totally unchallenged (57–59). Already Sommerfeld warned his students not to get into the "meaning behind" quantum mechanics, and, as mentioned by Clauser (60), not long ago scientists working in that field had to be very careful not to become discredited as "quacks." Richard Feynman (17, p. 129) once mentioned the

> *"... perpetual torment that results from [[the question]], 'But how can it be like that?' which is a reflection of uncontrolled but utterly vain desire to see [[quantum mechanics]] in terms of an analogy with something familiar ... Do not keep saying to yourself, if you can possibly avoid it, 'But how can it be like that?' because you will get 'down the drain', into a blind alley from which nobody has yet escaped."*

In what follows, we shall discuss three main quantum unknowables: (i) randomness of single events, (ii) complementarity, and (iii) value indefiniteness.

A. Random events

The quantum formalism does not predict the outcome of single events when there is a mismatch between the context in which a state was prepared, and the context in which it is measured. Here, context means maximal observable, or more technically, the maximal operator from which all commuting operators can be functionally derived (61, §84).

In the absence of other explanations, one is thus lead to the conclusion that these single events occur without any causation and thus at random. Such random "quantum coin toss" (62) have been used for various purposes, among them delayed choice experiments (63; 64). Commercial interface cards (65) perform at a rate of 4 to 16 Mbit/s.

Note that randomness of this type (66; 67) is postulated rather than proven. This is necessarily so, for any claim of randomness can only be corroborated with respect to a more or less large class of laws or behaviors; it is impossible to inspect the hypothesis against an infinity of conceivable laws. How can we ever exclude the possibility of our presented, some day (perhaps by some extraterrestrial visitors), with a (perhaps extremely complex) device that "computes" and "predicts" a certain type of hitherto "random" physical behavior?

B. Complementarity

Another quantum indeterminism is complementarity. Complementarity is the principal impossibility to measure two or more complementary observables with arbitrary precision simultaneously.

Complementarity was first encountered in quantum mechanics, but it is a phenomenon also observable in the classical world. To get a better feeling for complementarity, we shall consider generalized urn models (68; 69) or, equivalently (70), finite Moore and Mealy automata (71–74). Both quasi-classic examples mimic complementarity and even quasi-quantum cryptography (75).

A generalized urn model is characterized by an ensemble of balls with black background color. Printed on these balls are some color symbols from a symbolic alphabet. The colors are elements of a set of colors. A particular ball type is associated with a unique combination of mono-spectrally (no mixture of wavelength) colored symbols printed on the black ball background. Every ball contains just one single symbol per color.

Assume further some mono-spectral filters or eyeglasses which are "perfect" by totally absorbing light of all other colors but a particular single one. In that way, every color can be associated with a particular eyeglass and vice versa.

When a spectator looks at a particular ball through such an eyeglass, the only operationally recognizable symbol will be the one in the particular color which is transmitted through the eyeglass. All other colors are absorbed, and the symbols printed in them will appear black and therefore cannot be differentiated from the black background. Hence the ball appears to carry a different "message" or symbol, depending on the color at which it is viewed. An explicit example is enumerated in Table I.

The difference between the balls and the quanta is the possibility to view all the different symbols on the balls in all different colors by taking off the eyeglasses. Quantum mechanics does not provide us with such a possibility. On the contrary, there are strong formal arguments suggesting that the assumption of a simultaneous physical existence of such complementary observables yields a complete contradiction. These issues will be discussed next.

ball type	red	green
1	0	0
2	0	1
3	1	0
4	1	1

TABLE I Schema of imprinting of four ball types filling a generalized urn. Whenever the spectator looks through the red eyeglass, the red symbols printed on the balls appear, whereas the green symbols merge in their black background. Conversely, the spectator may choose to look at the green symbols through the green eyeglass. In the latter case, the red symbols become unrecognizable.

C. Value indefiniteness versus omniscience

Still another quantum unknowable results from the fact that no "global" classical truth assignment exists which is consistent with even a finite number of "local" ones. That is, no consistent classical truth table can be given by pasting together commeasurable observables. This phenomenon is also known as value indefiniteness or contextuality.

Already scholastic philosophy (76), for instance Thomas Aquinas Ref. (77), considered questions such as whether God has knowledge of non-existing things (Part 1, Question 14, Article 9) or things that are not yet (Part 1, Question 14, Article 13). Classical omniscience, at least its naive expression that "if a proposition is true, then an omniscient agent (such as God) knows that it is true" is plagued by paradoxical self-referential.

The empirical sciences implement classical omniscience by assuming that in principle, all observables of classical physics are (co-)measurable without any restrictions, and regardless of whether they are actually measured or not. No distinction is made between an observable obtained by an "actual" and a "potential" measurement. (In contrast compare Schrödinger's own interpretation of the wave function (78, §7) as a "catalogue of expectations.") Precision and (co-)measurability are limited only by the technical capacities of the experimenter. The principle of empirical classical omniscience has given rise to the realistic believe that all observables "exist" regardless of their observation; i.e., regardless and independent of any particular measurement. Physical (co-)existence is thereby related to the realistic assumption (79) (sometimes referred to as the "ontic" (80) viewpoint) that such physical entities exist even without being experienced by any finite mind.

The formal expression of classical omniscience is the Boolean algebra of observable propositions (81), the distributive law satisfied by the classical logical operations, and in particular the "abundance" of two-valued states interpretable as omniscience about the system. Thereby, any such "dispersionless" two-valued state — associated with a "truth table" — can be defined on all observables, regardless of whether they have been actually observed or not.

Historically, the discovery of the uncertainty principle and complementarity seem to have been first indications of the decline of classical omniscience. A formal expression of complementarity is the nondistributive algebra of quantum observables. Alas, nondistributivity of the empirical logical structure is no sufficient condition for the impossibility of omniscience. The generalized urn as well as equivalent finite automaton models discussed above possess two-valued states interpretable as omniscience.

A further blow to quantum omniscience came from Boole's "conditions of possible experience" (82; 83) for quantum probabilities and expectation functions. In particular, Bell was the first to point to experiments which, based on counterfactually inferred elements of physical reality discussed by Einstein, Podolsky and Rosen (84), seemed to indicate the impossibility to faithfully embed quantum observables into classical Boolean algebras. To state the issues pointedly, under some (presumably mild) side assumptions, "unperformed experiments have no results" (85) — there cannot exist a table enumerating all actual and hypothetical experimental outcomes consistent with the observed quantum frequencies (86). Any such table could be interpreted as omniscience with respect to the observables in the Bell-type experiments. The impossibility to construct such tables appears to be a very serious indication against quantum omniscience.

The reason for the impossibility to describe all quantum observables simultaneously by classical tables of experimental outcomes can be understood in terms of a "stronger" result stating that, for quantum systems whose Hilbert space is of dimension greater than two, there does not exist any dispersionless, two-valued state interpretable as truth table. This result, which is known as the Kochen-Specker theorem (76; 87–93), has a finitistic proof by contradiction. To get a flavor of the argument, a short version of the proof is depicted in Fig. 1. It is a brain teaser to argue that no coloring of the points in this diagram exists which would include only one red point per smooth, unbroken curve; the other three points all remaining green.

The violations of conditions of possible classical experience or the Kochen-Specker theorem do not exclude realism restricted to a single context, but realistic omniscience beyond it. It might be a classical anachronism to assume that outside of a single context in which the particle was prepared, all observables are (pre-)defined.

V. MIRACLES DUE TO GAPS IN CAUSAL DESCRIPTION

A different issue, discussed by Philipp Frank, is the possible occurrence of miracles in the presence of gaps of physical determinism. One might perceive singular events occurring within the bounds of classical and quantum physics without any apparent cause as miracles. For, if there is no cause to an event, why should such an event occur altogether rather than not occur?

Although such thoughts remain highly speculative, miracles, if they exist, could be the basis for a directed evolution in otherwise deterministic physical systems. Similar models have also been applied to dualistic models of the mind (95; 96).

There exist bounds on miracles and on behavioral patterns in general due to the self-referential perception of intrinsic observers endowed with free will: if such an observer is omniscient and has absolute predictive power, then free will could counteract omniscience, and in particular the own predictions. The only consistent alternative seems either to abandon free will, stating that it is an idealistic illusion, or to accept that omniscience and absolute predictive power is bound by paradoxical self-reference.

VI. SUMMARY

Hilbert's 6th problem is about the axiomatization of all of physics. We still do not know whether or not this goal is achievable. All we know is that even if it could be achieved, omniscience cannot be gained via the formalized, syntactic route to infer and predict physical behaviors. It will remain blocked forever by paradoxical self-reference which intrinsic observers and operational methods are bound to, It remains to be seen whether or not these Gödelian-type physical unknowables are relevant for the practical development of physics proper.

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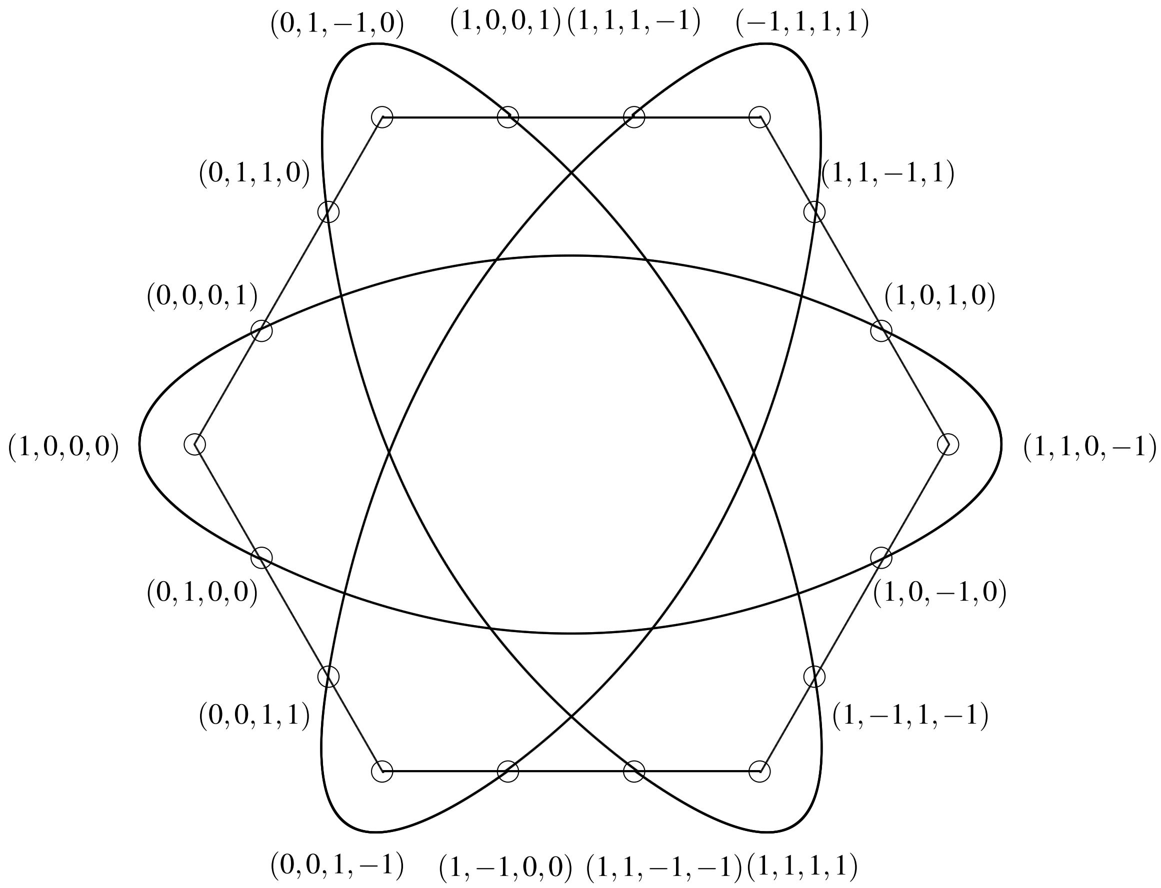

FIG. 1 Proof of the Kochen-Specker theorem (93; 94) in four-dimensional real vector space. Nine interconnected contexts (or four-pods) are represented by smooth, unbroken curves. The graph contains possible quantum observables represented by 18 points, which are explicitly enumerated. It cannot be colored by the two colors red (associated with truth) and green (associated with falsity) such that every context contains exactly one red and three green points. For, by construction, on the one hand, every red point occurs in exactly two contexts (four-pods), and hence there is an even number of red points in a table containing the points of the contexts as columns and the enumeration of contexts as rows. On the other hand, there are nine contexts involved; thus by the rules it follows that there is an odd number of red points in this table (exactly one per context). Thus, our assumption about the colorability and therefore about possible consistent truth assignments for this finite set of quantum observables leads to a complete contradiction.