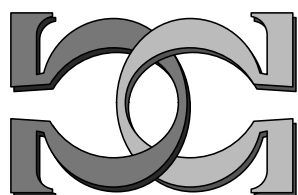
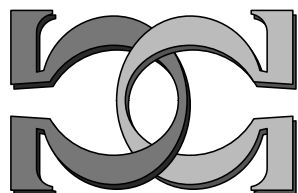
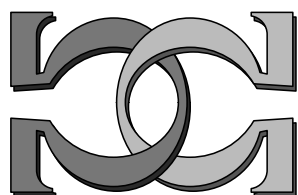


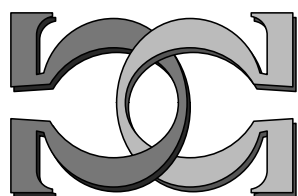
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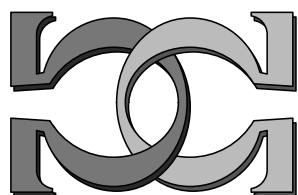
**Is Incompleteness A Serious
Problem?**



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Is Incompleteness A Serious Problem?

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In 1931 Kurt Gödel astonished the mathematical world by showing that no finite set of axioms can suffice to capture all of mathematical truth. He did this by constructing an assertion G_F about the whole numbers that manages to assert that it itself is unprovable (from a given finite set F of axioms using formal logic).¹

G_F : “ G_F cannot be proved from the finite set of axioms F .”

This assertion G_F is therefore true if and only if it is unprovable, and the formal axiomatic system F in question either proves falsehoods (because it enables us to prove G_F) or fails to prove a true assertion (because it does not enable us to prove G_F). If we assume that the former situation is impossible, we conclude that F is necessarily incomplete since it does not permit us to establish the true statement G_F .

**Either G_F is provable and F proves false statements,
or G_F is unprovable and therefore true, and F is incomplete.**

Today, a century after Gödel’s birth, the full implications of this “incompleteness” result are still quite controversial.²

An important step forward was achieved by Alan Turing in 1936. He showed that incompleteness could be derived as a corollary of uncomputability. Because if there are things that cannot be computed (Turing’s halting problem), then these things also cannot be proven. More precisely, if there were a finite set of axioms F that always enabled us to prove whether particular programs P halt or fail to halt, then we could calculate whether a given program P halts or not by running through the tree of all possible deductions from the axioms F until we either find a proof that P halts or we find a proof that P never halts. But, as Turing showed in his famous 1936 paper “On Computable Numbers with an Application to the *Entscheidungsproblem*,” there cannot be an algorithm for deciding whether or not individual programs P halt.³

¹Gödel’s paper is included in the well-known anthology [1].

²Compare for example the attitude in Franzén [2,3] with that in Chaitin [4,5,6].

³Turing’s paper is also included in the collection [1].

If we can always prove whether or not P halts,
then we can always calculate whether or not P halts
(by systematically running through the tree of all possible proofs).

Now let's combine Turing's approach with ideas from Sections V and VI of Leibniz's *Discours de métaphysique* (1686). Consider the following toy model of what physicists do:

Theory (program) \rightarrow *COMPUTER* \rightarrow *Experimental Data* (output).

In other words, this is a software model of science, in which theories are considered to be programs for computing experimental data. In this toy model, the statement that the simplest theory is best corresponds to choosing the smallest, the most concise program for calculating the facts that we are trying to explain. And a key insight of Leibniz [7] is that if we allow arbitrarily complicated theories then the concept of theory becomes vacuous because there is always a theory. More precisely, in our software model for science this corresponds to the observation that if we have N bits of experimental data then our theory must be a program that is much less than N bits in size, because if the theory is allowed to have as many bits as the data, then there is always a theory.

Understanding = Compression!

Now let's abstract from this the concept of an "elegant" program:

P is an elegant program if and only if
no smaller program Q written
in the same programming language
produces exactly the same output that P does.

In our software model for science, the best theory is always an elegant program. Furthermore, there are infinitely many elegant programs, since for any computational task there is always at least one elegant program, and there are infinitely many computational tasks. However, what if we want to **prove** that a particular program P is elegant? Astonishingly enough, any finite set of axioms F can only enable us to prove that **finitely** many individual programs P are elegant!

Why is this the case? Consider the following paradoxical program P_F :

**P_F : The output of P_F is the same as
the output of the first provably elegant program Q
that is larger than P_F is.**

P_F runs through the tree of all possible deductions from the finite set of axioms F until it finds the first provably elegant program Q that is larger than P_F is, and then P_F simulates the computation that Q performs and then produces as its output the same output that Q produces. But this is impossible because P_F is too small to be able to produce that output! Assuming that F cannot enable us to prove false theorems, we must conclude that Q cannot exist. Thus if Q is an elegant program that is larger than P_F is, then the axioms F cannot enable us to prove that Q is elegant. Therefore F can only enable us to prove that **finitely** many individual programs Q are elegant. ***Q.E.D.***⁴

My personal belief, which is not shared by many in the mathematics community, is that modern incompleteness results such as this one push us in the direction of a “quasi-empirical” view of mathematics, in which we should be willing to accept new mathematical axioms that are not at all self-evident but that are justified pragmatically, because they enable us to explain vast tracts of mathematical results. In other words, I believe that in mathematics, just as in physics, the function of theories is to enable us to compress many observations into a much more compact set of assumptions.⁵

So, in my opinion, incompleteness is extremely serious: It forces us to realize that *perhaps mathematics and physics are not as different as most people think*.⁶

Mathematics \approx Physics?!

References

- [1] Martin Davis, *The Undecidable*, Dover Publications, Mineola, New York, 2004.
- [2] Torkel Franzén, *Gödel’s Theorem*, A. K. Peters, Wellesley, Massachusetts, 2005.
- [3] Torkel Franzén, “The Popular Impact of Gödel’s Incompleteness Theorem,” *Notices of the AMS*, April 2006, pp. 440–443.

⁴An immediate corollary is that the halting problem is unsolvable. For if we could determine all the programs that halt, then by running them and seeing their output we could also determine all the elegant programs, which we have just shown to be impossible. This program-size complexity argument for deriving the unsolvability of the halting problem is completely different from Turing’s original 1936 proof, which is basically just an instance of Cantor’s diagonal argument—from set theory—applied to the set of all computable real numbers.

⁵“Quasi-empirical” is a term invented by Lakatos. See Tymoczko [8].

⁶In this connection, see Borwein and Bailey [9] on the use of experimental methods in mathematics.

- [4] Gregory Chaitin, *Meta Math!*, Pantheon Books, New York, 2005.⁷
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⁷Italian translation to be published by Adelphi.

⁸Italian translation published in the May 2006 issue of *Le Scienze*.