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Combinatorial Isomorphisms Beyond a Simion-Schmidt's Bijection

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Abstract

In 1985 Simion and Schmidt gave a constructive bijection φ from F_{n-1} to $S_n(123, 132, 213)$, where F_{n-1} is the set of all length $(n-1)$ binary strings having no two consecutive 1s and $S_n(123, 132, 213)$ is the set of all permutations of $\{1, 2, \ldots, n\}$ that avoid all patterns in the set $\{123, 132, 213\}$.

In this paper we generalize φ to an injective but not surjective function from $\{0, 1\}^{n-1}$ to the set S_n of all length n permutations. From this *not bijective* function we derive three bijections $\varphi : P \to Q$ where $P \subseteq \{0, 1\}^{n-1}$ and $Q \subset S_n$; the domains are sets of restricted binary strings and the codomains are sets of pattern-avoiding permutations. As particular case of one of these bijections, we retrieve back the original Simion-Schmidt bijection.

We also show that the obtained bijections are actually combinatorial isomorphisms, i.e., closeness preserving bijections. Each domain of them has known Gray code and generating algorithm, hence we propose similar results for each of corresponding codomains, through the corresponding combinatorial isomorphism.
Keywords: pattern-avoiding permutations, Fibonacci strings, constructive bijections, combinatorial isomorphisms, Gray codes.

1 Introduction and Motivation

A permutation π of the set of integers $[n] = \{1, 2, \ldots, n\}$ is a bijection from $[n]$ onto itself and we denote by S_n the set of all such permutations. For two permutations $\tau \in S_k$ and $\pi \in S_n$, with $k < n$, we say that a subsequence $\pi_{\ell_1}, \pi_{\ell_1}, \ldots, \pi_{\ell_k}$ of π has *type* τ whenever $\pi_{\ell_i} < \pi_{\ell_j}$ if and only if $\tau_i < \tau_j$ for all i, j, $1 \leq i, j \leq k$. In this context the permutation τ is called *pattern*. For example, the subsequence 523 of the permutation 15423 has type 312. Now, let $T = \{\tau_1, \tau_2, \ldots, \tau_k\}$ be a set of patterns. We say π *avoids* T whenever π contains no subsequence of type τ_i for all $\tau_i \in T$, and we will denote by $S_n(T)$ the set of all such permutations. For example, the permutation $15423 \in S_5$ avoids the set of patterns $\{231, 213\}$ because it has no subsequence of type 231 nor 213. So we have $15423 \in S_5(231, 213)$ but $15423 \notin S_5(312)$. Clearly $S_n(T_1) \subset S_n(T_2)$ if $T_2 \subset T_1$.

This paper is inspired by Simion-Schmidt's result [5, Proposition 15*] which gave a constructive bijection φ from F_{n-1} to $S_n(123, 132, 213)$. Here F_{n-1} is the set of all length $(n-1)$ binary

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strings having no two consecutive 1s (also known as the set of Fibonacci strings). In this paper we generalize φ to an *injective* but not surjective function φ from $\{0,1\}^{n-1}$ to S_n. From this not bijective function we derive three bijections $\varphi : P \to Q$ where $P \subseteq \{0,1\}^{n-1}$ and $Q \subset S_n$. The three pairs (P, Q) are:

1. $(\{0,1\}^{n-1}, S_n(123, 132))$,
2. $(F^{(p)}_{n-1}, S_n(123, 132, \sigma_p))$, where $F^{(p)}_{n-1}$ is the set of length $(n-1)$ binary strings with no p consecutive ones and σ_p is the length $(p+1)$ permutation $p(p-1)(p-2)\ldots 1(p+1)$ (when $p = 2$ the bijection φ becomes the original Simion-Schmidt's bijection), and
3. $(C_{n-1,k}, S'_{n,k}(123, 132))$, where $C_{n-1,k}$ is the set of binary strings in $\{0,1\}^{n-1}$ having exactly k 1s and $S'_{n,k}(123, 132)$ is the set of permutations in $S_n(123, 132)$ having exactly k *left-inversions.*

We denote by φ the Simion-Schmidt bijection and its extensions because they have same mapping rule but different domain-codomain pairs.

In this paper we also prove that the three bijections $\varphi : P \to Q$ are combinatorial isomorphisms, those are, closeness preserving bijections. Each domain of these bijections has known Gray code and generating algorithm, hence we propose Gray code and sketch generating algorithm for $S_n(123, 132)$, $S_n(123, 132, \sigma_p)$, and $S'_{n,k}(123, 132)$, each of them obtained through the corresponding combinatorial isomorphism.

The structure of this paper is as follows. After this introduction, Section 2 presents the generalization of the Simion-Schmidt's bijection and the derivation of the three bijections. Section 3 shows that these bijections are actually combinatorial isomorphisms, Section 4 proposes a Gray code for each codomain of these bijections, and Section 5 delivers some graph theoretical consequences and sketches generating algorithm for each Gray code. The final section gives some concluding remarks.

2 The Simion-Schmidt Generalized Injection

For any $b = b_1 b_2 \cdots b_{n-1} \in \{0,1\}^{n-1}$ we construct a permutation $\pi \in S_n$ which has its ith entry, π_i, given by the following rule. If $X_i = \{1, 2, \cdots, n\} - \{\pi_1, \pi_2, \cdots, \pi_{i-1}\}$, then set

$$\pi_i = \begin{cases} \text{the largest element in } X_i \text{ if } b_i = 0 \\ \text{the second largest element in } X_i \text{ if } b_i = 1 \end{cases} \tag{1}$$

and finally π_n is the single element in X_n.

We denote by $\varphi(b)$ the unique image of $b \in \{0,1\}^{n-1}$ through this procedure. Furthermore, two different strings in $\{0,1\}^{n-1}$ are mapped into two different permutations in S_n, therefore $\varphi : \{0,1\}^{n-1} \to S_n$ is an injective function. Moreover, since for $n \geq 3$, $|S_n| > |\{0,1\}^{n-1}| = 2^{n-1}$, hence $\varphi : \{0,1\}^{n-1} \to S_n$ is not surjection, and so φ is not bijection.

The construction above was already given by Simion and Schmidt [5] in a particular context, namely as a bijection between length $(n-1)$ binary strings with no two consecutive ones and permutations in $S_n(123, 132, 213)$. The next two lemmata generalize they result.

Lemma 1.

1. $\varphi : \{0,1\}^{n-1} \to S_n(123, 132)$ *is a bijection.*

2. $\varphi : F^{(p)}_{n-1} \to S_n(123, 132, \sigma_p)$ *is a bijection, where* $F^{(p)}_{n-1}$ *is the set of length* $(n-1)$ *binary strings having no* p *consecutive* 1s *and* σ_p *is the length* $(p+1)$ *permutation* $p(p-1)(p-2)\dots 1(p+1)$.

Proof. 1. Let $\pi \in S_n(123, 132)$ and k, $1 \le k \le n$, such that $\pi_k = n$. If $k > 1$ then $\pi_i > \pi_{i+1}$ for all i, $1 \le i < k-1$, else 123 could not be avoided. Moreover, $\pi_i > \pi_j$ for all $i < k$ and $j > k$, else $\pi_i\pi_k\pi_j$ is a sequence of type 132. Therefore, $\pi = \pi_1\pi_2\dots\pi_k\pi'$ with $\pi_i = \pi_{i+1}+1$ for $1 \le i < k-1$ and $\pi' \in S_{n-k}(123, 132)$.

By recurrence, if $\pi \in S_n(123, 132)$ then there exist integers $0 = k_0 < k_1 < \dots < k_r < \dots < k_m = n$ such that π is a sequence of m blocks

$$\pi = \underbrace{\pi_1\pi_2\dots\pi_{k_1}}\dots\underbrace{\pi_{k_{r-1}+1}\pi_{k_{r-1}+2}\dots\pi_{k_r}}\dots\underbrace{\pi_{k_{m-1}+1}\pi_{k_{m-1}+2}\dots\pi_{k_m}} \qquad (2)$$

with

- the rightmost elements of each block are in decreasing order: $n = \pi_{k_1} > \pi_{k_2} > \dots > \pi_{k_m}$,
- in each block containing more than one element
  - the first element equals the last one minus one: $\pi_{k_{r-1}+1} = \pi_{k_r} - 1$,
  - all elements, except the last one, are consecutive integers in decreasing order: $\pi_\ell = \pi_{\ell+1}+1$ for $k_{r-1}+1 \le \ell < k_r - 1$.

It is easy to check that $b \in \{0,1\}^{n-1}$ defined by

$$b_i = \begin{cases} 0 \text{ if } i = k_r \text{ for some } r, 1 \le r < m \\ 1 \text{ otherwise} \end{cases}$$

satisfies $\varphi(b) = \pi$.
2. In addition, if π avoids $\sigma_p = p(p-1)(p-2)\dots 1(p+1)$ then each block $\pi_{k_{r-1}+1}\pi_{k_{r-1}+2}\dots\pi_{k_r}$ has length at most p and b defined above has no p consecutive 1s. □

Figure 1 shows the permutations $\underline{9}7654\underline{8}21\underline{3}$ and $\underline{9}7\underline{8}54\underline{6}21\underline{3}$ in $S_9(123, 132)$ in array representation; the rightmost element of each block, as is mentioned in the proof above, is underlined. Table 1 gives the domains and codomains of the bijection $\varphi : \{0,1\}^{n-1} \to S_n(123, 132)$, and $\varphi : F^{(2)}_{n-1} \to S_n(123, 132, 213)$, for $n = 5$. The listing actually is in Gray code order; see Section 4 for more about Gray code.

Let now $C_{n-1,k}$ be the set of strings in the $\{0,1\}^{n-1}$ with exactly k occurrences of 1. Strings in $C_{n-1,k}$ are the usually binary string representation of the combinations of k objects chosen from $(n-1)$ so $|C_{n-1,k}| = \binom{n-1}{k}$.

Definition 1. *In a permutation* $\pi \in S_n$ *a pair* (i, j)*, with* $i < j$*, is called an* inversion *iff* $\pi_i > \pi_j$ *and* left inversion *iff* $\pi_i < \pi_j$.

We write $S_{n,k}(T)$ to denote the set of permutations in $S_n(T)$ having exactly k inversions, and similarly, $S'_{n,k}(T)$ for the set of permutations having exactly k left inversions.

Lemma 2. $\varphi : C_{n-1,k} \to S'_{n,k}(123, 132)$ *is a bijection.*

Table 1: (a) The set of permutations in $S_5(123,132)$ is constructed from the set of binary strings $\{0,1\}^4$ through the bijection φ. (b) The set of permutations in $S_5(123,132,213)$ is constructed from the set $F_4^{(2)}$ of Fibonacci strings through the bijection φ.

(a)

$\{0,1\}^4$	$S_5(123,132)$
0111	53214
0110	53241
0100	53421
0101	53412
0001	54312
0000	54321
0010	54231
0011	54213
1011	45213
1010	45231
1000	45321
1001	45312
1101	43512
1100	43521
1110	43251
1111	43215

(b)

$F_4^{(2)}$	$S_5(123,132,213)$
0100	53421
0101	53412
0001	54312
0000	54321
0010	54231
1010	45231
1000	45321
1001	45312

Proof. Let $b \in \{0,1\}^{n-1}$ and i, $1 \le i < n$, such that $b_i = 1$. Then i induces exactly one left inversion in $\pi = \varphi(b)$. Indeed, let j the position of the leftmost 0 bit in b at the right of i if any, and $j = n$ otherwise. In π, $\pi_i > \pi_\ell$, for all $\ell > i$, except for $\ell = j$, and so, (i,j) is a left inversion and the number of left inversions in π equals the number of 1s in b. □

Table 2 shows the domain and codomain of the bijection $\varphi : C_{n-1,k} \to S'_{n,k}(123,132)$ for $n = 5$ and $k = 2$. $C_{4,2}$ is listed such that consecutive strings differ in two positions; when these positions are consecutive the corresponding permutations differ in 3 positions and in 4 positions otherwise. As we will see in Section 3 this is valid for all n.

By considering the definition (1) of the function φ it is easy to check the following

Table 2: The set of all length 5 permutations of $S'_{5,2}(123,132)$ is constructed from the set of all length 4 and 2 occurrences of 1s strings in $C_{4,2}$ through the bijection φ.

$C_{4,2}$	$S'_{5,2}(123,132)$
0110	53241
0101	53412
0011	54213
1010	45231
1001	45312
1100	43521

Remark 1. *Let i, $1 \le i \le n-1$, $b, b' \in \{0,1\}^{n-1}$ and suppose $b_\ell = b'_\ell$ except for $\ell = i$. Then $\pi = \varphi(b)$ and $\pi' = \varphi(b')$ are such that $\pi_\ell = \pi'_\ell$ except for $\ell \in \{i, j\}$ with j is as follow: the leftmost position at the right of i where $b_j = 0$ if any, and n otherwise.*

3 The Isomorphism of φ

In a combinatorial class we say that two objects are *close* if they differ in some pre-specified, usually small, way; the Hamming distance is a customary specification. A (*combinatorial*) *isomorphism* between two combinatorial classes is a *closeness* preserving bijection, i.e., two objects in a class are close if and only if their images through this bijection are also close. In this section we show that the bijections in Lemmata 1 and 2 are actually isomorphisms.

Definition 2.

1. *Two binary strings in $\{0,1\}^{n-1}$ are close if they differ in a single position.*
2. *Two permutations in $S_n(123, 132)$ are close if they differ by the transposition of two entries.*

For example, the binary strings 0111 and 0110 are close and so are their images through φ, i.e., the permutations 53214 and 53241.

Lemma 3. *Let $b, b' \in \{0,1\}^{n-1}$ and $\pi = \varphi(b), \pi' = \varphi(b') \in S_n(123, 132)$. The followings are equivalent:*

1. *b and b' are close in $\{0,1\}^{n-1}$,*
2. *π and π' are close in $S_n(123, 132)$,*
3. *the decomposition in blocks of π' (as in relation (2)) is obtained from the one of π either by splitting a block (into two adjacent blocks) or by merging tow adjacent blocks.*

Proof. '1 $\Rightarrow$ 2' Results directly from Remark 1.
'2 $\Rightarrow$ 3' Let $\pi \in S_n(123, 132)$ and suppose π' is obtained from π by transposing the entries in positions i and j, $1 \le i < j \le n$. If π' avoids 123 and 132 then, in the permutation π (with the notations in relation (2)), j must be the rightmost entry in its block and i is either (a) in the same block as j, or (b) the rightmost entry of the precedent block. In the case (a) π' is obtained from π by splitting the block containing j into two blocks and in the case (b) by merging tow adjacent blocks.
'3 $\Rightarrow$ 1' By considering the definition of the function φ, with the notations in the previous point, $b_\ell = b'_\ell$ except for $\ell = i$. In the case (a) $b_i = 1$ and $b'_i = 0$; and in the case (b) $b_i = 0$ and $b'_i = 1$. See Figure 1 for an example. □

By the above lemma and since the restriction of a combinatorial isomorphism to a subclass remains a combinatorial isomorphism we have:

Corollary 1. *The bijections*

- $\varphi : \{0,1\}^{n-1} \to S_n(123, 132)$, *and*
- $\varphi : F^{(p)}_{n-1} \to S_n(123, 132, \sigma_p)$

are combinatorial isomorphisms.

Under Definition 2, $C_{n-1,k}$ does not contain close strings and now we relax this definition.

Definition 3.

1. *Two binary strings in* $C_{n-1,k}$ *are* close *if they differ by the transposition of two bits.*
2. *Two permutations in* $S'_{n,k}(123, 132)$ *are* close *if they differ by two transpositions.*

Corollary 2. *The bijection* $\varphi : C_{n-1,k} \to S'_{n,k}(123, 132)$ *is a combinatorial isomorphism under Definition 3.*

Proof. Let b and b' two close strings in $C_{n-1,k}$, and π and π' their images in $S'_{n,k}(123, 132)$ through the bijection φ. Consider a binary string $c \in \{0,1\}^{n-1}$ such that c differs from b and from b' in a single position and $\tau \in S_n(123, 132)$ the image of c through φ. Notice that $c \notin C_{n-1,k}$ and so $\tau \notin S'_{n,k}(123, 132)$, and there are two such strings c. By Lemma 3, τ differs from π and from π' by a transposition thus π differs from π' by two transpositions. Similarly, if π and π' are close in $S'_{n,k}(123, 132)$ then so are they preimages in $C_{n-1,k}$. ☐

Notice that, when the two transpositions in the previous proof have no disjoint domains then π and π' differ by a three length cycle. For example, the transition from the first to the second permutation in $S'_{5,2}(123, 132)$ as shown in Table 2, namely from 53241 to 53412, is done via a three length cycle.

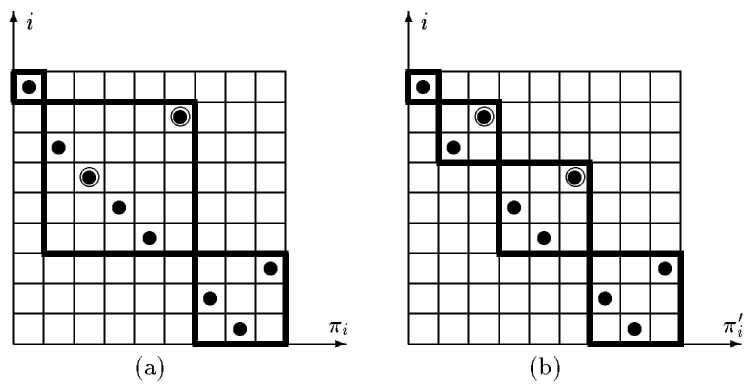

Figure 1: The permutations 976548213 and 978546213 in $S_9(123, 132)$ in array representation. Transposing two entry results block-splitting (from (a) to (b)), or block-merging (from (b) to (a)).

4 Gray Codes

A list $\mathcal{L}$ for a string set L is an ordered list of the elements of L. If the elements of $\mathcal{L}$ are in some order such that two consecutive elements are close, the list is called a Gray code list.

Let α be an integer or a string and $\mathcal{L}$ a list of strings, then $\alpha \cdot \mathcal{L}$ denotes the list obtained by concatenating α to each string of $\mathcal{L}$, e.g., if $\alpha = 4$ and $\mathcal{L} = \{123, 132, 213\}$, then $\alpha \cdot \mathcal{L} = \{4123, 4132, 4213\}$. If $\mathcal{L}'$ is an other list then $\mathcal{L} \circ \mathcal{L}'$ is the concatenation of the two lists, e.g., if $\mathcal{L}' = \{231, 312, 321\}$ then $\mathcal{L} \circ \mathcal{L}' = \{123, 132, 213, 231, 312, 321\}$. Furthermore, by $\overline{\mathcal{L}}$ we denote the reverse of the list $\mathcal{L}$ and $\mathcal{L}^*$ is the list $\mathcal{L}$ after increasing the largest entry in all strings of $\mathcal{L}$ by one. So, with $\mathcal{L}$ as above, $\overline{\mathcal{L}} = \{213, 132, 123\}$ and $\mathcal{L}^* = \{124, 142, 214\}$.

In this section we construct Gray codes for $S_n(123,132)$, $S_n(123,132,\sigma_p)$, and $S'_{n,k}(123,132)$ by considering Gray codes of their pre-images under the bijection φ. We begin this section with the concept of *dual reflected order* which will be used in constructing a Gray code for each pre-images of the bijection φ. The dual reflected order, defined below, is a slight modification of reflected order [1] and like lexicographical order, both of them are particular cases of *genlex order* [8], that is, any set of strings listed in such an order has the property that strings with a common prefix are contiguous.

Definition 4 ([7]). *For two strings $b = b_1b_2\ldots b_n$ and $b' = b'_1b'_2\ldots b'_n$ in $\{0,1\}^n$ we say that b is less than b' in* dual reflected order *if $b_1b_2\ldots b_k$, the length k prefix of b, contains an odd number of 0s, where k is the leftmost position with $b_k \neq b'_k$.*

In [7] is noticed that: (1) b is less than b' in dual reflected order iff $\overline{b'}$ is less than $\overline{b}$ in reflected order, with $\overline{b}$ and $\overline{b'}$ the bitwise complement of b and b'; (2) Like reflected order, dual reflected order induces a Gray code on $\{0,1\}^n$ and $C_{n,k}$, but only the last one yields a Gray code on $F_n^{(p)}$. Here we adopt this order relation in constructing Gray codes for $S_n(123,132)$, $S_n(123,132,213)$, and $S'_{n,k}(123,132)$.

4.1 Gray code for $S_n(123,132)$

The following Gray code for the set $\{0,1\}^n$ can be obtained from the famous Binary Reflected Gray Code [1] by replacing in it all 0 bits in each string by 1 bits and vice-versa, and then reversing the obtained list; two consecutive strings differ in a single position and the listing order is the dual reflected order [6].

$$\mathcal{B}_n = \begin{cases} \emptyset & \text{if} \quad n = 0 \\ 0 \cdot \overline{\mathcal{B}}_{n-1} \circ 1 \cdot \mathcal{B}_{n-1} & \text{if} \quad n \geq 1. \end{cases} \tag{3}$$

By considering φ, the list $\mathcal{B}_{n-1}$ is transformed to the following list for the set $S_n(123,132)$:

$$\mathcal{S}_n(123,132) = \begin{cases} \{1\} & \text{if} \quad n = 0 \\ n \cdot \overline{\mathcal{S}}_{n-1}(123,132) \circ (n-1) \cdot \mathcal{S}^*_{n-1}(123,132) & \text{if} \quad n \geq 1. \end{cases} \tag{4}$$

Through the isomorphism φ, two consecutive permutations in the list (4) differ by a transposition, and so $\mathcal{S}_n(123,132)$ is a Gray code. See Table 1 (a) for $\mathcal{B}_4$ and $\mathcal{S}_5(123,132)$.

4.2 Gray code for $S_n(123,132,\sigma_p)$

The following list is a Gray code for the set $F_n^{(p)}$ [6]:

$$\mathcal{F}_n^{(p)} = \begin{cases} \emptyset & \text{if} \quad n = 0 \\ \{0,1\} & \text{if} \quad n = 1 \\ 0 \cdot \overline{\mathcal{F}}_{n-1}^{(p)} \circ 10 \cdot \overline{\mathcal{F}}_{n-2}^{(p)} \circ \ldots \circ 1^{p-1}0 \cdot \overline{\mathcal{F}}_{n-p}^{(p)} & \text{if} \quad n > 1 \end{cases} \tag{5}$$

with two conventions: (1) the list $\alpha \cdot \mathcal{F}_{-1}^{(p)}$ consists of the single string list obtained from α by deleting its last bit, and (2) $\mathcal{F}_{-t}^{(p)}$ is the empty list for $t > 1$. In the list above, two consecutive strings differ in a single position and the listing order is the dual reflected order. By considering φ, the list $\mathcal{F}_{n-1}^{(p)}$ is transformed to the following list for the set $S_n(123,132,\sigma_p)$:

$$\mathcal{S}_n(123,132,\sigma_p) = \begin{cases} \{1\} & \text{if} \quad n=0 \\ \{21,12\} & \text{if} \quad n=1 \\ n\cdot\overline{\mathcal{S}}_{n-1}(123,132,\sigma_p) & \\ \circ(n-1)n\cdot\overline{\mathcal{S}}_{n-2}(123,132,\sigma_p) & \\ \ldots & \\ \circ(n-1)(n-2)\ldots(n-p+1)n\cdot\overline{\mathcal{S}}_{n-p}(123,132,\sigma_p) & \text{if} \quad n>1. \end{cases} \tag{6}$$

with the conventions: (1) the list $\alpha\cdot\mathcal{S}_0(123,132,\sigma_p)=\alpha$, and (2) $\mathcal{S}_{-t}(123,132,\sigma_p)$ is the empty list for $t>0$.
Through the isomorphism φ, two consecutive permutations in the list (6) differ by a transposition, and so $\mathcal{S}_n(123,132,\sigma_p)$ is a Gray code. See Table 1 (b) for $\mathcal{F}_4^{(2)}$ and $\mathcal{S}_5(123,132,213)$.

4.3 Gray code for $S'_{n,k}(123,132)$

The following list is the restriction of $\mathcal{B}_n$ defined in (3) to the set $C_{n,k}$. Two consecutive strings differ in two positions and this list is similar to Liu-Tang Gray code [2] except it lists strings in dual reflected order.

$$\mathcal{C}_{n,k} = \begin{cases} \emptyset & \text{if } n=0 \\ \{0^n\} & \text{if } n\geq 1 \text{ and } k=0 \\ \{1^n\} & \text{if } n\geq 1 \text{ and } k=n \\ 0\cdot\overline{\mathcal{C}}_{n-1,k}\circ 1\cdot\mathcal{C}_{n-1,k-1} & \text{if } n\geq 1 \text{ and } 0<k<n. \end{cases} \tag{7}$$

Through function φ, $\mathcal{C}_{n-1,k}$ is transformed to the following list for the set $S'_{n,k}(123,132)$:

$$\mathcal{S}'_{n,k}(123,132) = \begin{cases} \{1\} & \text{if } n=0 \\ \{n(n-1)\ldots 21\} & \text{if } n\geq 1 \text{ and } k=0 \\ \{(n-1)(n-2)\ldots 21n\} & \text{if } n\geq 1 \text{ and } k=n \\ n\cdot\overline{\mathcal{S}}'_{n-1,k}(123,132)\circ(n-1)\cdot\mathcal{S}'^{*}_{n-1,k-1}(123,132) & \text{if } n\geq 1 \text{ and } 0<k<n. \end{cases} \tag{8}$$

Through the isomorphism φ, two consecutive permutations in the list (8) differ by two transpositions, and so $\mathcal{S}'_{n,k}(123,132)$ is a Gray code. See Table 2 for $\mathcal{C}_{4,2}$ and for $\mathcal{S}'_{5,2}(123,132)$.

5 Graph Theoretical and Algorithm Considerations

5.1 Graphes

Here we deliver some graph theoretical interpretations of the previous results. The graph *induced* by a combinatorial class is that where the objects of the class act as its vertices. Two vertices of this graph are connected if the associated two combinatorial objects are close. We denote by $G(X)$ the graph induced by the combinatorial class X and the hypercube Q_n is the graph $G(\{0,1\}^n)$.

Two graphs $G(X)$ and $G(Y)$ are isomorphic if there is a bijection $p : X \to Y$ such that two vertices a and b are connected in $G(X)$ if and only if the vertices $p(a)$ and $p(b)$ are connected in $G(Y)$, and so, combinatorial and graph isomorphism are equivalent notions, and in this case $G(p(X)) = p(G(X))$.

A graph is connected if there exists a path between any two vertices. A Hamiltonian path is a path between two vertices of a graph which visits each vertex exactly once. A Hamiltonian path corresponds to a Gray code for the related class.

Figure 2 (a) and (b) show the isomorphic graphs Q_3 and $G(S_4(123,132))$. Hamiltonian paths—or equivalently, Gray codes for the corresponding combinatorial classes—are in bold. The graph $G(F_{n-1}^{(p)})$ is the restriction of Q_{n-1} to the set $F_{n-1}^{(p)}$, and in Figure 2 (c) the subgraph $G(F_3^{(2)})$ is in bold. Similarly, through the isomorphism $\varphi : F_{n-1}^{(p)} \to S_n(123,132,\sigma_p)$, the graph $G(S_n(123,132,\sigma_p))$ is the restriction of $G(S_n(123,132))$ to the set $S_n(123,132,\sigma_p)$.

For all k, $1 \le k \le n-1$, and under Definition 2, the restriction of Q_{n-1} to the set $C_{n-1,k}$ is not connected because the Hamming distance between any two strings in $C_{n-1,k}$ is at least 2. Now, let G^m be the m-th power of the graph G, i.e., the graph where two vertices are connected if there is a path in G of length at most m between these vertices. In this context the graph $G(C_{n-1,k})$, with respect to the closeness Definition 3 is the restriction of Q_{n-1}^2 to the set $C_{n-1,k}$, and through the isomorphism $\varphi : C_{n-1,k} \to S'_{n,k}(123,132)$, $G(S'_{n,k}(123,132))$ is the restriction of $G^2(S_n(123,132))$ to the set $S_{n,k}(123,132)$. In Figure 2 (d) is depicted Q_3^2 and its restriction to $C_{3,2}$.

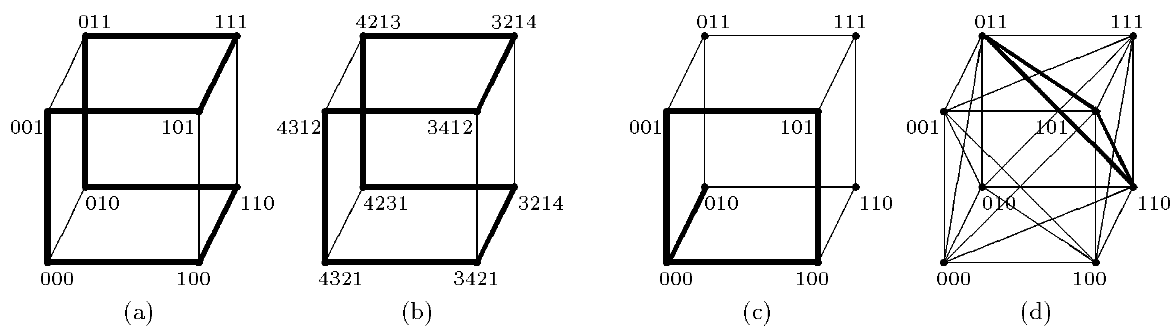

Figure 2: (a) The graph Q_3, and (b) that induced by $S_4(123,132)$; A Hamiltonian cycles in each graph are in bold. (c) The graph Q_3 and, in bold, its restricted to $F_3^{(2)}$. (d) Q_3^2, the square of the cube, and in bold, its restricted to $C_{3,2}$.

5.2 Algorithm considerations

Here we show how the isomorphism of φ allows us to construct efficient generating algorithms for the lists defined in (4), (6) and (8).

Let b and b' be two successive strings in $\mathcal{B}_n$ defined by (3) and suppose that b and b' differ in position i. Dual-reflected order has the following consequence: either $i = n$ or $b_{i+1} = 0$. Indeed, if $b_{i+1} = 1$ then the string $b_1 \dots b_i 0 b_{i+2} \dots b_n$ is larger than b and smaller than b', in dual reflected order, so b' is not the successor of b. This remark remains true if b and b' are successive strings in $\mathcal{F}_n^{(p)}$. Let now `next` be a procedure which computes the position i where a given string b differs from its successor in the list $\mathcal{X} = \mathcal{B}_{n-1}$ or $\mathcal{X} = \mathcal{F}_{n-1}^{(p)}$. When $\mathcal{X} = \mathcal{B}_{n-1}$ i is alternatively $n-1$ and the rightmost position in b with $b_{i+1} = 0$. When $\mathcal{X} = \mathcal{F}_{n-1}^{(p)}$, `next` is a little more complicate, and is given in [6]. The following algorithm results from the isomorphism of φ and generates the list $\mathcal{S}_n(123,132)$ when $\mathcal{X} = \mathcal{B}_{n-1}$ and the list $\mathcal{S}_n(123,132,\sigma_p)$ when $\mathcal{X} = \mathcal{F}_{n-1}^{(p)}$.

- Initialize b by the first string in $\mathcal{X}$ and π by $\varphi(b)$. The first string in $\mathcal{B}_{n-1}$ is 01^{n-2} and the first one in $\mathcal{F}_{n-1}^{(p)}$ is given in [6].
- Run `next`. If b differs from its successor in $\mathcal{X}$ in position i then the successor of π in $\varphi(\mathcal{X})$ is obtained by transposing the entries in position i and $i+1$.
- Stop when the last string in $\mathcal{X}$ is reached. The last string in $\mathcal{B}_{n-1}$ is 1^{n-1} and in the case of the list $\mathcal{F}_{n-1}^{(p)}$ `next` detects its last string in constant time [6].

Now let discuss the generation of $S'_{n,k}(123,132)$ defined in (8), the image of $\mathcal{C}_{n-1,k}$ through the function φ. Let $b = b_1 b_2 \dots b_n$ be a binary string in $C_{n,k}$ which is not the last one in dual reflected order. Suppose that b differs from its successor, in dual reflected order, in positions i and j, $i < j$. Again, dual-reflected order has the following consequences: (1) either $b_{i+1} = 0$ or $b_{i+1} = b_{i+2} = \dots = b_n = 1$ (in the latter case $j = i+1$), and (2) either $j = n$ or $b_{j+1} = 0$ or $b_{j+1} = b_{j+2} = \dots = b_n = 1$. Let `next` be a procedure which computes the positions i and j where a given string b differs from its successor in $\mathcal{C}_{n-1,k}$. Such a procedure can be obtained by a direct implementation of definition (7) or by a slight modification of Liu-Tang algorithm [2]. The following algorithm results from the considerations above and the isomorphism of φ and generates the list $S'_{n,k}(123,132)$.

- Initialize b by 01^k0^{n-k-2}, the first string in $\mathcal{C}_{n-1,k}$, and π by $\varphi(b)$.
- Run `next` and let i and j, $i < j$, the positions where b differs from its successor in $\mathcal{C}_{n-1,k}$.
  - if $b_{i+1} = 0$ then transpose π_i and π_{i+1}, else transpose π_i and π_n
  - if $j = n-1$ or $b_{j+1} = 0$ then transpose π_j and π_{j+1} else transpose π_j and π_n.
- Stop when $b = 1^k0^{n-k-1}$, that is, when the last string in $\mathcal{C}_{n-1,k}$ is reached.

Each of the procedures `next` above has a constant time implementation and so are the obtained generating algorithms.

6 Concluding Remarks

The bijections $\varphi : B_{n-1} \to S_n(123,132)$, $\varphi : F_{n-1} \to S_n(123,132,213)$, and $\varphi : C_{n-1,k} \to S_{n,k}(123,132)$ are isomorphisms. Since the lists defined in (3), (5), and (7) are Gray codes so are their images under φ, namely the lists defined by (4), (6), and (8). Tables 1 and 2 actually are the mentioned Gray codes for $n = 5$ (and with $k = 2$ in Table 2).

B_n is a superset of F_n and $C_{n,k}$ as well as $S_n(123,132)$ is a superset of $S_n(123,132,213)$ and $S'_{n,k}(123,132)$. Our choice of Gray codes (3), (5), and (7) induced some interested properties to they images through φ; following are two of them.

1. The restriction of the list $\mathcal{B}_{n-1}$ to the set F_{n-1} (resp. $C_{n-1,k}$) is exactly the list $\mathcal{F}_{n-1}$ (resp. $\mathcal{C}_{n-1,k}$), or equivalently $\mathcal{F}_{n-1}$ and $\mathcal{C}_{n-1,k}$ are *(scattered) sublists* of $\mathcal{B}_{n-1}$. For instance, deleting all elements of $\mathcal{B}_4$ having two consecutive 1s from the list in Table 1(a) produces $\mathcal{F}_4$ in Table 1(b), deleting all elements of $\mathcal{B}_4$ having no exactly two 1s from the list in Table 1(a) produces $\mathcal{C}_{4,2}$ in Table 2; similarly
2. The restriction of the list $\mathcal{S}_n(123,132)$ to the set $S_n(123,132,213)$ (resp. $S'_{n,k}(123,132)$) is exactly the list $\mathcal{S}_n(123,132,213)$ (resp. $\mathcal{S}'_{n,k}(123,132)$).

References

[1] F. Gray. *Pulse Code Communication.* U.S. Patent 2632058 (1953).

[2] C.N. Liu and D.T. Tang. *Algorithm 452: enumerating combinations of m out of n objects [G6]* Communications of the ACM, **16**(8), pp. 485, 1973.

[3] F; Ruskey. *Adjacent interchange generation of combinations* Journal of Algorithms, **9**, pp. 162-180, 1988.

[4] F. Ruskey. *Combinatorial Generation, Section 5.3.* http://www.csc.uvic.ca/∼csc528/bookOnline.html, 2001.

[5] R. Simion and F.W. Schmidt. *Restricted Permutations.* Europ. J. Combinatorics, (**6**), pp. 383-406, 1985.

[6] V. Vajnovszki. *A Loopless Generation of Bitstring without p Consecutive Ones.* Proceeding of 3-rd Conference on Discrete Mathematics and Theoretical Computer Science, pp. 227-239, 2001.

[7] V. Vajnovszki. *Gray viziting Lyndons.* Journées Montoises d'Informatique Théorique à Liège, 8-11 Septembre 2004.

[8] T. Walsh. *Generating Gray Codes in O(1) Worst-Case Time per Word.* Proceeding of 4-rd Conference on Discrete Mathematics and Theoretical Computer Science, pp. 73-78, 2003.