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Quasi-Apartness and Neighbourhood Spaces

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Abstract

We extend the concept of apartness spaces to the concept of quasi-apartness spaces. We show that there is an adjunction between the category of quasi-apartness spaces and the category of neighbourhood spaces, which indicates that quasi-apartness is a more natural concept than apartness. We also show that there is an adjoint equivalence between the category of apartness spaces and the category of Grayson's separated spaces.

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1 Introduction

Although Bishop [3, 4] proposed neighbourhood spaces as a constructive substitute for topological spaces, he did not develop the theory. About thirty years later, Bridges and Vîţă [6, 7] proposed the theory of apartness spaces as an alternative approach to topology from a constructive point of view.

In this paper, we connect these proposals by introducing the weaker notion of quasi-apartness, and construct an adjunction between the category of quasi-apartness spaces and the category of neighbourhood spaces. We introduce the notion of a "decent open base" in a neighbourhood space, which corresponds to the notion of "weak nested neighbourhood" in (quasi-) apartness spaces [7]. We construct an adjoint equivalence between the category of quasi-apartness spaces with the weak nested neighbourhood propertiy and the category of neighbourhood spaces with decent open bases. Finally, we construct an adjoint equivalence between the category of apartness spaces with the weak nested nighbourhood property and the category of Grayson's separated spaces [8] with decent open bases.

There are other constructive treatments of topology: see Grayson [8, 9] for general topology, Sambin [15, 16] for formal topology in Martin-Löf type theory [11], and Aczel [1] for topology in constructive set theory (CZF) [2]. Palmgren and Schuster [13] deal with relations between apartness and formal topology.

2 From neighbourhood to quasi-apartness

In this section, we deal with constructing quasi-apartness spaces from neighbourhood spaces. We start with our notion of quasi-apartness space. For a susbet S of a set X, let $\neg S$ denote a set $\{x \in X \mid \neg(x \in S)\}$, called the *logical complement of* S.

Definition 2.1. A *quasi-apartness space*, $\langle X, -\rangle$, is a set X with an operation, $-$, on the subsets of X such that for all sets $S, T \subset X$,

QA1. $-\emptyset = X$,

QA2. $-S \subset \neg S$,

QA3. $-(S \cup T) = -S \cap -T$,

QA4. $-S \subset \neg T \implies -S \subset -T$.

We say that $-$ is a *quasi-apartness* on X.

Note that, under classical logic, we have $\neg\neg S = S$ for any set S. Constructively though, we can only prove that $S \subset \neg\neg S$. To see this, let P be a predicate, and let $X := \{0, 1\}$ with the equality defined by $0 = 1 \iff P \vee \neg P$. If $1 \in \neg\{0\}$, then assuming $P \vee \neg P$, we have $1 \in \{0\}$, a contradiction, and hence $\neg(P \vee \neg P)$, a contradiction again. Thus $1 \in \neg\neg\{0\}$. If we could prove that $\neg\neg\{0\} \subset \{0\}$, we would have $0 = 1$, and hence $P \vee \neg P$.

Lemma 2.2. *Let* $\langle X, -\rangle$ *be a quasi-apartness. Then* $-\neg - S = -S$

Proof. Since $S \subset \neg - S$, by QA2, we have $-\neg - S \subset -S$, by QA3. On the other hand, since $-S \subset \neg\neg - S$, we have $-S \subset -\neg - S$, by QA4. $\square$

Bishop [3] introduced the notion of neighbourhood space as a constructive substitute of topological spaces.

Definition 2.3. A *neighbourhood space* is a pair (X, τ) consisting of a set X and a set τ of subsets of X such that

NS1. $\forall x \in X \exists U \in \tau (x \in U)$,

NS2. $\forall x \in X \forall U, V \in \tau [x \in U \cap V \implies \exists W \in \tau (x \in W \subset U \cap V)]$.

We say that τ is an *open base* on X. The interior, S°, of a subset S of X is $\{x \in S \mid \exists U \in \tau (x \in U \subset S)\}$. A subset S of X is *open* if $S = S^\circ$.

It turns out that with every neighbourhood space we can associate a quasi-apartness structure.

Definition 2.4. For a neighbourhood space (X, τ), define an operation $-_\tau$ on the subsets of X by

$$-_\tau S := \{x \in X \mid \exists U \in \tau (x \in U \subset \neg S)\} = (\neg S)^\circ.$$

Proposition 2.5. *Let (X, τ) be a neighbourhood space. Then $\langle X, -_\tau \rangle$ is a quasi-apartness space.*

Proof. (QA1): Trivially $x \in X \subset \neg\emptyset$ for all $x \in X$, and hence $-_\tau \emptyset = X$.

(QA2): Trivial.

(QA3): $x \in -_\tau (S \cup T) \iff \exists U \in \tau (x \in U \subset \neg(S \cup T) = \neg S \cap \neg T) \iff \exists V, W \in \tau (x \in V \subset \neg S \wedge x \in W \subset \neg T) \iff x \in -_\tau S \cap -_\tau T$.

(QA4): Suppose that $-_\tau S \subset \neg T$ and $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$. Noting that $U \subset -_\tau S$, we have $x \in U \subset \neg T$, and hence $x \in -_\tau T$. $\square$

Definition 2.6. Let $-$ and $-'$ be two quasi-apartness operators on a set X. Then we say that $-$ *is weaker than* $-'$ (or $-'$ *is stronger than* $-$) and write $- \preceq -'$ if

$$\forall S \subset X (-S \subset -'S).$$

We say that $-$ *and* $-'$ *are equivalent* and write $- \simeq -'$ if $- \preceq -'$ and $-' \preceq -$.

Let τ and τ' be two open bases on a set X. Then we say that τ *is weaker than* τ' (or τ' *is stronger than* τ) and write $\tau \sqsubseteq \tau'$ if

$$\forall x \in X \forall U \in \tau [x \in U \implies \exists V \in \tau' (x \in V \subset U)].$$

We say that τ *and* τ' *are equivalent* and write $\tau \approx \tau'$ if $\tau \sqsubseteq \tau'$ and $\tau' \sqsubseteq \tau$. Note that if $\tau \approx \tau'$, τ and τ' give the same open sets.

Lemma 2.7. *Let τ and τ' be open bases on a set X. If $\tau \sqsubseteq \tau'$, then $-_\tau \preceq -_{\tau'}$.*

Proof. Suppose that $\tau \sqsubseteq \tau'$, and let $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$, and hence there exists $V \in \tau'$ such that $x \in V \subset U \subset \neg S$. Therefore $x \in -_{\tau'} S$. □

Lemma 2.8. *Let τ and τ' be open bases on a set X. Then $-_\tau \preceq -_{\tau'}$ if and only if for each $U \in \tau$ and $x \in U$ there exists $V \in \tau'$ such that $x \in V \subset \neg\neg U$.*

Proof. Suppose that $-_\tau \preceq -_{\tau'}$. Then given $U \in \tau$ and $x \in U$, since $U \subset \neg\neg U$, we have $x \in U \subset -_\tau \neg U \subset -_{\tau'} \neg U$, and hence there exists $V \in \tau'$ such that $x \in V \subset \neg\neg U$.

Conversely suppose that for each $U \in \tau$ and $x \in U$ there exists $V \in \tau'$ such that $x \in V \subset \neg\neg U$, and let $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$, and hence there exists $V \in \tau'$ such that $x \in V \subset \neg\neg U$. Therefore, since $U \subset \neg S$ implies $\neg\neg U \subset \neg S$, we have $x \in V \subset \neg\neg U \subset \neg S$, and so $x \in -_{\tau'} S$. □

Proposition 2.9. *Let τ and τ' be open bases on a set X. If $-_\tau \preceq -_{\tau'}$, then there exists an open base σ such that $\tau \sqsubseteq \sigma$ and $-_{\tau'} \simeq -_\sigma$.*

Proof. Suppose that $-_\tau \preceq -_{\tau'}$, and let $\sigma := \{U \cap V \mid U \in \tau \wedge V \in \tau'\}$. Then it is straightforward to show that σ is an open base, $\tau \sqsubseteq \sigma$, and $\tau' \sqsubseteq \sigma$; whence $-_{\tau'} \preceq -_\sigma$, by Lemma 2.7. To show that $-_\sigma \preceq -_{\tau'}$, let $x \in U \cap V$ for some $U \in \tau$ and $V \in \tau'$. Then by the hypothesis and Lemma 2.8, there exists $W \in \tau'$ such that $x \in W \subset \neg\neg U$, and hence $x \in W \cap V \subset \neg\neg U \cap V \subset \neg\neg U \cap \neg\neg V = \neg\neg(U \cap V)$ and $W \cap V \in \tau'$. Therefore $-_{\tau'} \preceq -_\sigma$, by Lemma 2.8. □

We next look at how continuity with respect to quasi-apartness spaces relates to continuity with respect to neighbourhood spaces.

Definition 2.10. A function f between quasi-apartness spaces $\langle X, -\rangle$ and $\langle Y, -'\rangle$ is *continuous* if for all $x \in X$ and $S \subset X$,

$$f(x) \in -'f(S) \implies x \in -S.$$

A function f between neighbourhood spaces (X, τ) and (Y, τ') is *continuous* if for all $x \in X$ and $V \in \tau'$, we have

$$f(x) \in V \implies \exists U \in \tau (x \in U \subset f^{-1}(V)).$$

Theorem 2.11. *Let f be a mapping between neighbourhood spaces (X, τ) and (Y, τ'). Then $f : \langle X, -_\tau\rangle \to \langle Y, -_{\tau'}\rangle$ is continuous if and only if there exists an open base σ with $-_\tau \simeq -_\sigma$ such that $f : (X, \sigma) \to (Y, \tau')$ is continuous.*

Proof. Suppose that $f : (X, \sigma) \to (Y, \tau')$ is continuous for some open base σ with $-_\tau \simeq -_\sigma$, and let $f(x) \in -_{\tau'} f(S)$. Then there exists $U \in \tau'$ such that $f(x) \in U \subset \neg f(S)$, and hence there exists $V \in \sigma$ such that $x \in V \subset f^{-1}(U)$. Therefore $x \in V \subset f^{-1}(U) \subset f^{-1}(\neg f(S)) = \neg f^{-1}(f(S)) \subset \neg S$, and so $x \in -_\sigma S = -_\tau S$.

Conversely suppose that $f : \langle X, -_\tau\rangle \to \langle Y, -_{\tau'}\rangle$ is continuous, and let $\tau_f := \{f^{-1}(U) \mid U \in \tau'\}$. Then τ_f is an open base on X. Note that if σ is an open base on X, then $f : (X, \sigma) \to (Y, \tau')$ is continuous if $\tau_f \sqsubseteq \sigma$. To show that $-_{\tau_f} \preceq -_\tau$, assume that $x \in f^{-1}(U)$ for some $U \in \tau'$. Then letting $T := f^{-1}(\neg U)$, since $f(T) \subset \neg U$, we have $f(x) \in U \subset \neg\neg U \subset \neg f(T)$, and hence $f(x) \in -_{\tau'} f(T)$. Therefore $x \in -_\tau T$, and hence there exists $V \in \tau$ such that $x \in V \subset \neg T = \neg\neg f^{-1}(U)$. Thus $-_{\tau_f} \preceq -_\tau$, by Lemma 2.8. Hence there exists an open base σ such that $\tau_f \sqsubseteq \sigma$ and $-_\tau \simeq -_\sigma$, by Proposition 2.9. □

Corollary 2.12. *If $f : (X, \tau) \to (Y, \tau')$ is continuous, then $f : \langle X, -_\tau\rangle \to \langle Y, -_{\tau'}\rangle$ is continuous.*

3 From quasi-apartness to neighbourhood

In this section, we show how to obtain two neighbourhood spaces from a quasi-apartness space. We also investigate the interplay between these neighbourhood spaces.

Proposition 3.1. *Let $(X, -)$ be a quasi-apartness space, and let*

$$\tau_{-}^{w} := \{-S \mid S \subset X\}.$$

Then (X, τ_{-}^{w}) is a neighbourhood space.

Proof. Straightforward. □

Proposition 3.2. *Let $-$ be an apartness on a set X. Then $- \simeq -_{\tau_{-}^{w}}$.*

Proof. Suppose that $x \in -S$. Then $x \in -S \subset \neg S$ and $-S \in \tau_{-}^{w}$, and hence $x \in -_{\tau_{-}^{w}} S$. Therefore $- \preceq -_{\tau_{-}^{w}}$.

Conversely suppose that $x \in -_{\tau_{-}^{w}} S$. Then there exists $-T \in \tau_{-}^{w}$ such that $x \in -T \subset \neg S$, and hence $x \in -T \subset -S$, by QA4. Therefore $-_{\tau_{-}^{w}} \preceq -$. □

Proposition 3.3. *Let σ be an open base on a set X. Then $\tau_{-_\sigma}^{w} \sqsubseteq \sigma$.*

Proof. Suppose that $-_\sigma S \in \tau_{-_\sigma}^{w}$ and $x \in -_\sigma S$. Then there exists $U \in \sigma$ such that $x \in U \subset \neg S$, and therefore, since $U \subset -_\sigma S$, we have $x \in U \subset -_\sigma S$. Thus $\tau_{-_\sigma}^{w} \sqsubseteq \sigma$. □

Proposition 3.4. *Let $(X, -)$ be a quasi-apartness space, and let*

$$\tau_{-}^{s} := \{U \subset X \mid U \subset -\neg U\}.$$

Then τ_{-}^{s} is an open base on X.

Proof. (NS1): $X \in \tau_{-}^{s}$, by QA1.

(NS2): Let $U, V \in \tau_{-}^{s}$. Then, since

$$\begin{aligned} U \cap V &\subset -\neg U \cap -\neg V = -(\neg U \cup \neg V) \subset \neg(\neg U \cup \neg V) \\ &= \neg\neg U \cap \neg\neg V = \neg\neg(U \cap V), \end{aligned}$$

we have $U \cap V \subset -(\neg U \cup \neg V) \subset -\neg(U \cap V)$, by QA4, and hence $U \cap V \in \tau_{-}^{s}$. □

Proposition 3.5. *Let $-$ be an apartness on a set X. Then $- \simeq -_{\tau_{-}^{s}}$.*

Proof. Suppose that $x \in -S$. Then $x \in -S \subset \neg S$ and $-S \in \tau_{-}^{s}$ as $-S = -\neg - S$, by Lemma 2.2, and hence $x \in -_{\tau_{-}^{s}} S$. Therefore $- \preceq -_{\tau_{-}^{s}}$.

Conversely suppose that $x \in -_{\tau_{-}^{s}} S$. Then there exists $U \in \tau_{-}^{s}$ such that $x \in U \subset \neg S$, and therefore, since $S \subset \neg U$, we have $x \in U \subset -\neg U \subset -S$. Thus $-_{\tau_{-}^{s}} \preceq -$. □

Proposition 3.6. *Let σ be an open base on a set X. Then $\sigma \sqsubseteq \tau^s_{-_\sigma}$.*

Proof. Let $U \in \sigma$. Then, since $U \subset \neg\neg U$, we have $U \subset -_\sigma \neg U$, and hence $U \in \tau^s_{-_\sigma}$. □

Corollary 3.7. *Let σ be an open base on a set X. Then $- \simeq -_\sigma$ if and only if $\tau^w_- \sqsubseteq \sigma \sqsubseteq \tau^s_-$.*

Proof. If $- \simeq -_\sigma$, then $\tau^w_{-_\sigma} = \tau^w_- \sqsubseteq \sigma \sqsubseteq \tau^s_- = \tau^s_{-_\sigma}$, by Proposition 3.3 and Proposition 3.6. Conversely if $\tau^w_- \sqsubseteq \sigma \sqsubseteq \tau^s_-$, then $- \simeq -_{\tau^w_-} \preceq -_\sigma \preceq -_{\tau^s_-} \simeq -$, by Proposition 3.2, Lemma 2.7, and Proposition 3.5. □

Theorem 3.8. *Let f be a function between quasi-apartness spaces $\langle X, -\rangle$ and $\langle Y, -'\rangle$. Then $f : \langle X, -\rangle \to \langle Y, -'\rangle$ is continuous if and only if $f : (X, \tau^s_-) \to (Y, \tau^s_{-'})$ is continuous.*

Proof. Suppose that $f : \langle X, -\rangle \to \langle Y, -'\rangle$ is continuous, and let $f(x) \in U \in \tau^s_{-'}$. For $y \in f^{-1}(U)$, letting $T := f^{-1}(\neg U)$, since $f(T) \subset \neg U$, we have $f(y) \in U \subset -'\neg U \subset -'f(T)$, and hence $y \in -T = -\neg f^{-1}(U)$. Thus $x \in f^{-1}(U) \subset -\neg f^{-1}(U)$, and so $f^{-1}(U) \in \tau^s_-$.

Conversely suppose that $f : (X, \tau^s_-) \to (Y, \tau^s_{-'})$ is continuous, and let $f(x) \in -'f(S)$. Then, since $-'f(S) = -'\neg -' f(S)$, we have $-'f(S) \in \tau^s_{-'}$, and hence there exists $U \in \tau^s_-$ such that $x \in U \subset f^{-1}(-'f(S))$. Since $U \subset f^{-1}(-'f(S)) \subset f^{-1}(\neg f(S)) = \neg f^{-1}(f(S)) \subset \neg S$, we have $S \subset \neg U$, and hence $x \in U \subset -\neg U \subset -S$. □

Let $\langle X, -\rangle$ be a quasi-apartness space, and let σ be an open base on X. Then $- \simeq -_\sigma$ if and only if $\tau^w_- \sqsubseteq \sigma \sqsubseteq \tau^s_-$, by Corollary 3.7. Classically, we have $\tau^w_- \approx \tau^s_-$: if $U \in \tau^s_-$, then $U \subset -\neg U$, and therefore, since $U \subset -\neg U \subset \neg\neg U = U$ by classical logic, we have $U \in \tau^w_-$. But the following example shows that we can not prove $\tau^w_- \approx \tau^s_-$ constructively.

Let P be a predicate, and let $X := \{0, 1\}$ with the equality defined by $0 = 1 \Longleftrightarrow P \vee \neg P$. Then the open base $\sigma := \{\{0\}, \{1\}\}$ induces a quasi-apartness $-_\sigma$ on X. It is straightforward to see that $-_\sigma S = \neg S$ and $\sigma \approx \tau^s_{-_\sigma}$. Suppose that $\sigma \sqsubseteq \tau^w_{-_\sigma}$. Then, since $0 \in \{0\} \in \sigma$, there exists $S \subset X$ such that $0 \in \neg S \subset \{0\}$, and therefore, since $\{0\} \subset \neg S$, we have $S \subset \neg\{0\}$. Hence $\neg\neg\{0\} \subset \neg S \subset \{0\}$, and therefore $P \vee \neg P$.

Proposition 3.9. *Let (X, σ) be a neighbourhood space. Then the open bases $\{(\neg\neg U)^\circ \mid U \in \sigma\}$ and $\tau^w_{-_\sigma}$ are equivalent.*

Proof. Since $(\neg\neg U)^\circ = -_\sigma\neg U \in \tau^w_{-_\sigma}$, we have $\{(\neg\neg U)^\circ \mid U \in \sigma\} \sqsubseteq \tau^w_{-_\sigma}$. Let $-_\sigma S \in \tau^w_{-_\sigma}$ and $x \in -_\sigma S = (\neg S)^\circ$. Then there exists $U \in \sigma$ such that $x \in U \subset \neg S$, and hence $x \in U \subset (\neg\neg U)^\circ \subset \neg\neg U \subset \neg S$. Therefore $x \in (\neg\neg U)^\circ \subset -_\sigma S$. □

Corollary 3.10. *Let (X, σ) be a neighbourhood space. Then $\tau^w_{-_\sigma} \approx \sigma$ if and only if*

$$\forall x \in X \forall U \in \sigma[x \in U \implies \exists V \in \sigma(x \in V \wedge (\neg\neg V)^\circ \subset U)].$$

Proof. Straightforward using Proposition 3.9. □

Proposition 3.11. *Let (X, σ) be a neighbourhood space. Then the open bases $\{S \subset X \mid S \subset (\neg\neg S)^\circ\}$ and $\tau^s_{-_\sigma}$ are equivalent.*

Proof. Since $(\neg\neg S)^\circ = -_\sigma\neg S$, we have $\{S \subset X \mid S \subset (\neg\neg S)^\circ\} = \tau^s_{-_\sigma}$. □

Corollary 3.12. *Let (X, σ) be a neighbourhood space. Then $\sigma \approx \tau^s_{-_\sigma}$ if and only if $S \subset X$ is open whenever $S \subset (\neg\neg S)^\circ$.*

Proof. Straightforward using Proposition 3.9. □

4 The fundamental adjunction

Let **Nbh** denote the category of neighbourhood spaces with neighbourhood spaces as objects and continuous functions as arrows, and let **Qap** denote the category whose objects are quasi-apartness spaces and whose arrows are continuous functions (for terminologies in category theory here, see [12]).

Theorem 4.1. *There exists an adjunction between* **Qap** *and* **Nbh**.

Proof. Define a functor $\mathbf{G}$ from **Nbh** to **Qap** by $\mathbf{G}(X, \tau) := \langle X, -_\tau\rangle$ and $\mathbf{G}f := f$. Then $\mathbf{G}$ is a faithful functor, by Corollary 2.12. Similarly, define the functor $\mathbf{F}_s$ from **Qap** to **Nbh** by $\mathbf{F}_s\langle X, -\rangle := (X, \tau^s_-)$ and $\mathbf{F}_s f := f$. Then $\mathbf{F}_s$ is a full and faithful functor, by Theorem 3.8.

Furthermore, using Propositions 3.5 and 3.6, we see that letting $\eta_{\langle X,-\rangle}$ and $\varepsilon_{(X,\tau)}$ denote the identity map on the set X, then $\eta_{\langle X,-\rangle} : \langle X, -\rangle \to \langle X, -_{\tau^s_-}\rangle$ and $\varepsilon_{(X,\tau)} : (X, \tau^s_{-_\tau}) \to (X, \tau)$ are arrows in the category **Qap** and **Nbh**, respectively. Hence $\eta : \mathbf{1_{Qap}} \to \mathbf{GF}_s$ is a natural isomorphism and $\varepsilon : \mathbf{F}_s\mathbf{G} \to \mathbf{1_{Nbh}}$ are natural transformations satisfying $\varepsilon_{\mathbf{F}_s} \circ \mathbf{F}_s\eta = 1_{\mathbf{F}_s}$ and $\mathbf{G}\varepsilon \circ \eta_{\mathbf{G}} = 1_{\mathbf{G}}$. Hence $\langle \mathbf{F}_s, \mathbf{G}, \eta, \varepsilon\rangle$ forms an adjunction between **Qap** and **Nbh**. □

In **Nbh**, we can construct subspaces, products, quotient spaces, and coproducts (even in CZF and type theory [10]). These constructions can be moved into quasi-apartness spaces by the adjunction. For example, to construct a product in **Qap** via one in **Nbh**, let A and B be objects in **Qap**. Then, since the product

$$\mathbf{F}_s A \xleftarrow{p_1} \mathbf{F}_s A \times \mathbf{F}_s B \xrightarrow{p_2} \mathbf{F}_s B$$

exists in **Nbh**, we have in **Qap**

$$A \xleftarrow{q_1} \mathbf{G}(\mathbf{F}_s A \times \mathbf{F}_s B) \xrightarrow{q_2} B,$$

where $q_i := \eta^{-1} \circ \mathbf{G} p_i$ for $i = 1, 2$. Let C be an object and $f : C \to A$ and $g : C \to B$ be arrows in **Qap**. Then there exists a unique arrow $\langle \mathbf{F}_s f, \mathbf{F}_s g \rangle : \mathbf{F}_s C \to \mathbf{F}_s A \times \mathbf{F}_s B$ such that $\mathbf{F}_s f = q_1 \circ \langle \mathbf{F}_s f, \mathbf{F}_s g \rangle$ and $\mathbf{F}_s g = q_2 \circ \langle \mathbf{F}_s f, \mathbf{F}_s g \rangle$. Hence there is a unique $\langle f, g \rangle : C \to \mathbf{G}(\mathbf{F}_s A \times \mathbf{F}_s B)$ such that $f = q_1 \circ \langle f, g \rangle$ and $g = q_2 \circ \langle f, g \rangle$.

5 Relations to other theories

5.1 Spaces with inequality

The definition of a (point-set) apartness space [6, 7] requires that an inequality be given on the underlying set. An *inequality* $\neq$ on a set X is a binary relation on X such that $x \neq y \implies y \neq x$ and $x \neq y \implies \neg(x = y)$. Fred Richman [14] showed that the given inequality is determined by the point-set apartness on the set, so it can be viewed as a derivable concept. But this is not the case for quasi-apartness spaces: in fact, for the quasi-apartness $-_\tau$ on the set $X := \{0, 1\}$ with $0 \neq 1$ induced by the open base $\tau := \{X\}$, the derived inequality $x \neq' y := x \in -_\tau\{y\} \vee y \in -_\tau\{x\}$ satisfies $\forall x, y \in X \neg(x \neq' y)$.

The following results show how to include inequality in quasi-apartness spaces.

Definition 5.1. A *quasi-apartness space with an inequality* $\neq$ is a quasi-apartness space $\langle X, - \rangle$ satisfying the following axiom instead of QA2.

QA2$_{\neq}$. $\quad -S \subset \sim S$,

where $\sim S$ is the complement $\{x \in X \mid \forall y \in S(x \neq y)\}$ of S.

Definition 5.2. A *neighbourhood space with an inequality* $\neq$ is a neighbourhood space (X, τ) satisfying

$$\mathrm{T}_0^{-1}. \quad \exists U \in \tau[(x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)] \implies x \neq y.$$

Lemma 5.3. *Let (X, τ) be a neighbourhood space with an inequality $\neq$. Then $x \in -_\tau S$ if and only if there exists $U \in \tau$ such that $x \in U \subset\sim S$.*

Proof. The "if part" is trivial. Let $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$. Since $S \subset \neg U$, we have $z \neq y$ for all $z \in U$ and $y \in S$, by T_0^{-1}, and hence $x \in U \subset\sim S$. □

Proposition 5.4. *Let (X, τ) be a neighbourhood space with an inequality $\neq$. Then $\langle X, -_\tau \rangle$ is a quasi-apartness space with the inequality $\neq$.*

Proof. ($\mathrm{QA2}_{\neq}$): If $x \in -_\tau S$, then there exists $U \in \tau$ such that $x \in U \subset\sim S$, by Lemma 5.3. □

Proposition 5.5. *Let $\langle X, - \rangle$ be a quasi-apartness space with an inequality $\neq$. Then (X, τ_-^s) is a neighbourhood space with the inequality $\neq$.*

Proof. Suppose that there exists $U \in \tau_-^s$ such that $x \in U \wedge y \notin U$ or $x \notin U \wedge y \in U$. Then, in the former case, we have $x \in U \subset -\neg U \subset\sim \neg U$, by $\mathrm{QA2}_{\neq}$, and hence $x \neq y$. Similarly, in the latter case we have $x \neq y$. □

Corollary 5.6. *Let $\langle X, - \rangle$ be a quasi-apartness space with an inequality $\neq$, and let τ be an open base on X with $- \simeq -_\tau$. Then (X, τ) is a neighbourhood space with the inequality $\neq$.*

Proof. Suppose that there exists $U \in \tau$ such that $x \in U \wedge y \notin U$ or $x \notin U \wedge y \in U$. Then, in the former case, since $\tau \sqsubseteq \tau_-^s$, by Corollary 3.7, there exists $V \in \tau_-^s$ such that $x \in V \subset U$, and hence $y \notin V$. Therefore $x \neq y$, by Proposition 5.5. In the latter case, by a similar argument, we have $x \neq y$. □

5.2 Weak nested neighbourhood property

The following notion of weak nested neighbourhood was introduced in Bridges and Vîţă for apartness spaces [7] to establish connections between certain continuity properties.

Definition 5.7. A quasi-apartness space $\langle X, - \rangle$ has the *weakly nested neighbourhood property* if for all $x \in X$ and $S \subset X$, WNN. $x \in -S \implies \exists T \subset X(x \in -T \wedge \neg T \subset -S)$.

As we shall see, the corresponding notion in neighbourhood spaces is that of a decent open base. (Classically every open base is decent.)

Definition 5.8. An open base τ on X is *decent* if

DOB. $\forall x \in X \forall U \in \tau[x \in U \implies \exists V \in \tau(x \in V \wedge \neg\neg V \subset U)]$.

Proposition 5.9. *Let τ be a decent open base on a set X. Then $\langle X, -_\tau\rangle$ has the weakly nested neighbourhood property.*

Proof. Let $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$, and hence there exists $V \in \tau$ such that $x \in V \wedge \neg\neg V \subset U \subset \neg S$. Therefore, letting $T := \neg V$, we have $\neg T = \neg\neg V \subset -_\tau S$, and, since $x \in V \subset -_\tau \neg V$, we have $x \in -_\tau T$. □

Proposition 5.10. *Let σ be a decent open base on a set X. Then $\tau^w_{-_\sigma} \approx \sigma$.*

Proof. By Corollary 3.10. □

Theorem 5.11. *Let f be a function between neighbourhood spaces (X, τ) and (Y, τ') such that τ' is decent. Then $f : (X, \tau) \to (Y, \tau')$ is continuous if and only if $f : \langle X, -_\tau\rangle \to \langle Y, -_{\tau'}\rangle$ is continuous.*

Proof. Suppose that $f : (X, \tau) \to (Y, \tau')$ is continuous, and let $f(x) \in -_{\tau'} f(S)$. Then there exists $U \in \tau'$ such that $f(x) \in U \subset \neg f(S)$, and hence there exists $V \in \tau$ such that $x \in V \subset f^{-1}(U)$. Therefore $x \in V \subset f^{-1}(U) \subset f^{-1}(\neg f(S)) = \neg f^{-1}(f(S)) \subset \neg S$, and so $x \in -_\tau S$.

Conversely, suppose that $f : \langle X, -_\tau\rangle \to \langle Y, -_{\tau'}\rangle$ is continuous, and let $f(x) \in U \in \tau'$. Then there exists $V \in \tau'$ such that $f(x) \in V \subset \neg\neg V \subset U$. Letting $S := f^{-1}(\neg V)$, we have $f(S) \subset \neg V$, and so $f(x) \in V \subset \neg\neg V \subset \neg f(S)$. Hence $f(x) \in -_{\tau'} f(S)$, and therefore $x \in -_\tau S$. Thus there exists $W \in \tau$ such that $x \in W \subset \neg S = \neg f^{-1}(\neg W) = f^{-1}(\neg\neg W) \subset f^{-1}(U)$. □

Proposition 5.12. *Let $\langle X, -\rangle$ be a quasi-apartness space. Then $\langle X, -\rangle$ has the weakly nested neighbourhood property if and only if τ^w_- is decent.*

Proof. Suppose that $\langle X, -\rangle$ has the weakly nested neighbourhood property, and let $x \in -S \in \tau^w_-$. Then there exists $-T \in \tau^w_-$ such that $x \in -T \wedge \neg T \subset -S$, and hence $x \in -T \subset \neg\neg - T \subset \neg\neg\neg T = \neg T \subset -S$. Conversely, suppose that τ^w_- is decent, and let $x \in -S$. Then, since $-S \in \tau^w_-$, there exists $-T' \in \tau^w_-$ such that $x \in -T' \subset \neg\neg - T' \subset -S$. Let $T := \neg - T'$. Then, since $-T = -\neg - T' = -T'$, by Lemma 2.2, we have $x \in -T$ and $\neg T = \neg\neg - T' \subset -S$. □

Theorem 5.13. *Let f be a function between quasi-apartness spaces $\langle X, -\rangle$ and $\langle Y, -'\rangle$ such that $\langle Y, -'\rangle$ has the weakly nested neighbourhood property. Then $f : \langle X, -\rangle \to \langle Y, -'\rangle$ is continuous if and only if $f : (X, \tau_{-}^{w}) \to (Y, \tau_{-'}^{w})$ is continuous.*

Proof. Suppose that $f : \langle X, -\rangle \to \langle Y, -'\rangle$ is continuous, and let $f(x) \in -'S \in \tau_{-'}^{w}$. Then there exists $T \subset Y$ such that $f(x) \in -'T$ and $\neg T \subset -'S$. Letting $R := f^{-1}(\neg -' T)$, we have $f(R) \subset \neg -' T$, and therefore $f(x) \in -'T = -'\neg -' T \subset -'f(R)$, by Lemma 2.2. Thus $x \in -R$, and

$$\begin{aligned} x \in -R &\subset \neg R = \neg f^{-1}(\neg -' T) \subset f^{-1}(\neg\neg -' T) \subset f^{-1}(\neg\neg\neg T) \\ &= f^{-1}(\neg T) \subset f^{-1}(-'S). \end{aligned}$$

Conversely, suppose that $f : (X, \tau_{-}^{w}) \to (Y, \tau_{-'}^{w})$ is continuous, and let $f(x) \in -'f(S)$. Then, since $-'f(S) \in \tau_{-'}^{w}$, there exists $-T \in \tau_{-}^{w}$ such that $x \in -T \subset f^{-1}(-'f(S))$, and hence

$$x \in -T \subset f^{-1}(-'f(S)) \subset f^{-1}(\neg f(S)) = \neg f^{-1}(f(S)) \subset \neg S.$$

Therefore $x \in -T \subset -S$, by QA4. □

Let **Dob** denote a full subcategory of **Nbh** comprising the objects with decent open bases, and let **Wnn** denote a full subcategory of **Qap** comprising the objects that have the weak nested neighbourhodd property.

Theorem 5.14. *There exists an adjoint equivalence between* **Wnn** *and* **Dob**.

Proof. Define the functor $\mathbf{G}$ as before, and define the functor $\mathbf{F}_w$ from **Wnn** to **Dob** by $\mathbf{F}_w\langle X, -\rangle := (X, \tau_{-}^{w})$ and $\mathbf{F}_w f := f$. Then $\mathbf{G}$ and $\mathbf{F}_w$ are full and faithful functors, by Theorem 5.13 and Theorem 5.11.

Furthermore, using Proposition 3.2 and Proposition 5.10 we see that $\eta : \mathbf{1_{Wnn}} \to \mathbf{GF}_w$ and $\varepsilon : \mathbf{F}_w\mathbf{G} \to \mathbf{1_{Dob}}$ defined by the identity map as before are natural isomorphisms which satisfy $\varepsilon_{\mathbf{F}_w} \circ \mathbf{F}_w\eta = 1_{\mathbf{F}_w}$ and $\mathbf{G}\varepsilon \circ \eta_{\mathbf{G}} = 1_{\mathbf{G}}$. Hence $\langle \mathbf{F}_w, \mathbf{G}, \eta, \varepsilon\rangle$ forms an adjoint equivalence between **Wnn** and **Dob**. □

5.3 Separated spaces

The notion of separated space was introduced by Grayson [8, 3.2].

Definition 5.15. A neighbourhood space (X, τ) with an inequality $\neq$ is *separated* if it satisfies

SEP1. $\forall x \in X(\sim \{x\}$ is open),

SEP2. $\forall x \in X \forall U \in \tau[x \in U \implies \forall y(x \neq y \vee y \in U)]$.

The notion of apartness space [6, 7] is obtained by adding two axioms to those of a quasi-apartness space with an inequality.

Definition 5.16. A quasi-apartness space $\langle X, - \rangle$ with an inequality $\neq$ is an *apartness space* if it satisfies

A1. $x \neq y \implies x \in -\{y\}$,

A5. $x \in -S \implies \forall y \in X(x \neq y \vee y \in -S)$

Proposition 5.17. *Let (X, τ) be a separated neighbourhood space. Then $\langle X, -_\tau \rangle$ is an apartness space.*

Proof. (A1): Suppose that $x \neq y$. Then $y \in \sim \{x\}$, and therefore, since $\sim \{x\}$ is open, by SEP1, there exists $U \in \tau$ such that $y \in U \subset \sim \{x\}$. Hence $y \in -_\tau\{x\}$, by Lemma 5.3.

(A5): Suppose that $x \in -_\tau S$. Then there exists $U \in \tau$ such that $x \in U \subset \neg S$, and hence for each $y \in X$ either $x \neq y$ or $y \in U$, by SEP2. In the latter case, since $U \subset -_\tau S$, we have $y \in -_\tau S$. ☐

Proposition 5.18. *Let $\langle X, - \rangle$ be an apartness space. Then (X, τ_-^w) is separated.*

Proof. (SEP1): Suppose that $y \in \sim \{x\}$. Then $x \neq y$, and hence $y \in -\{x\} \subset \sim \{x\}$, by A1 and $\text{QA2}_{\neq}$. Therefore $\sim \{x\}$ is open in (X, τ_-^w).

(SEP2): Suppose that $x \in -S \in \tau_-^w$. Then for each $y \in X$ either $x \neq y$ or $y \in -S$, by A5. ☐

Corollary 5.19. *There exists an adjoint equivalence between the full subcategory of the separated spaces with decent open bases in* **Dob** *and the full subcategory of apartness spaces with the weak nested neighbourhood property in* **Wnn**.

Proof. Propositions 5.17 and 5.18 show that $\mathbf{G}$ and $\mathbf{F}_w$ defined above are functors between the separated spaces with decent open bases in **Dob** and the full subcategory of apartness spaces with the weak nested neighbourhood property in **Wnn**. ☐

6 Examples

Our set of axioms for a quasi-apartness space differs from that for a (point-set) apartness space inasmuch as there is nothing that parallels the axioms A1 and A5 from [6, 7]. (Axioms QA2 and QA3 coincide with the corresponding axioms A2 and A3, and QA4 is A4 for the denial inequality.) Accordingly, we require QA1 as the nullary version of QA3.

We include neither A1 nor A5 because A1 forces spaces to be T_1, and, as we shall see, A5 forces us to exclude the discrete neighbourhood space on the real line $\mathbf{R}$.

To start with A1, let $\Sigma := \{0, 1\}$ with $0 \neq 1$ be the Sierpinski space whose open base is $\sigma := \{\{0\}, \Sigma\}$; this neighbourhood space is T_0 but not T_1. The quasi-apartness $-_\sigma$ induced by σ does not satisfy A1. Indeed, since $0 \neq 1$, by A1 we would have $1 \in -_\sigma\{0\} = \emptyset$, which is impossible.

As for A5, let δ be the discrete neighbourhood space on a set X. (The discrete neighbourhood space has the singleton subsets of X as an open base.) Clearly, the quasi-apartness $-_\delta$ induced by δ is nothing but the logical complement: that is, $-_\delta S = \neg S$ for all $S \subset X$. If A5 holds for this quasi-apartness, and with $x \neq y := \neg(x = y)$, then

$$(*) \quad \forall x, y \in X(\neg(x = y) \vee \neg\neg(x = y)).$$

To see this, let $x, y \in X$, and set $S := \neg\{x\}$, for which $x \in -S = -\neg\{x\} = \neg\neg\{x\}$. By A5, we either have $x \neq y$ or $y \in -S$: that is, either $\neg(x = y)$ or $\neg\neg(x = y)$.

Consider the case that X is the set $\mathbf{R}$ of real numbers, and $\neq$ is the natural inequality on $\mathbf{R}$. In this case, since $x = y$ if and only if $\neg(x \neq y)$, we have $x = y$ if and only if $\neg\neg(x = y)$ for elements $x, y \in \mathbf{R}$. Then condition $(*)$ implies for all $x \in \mathbf{R}$, either $x = 0$ or $\neg(x = 0)$. But this is the weak limited principle of omniscience (WLPO) [5]. Since it is doubtful that we can achieve a constructive proof of WLPO, we cannot expect to find one of (A5) for $-_\delta$.

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