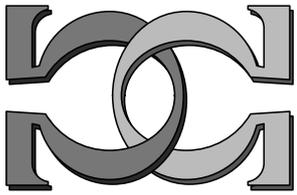


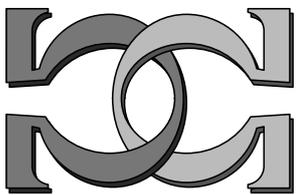
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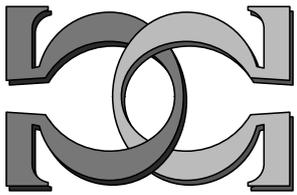
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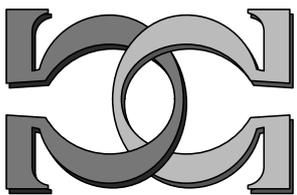


**Communication cost of breaking
the Bell barrier**



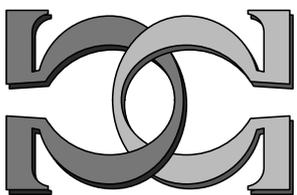
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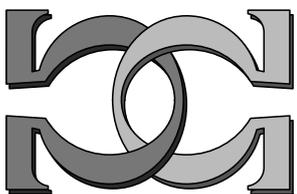


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Communication cost of breaking the Bell barrier

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Abstract

Adaptive as well as nonadaptive, memoryless protocols are presented which give rise to stronger than quantum correlations at the cost of the exchange of a single classical bit.

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From an operational point of view, the so-called nonlocal quantum correlations giving rise to violations of Bell-type inequalities amount to the fact that certain joint events occur with greater or smaller frequencies than can possibly be expected from classical, local realistic models. Two detectors at different locations register pairs of particles or particle properties more frequently or infrequently as can be explained by the usual classical assumption such as value definiteness.

With the rise of quantum algorithms and quantum information theory [1], the emphasis shifted to the communication cost and to the quantum communication complexity related to those quantum correlations. The question of the expense of obtaining quantum-type correlations from classical systems was stimulated by quantum [2] and classical [3–6] teleportation. In a recent Letter [7], Toner and Bacon, based on Refs. [4, 8], argue that classical systems could mimic quantum systems by reproducing the cosine law for correlation functions with the exchange of just one bit of classical information. (Classical noiseless correlation functions are linear.) They also raise the question whether or not their protocol is an indication of a deep structure in quantum correlations. In what follows this question will be answered to the negative by enumerating a protocol which yields stronger-than-quantum correlations with the exchange of a single classical bit.

Consider two correlated and spatially separated classical subsystems sharing common directions $\hat{\lambda}_i$, $i = 1, \dots$ which are chosen independently of each other and are distributed uniformly. All parameters $\hat{\lambda}_i$ are assumed to be identical on each one of the two subsystems. There are two measurement directions \hat{a} and \hat{b} of two dichotomic observables with values “-1” and “1” at two spatially separated locations. The measurement direction \hat{a} at “Alice’s location” is unknown to an observer “Bob” measuring \hat{b} and *vice versa*. A two-particle correlation function $E(\theta)$ with $\theta = \cos^{-1}(\hat{a} \cdot \hat{b})$ is defined by the summation of the product outcomes $O(\hat{a})_i, O(\hat{b})_i \in -1, 1$ in the i th experiment, divided by the number N of experiments for large N ; i.e., $E(\theta) = (1/N) \sum_{i=1}^N O(\hat{a})_i O(\hat{b})_i$.

We begin with a discussion of nonadaptive, memoryless protocols which could give rise to stronger-than-quantum correlations with the exchange of a single bit per experiment. Thereby, planar configurations will be considered. The protocols are similar to the one discussed by Toner and Bacon [7], but require only a single share $\hat{\lambda}$, and an additional direction $\hat{\Delta}(\delta)$, which is obtained by rotating $\hat{\lambda}$ clockwise around the origin by an angle δ which is a constant shift for all experiments. That is, suppose λ stands for the angle characterizing $\hat{\lambda}$ in polar coordinates, then $\hat{\Delta}(\delta) = (\cos(\lambda + \delta), \sin(\lambda + \delta))$ stands for the unit vector at angle $\lambda + \delta$. The bit communicated by Alice is given by $c(\delta) = \text{sgn}(\hat{a} \cdot \hat{\lambda}) \text{sgn}(\hat{a} \cdot \hat{\Delta}(\delta))$. Bob’s observable is defined by

$\beta(\delta) = \text{sgn}[\hat{b} \cdot (\hat{\lambda} + c(\delta)\hat{\Delta}(\delta))]$. This protocol reduces to Toner and Bacon's protocol if $\hat{\Delta}$ and thus δ is chosen uniformly and randomly over the entire circle.

The strongest correlations are obtained for $\delta = \pi/2$; i.e., in the case where the two directions $\hat{\lambda}$ and $\hat{\Delta}(\pi/2) = \hat{\lambda}^\perp$ are orthogonal and the information obtained by $c(\pi/2)$ is about the location of \hat{a} within two opposite quadrants. The effective shift in the parameter direction $\hat{\lambda} \pm \hat{\lambda}^\perp$ yields a correlation function of the form

$$E\left(\theta, \delta = \frac{\pi}{2}\right) = H\left(\theta - \frac{3\pi}{4}\right) - H\left(\frac{\pi}{4} - \theta\right) - 2\left(1 - \frac{2}{\pi}\theta\right)H\left(\theta - \frac{\pi}{4}\right)H\left(\frac{3\pi}{4} - \theta\right). \quad (1)$$

To obtain a better understanding for the shift mechanism, in Fig. 1 a configuration is drawn which, without the shift $\hat{\lambda} \rightarrow \hat{\lambda} - \hat{\lambda}^\perp$, $\text{sgn}(\hat{b} \cdot \hat{\lambda})$ would have contributed the factor -1 . The shift results in a positive contribution $\text{sgn}[\hat{b} \cdot (\hat{\lambda} - \hat{\lambda}^\perp)]$ to the expectation value. This shift mechanism always yield the strongest correlation ± 1 as long as the angle θ between \hat{a} and \hat{b} is not greater than $\pi/4$ or smaller than $3\pi/4$.

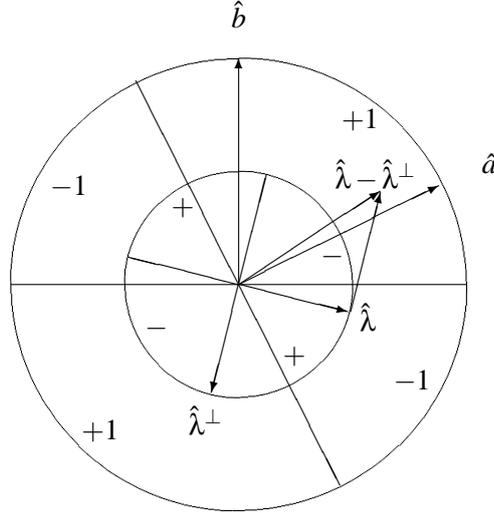


FIG. 1: Demonstration of the shift mechanism. Concentric circles represent the measurement directions \hat{a} and \hat{b} (outer circle), as well as $\hat{\lambda}$ and $\hat{\lambda}^\perp$ (inner circle) and their associated projective sign regions. The four measurement regions spanned by \hat{a} and \hat{b} are indicated by “ ± 1 ,” respectively. Positive and negative octants spanned by $\hat{\lambda}$ and $\hat{\lambda}^\perp$ are indicated in the inner circle by “ \pm ,” respectively. In this configuration, the shift $\hat{\lambda} \rightarrow \hat{\lambda} - \hat{\lambda}^\perp$ pushes $\hat{\lambda}$ into a positive region.

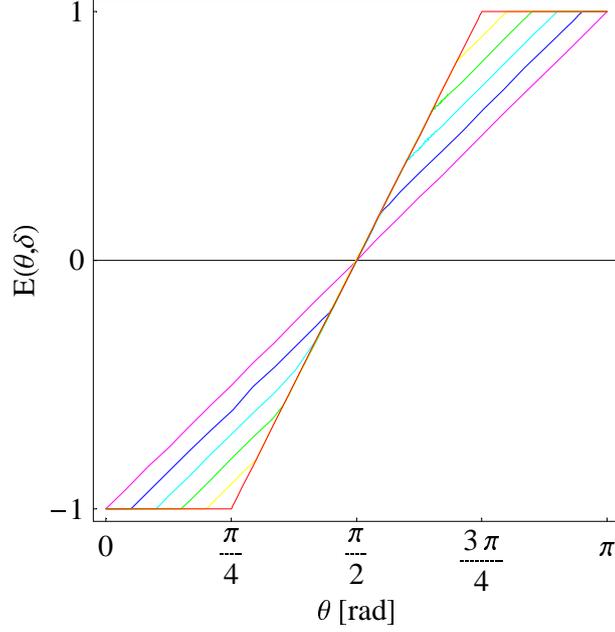


FIG. 2: Classical and stronger-than-quantum correlation functions obtained through the exchange of a single bit in the memoryless regime for values of $\delta \in \{0, \frac{\pi}{10}, \frac{\pi}{5}, \frac{3\pi}{10}, \frac{2\pi}{5}, \frac{\pi}{2}\}$ between 0 (straight line) and $\pi/2$ [cf. Eq (1)].

For general $0 \leq \delta \leq \pi/2$, Fig. 2 depicts numerical evaluations which fit the correlation function

$$E(\theta, \delta) = \begin{cases} -1 & \text{for } 0 \leq \theta \leq \frac{\delta}{2}, \\ -1 + \frac{2}{\pi}(\theta - \frac{\delta}{2}) & \text{for } \frac{\delta}{2} < \theta \leq \frac{1}{2}(\pi - \delta), \\ -2(1 - \frac{2}{\pi}\theta) & \text{for } \frac{1}{2}(\pi - \delta) < \theta \leq \frac{1}{2}(\pi + \delta), \\ 1 + \frac{2}{\pi}(\theta - \pi + \frac{\delta}{2}) & \text{for } \frac{1}{2}(\pi + \delta) < \theta \leq \pi - \frac{\delta}{2}, \\ 1 & \text{for } \pi - \frac{\delta}{2} < \theta \leq \pi. \end{cases} \quad (2)$$

Its domains are depicted in Fig. 3. For all nonzero δ , $E(\theta, \delta)$ correlates stronger than quantized systems for some values of θ . For $\delta = \pi/2$, the Clauser-Horne-Shimony-Holt (CHSH) inequality $|E(\hat{a}, \hat{b}) + E(\hat{a}, \hat{b}') + E(\hat{a}', \hat{b}) - E(\hat{a}', \hat{b}')| \leq 2$ for $\hat{a}(\pi), \hat{a}'(3\pi/4), \hat{b}(0), \hat{b}'(\pi/4)$ is violated by 3, a larger value than the Tsirelson bound for quantum violations $2\sqrt{2}$. This is due to the fact that, phenomenologically, the strategy allows for certain joint events to occur with greater or smaller frequencies as can possibly be expected from quantum entangled state measurements. For $\delta = 0$, the classical linear correlation function $E(\theta) = 2\theta/\pi - 1$ is recovered, as can be expected.

The average over all $0 \leq \delta \leq \pi/2$ yields a very similar, but not identical behaviour as the quantum cosine correlation function. More precisely, assume two independent, uniformly distributed shares $\hat{\lambda}_1$ and $\hat{\lambda}_2$, and a single communicated bit $H(c(\hat{\lambda}_1, \hat{\lambda}_2))$ with $c(\hat{\lambda}_1, \hat{\lambda}_2) = \text{sgn}(\hat{a} \cdot \hat{\lambda}_1) \text{sgn}(\hat{a} \cdot$

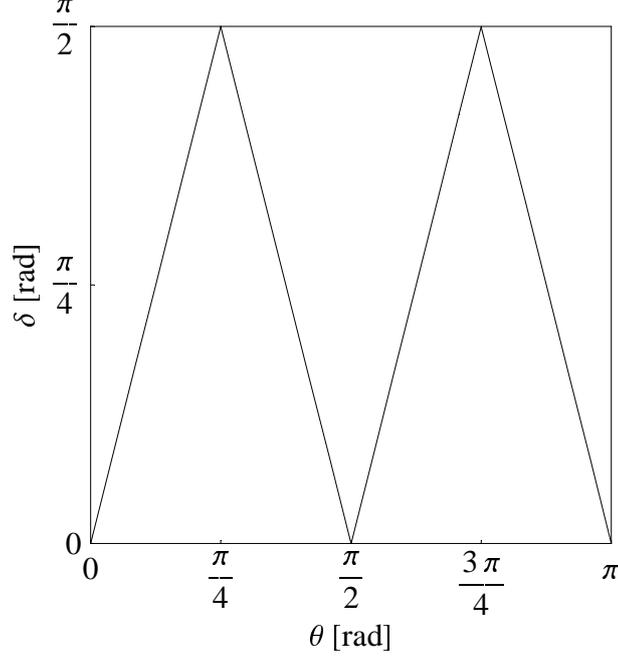


FIG. 3: Domains of the correlation function $E(\theta, \delta)$ of Eq. (2).

$\hat{\lambda}_2$) per measurement of $\text{sgn}(\hat{a} \cdot \hat{\lambda}_1) \text{sgn}[\hat{b} \cdot (\hat{\lambda}_2 - c\hat{\lambda}_1)]$. The associated correlation function is

$$\begin{aligned} E(\theta) &= \frac{1}{(2\pi)^2} \int d\hat{\lambda}_1 d\hat{\lambda}_2 \text{sgn}(\hat{a} \cdot \hat{\lambda}_1) \text{sgn}[\hat{b} \cdot (\hat{\lambda}_2 - c\hat{\lambda}_1)] \\ &= \frac{1}{2\pi^2} \int d\hat{\lambda}_1 \text{sgn}(\hat{a} \cdot \hat{\lambda}_1) \int d\hat{\lambda}_2 \text{sgn}[\hat{b} \cdot (\hat{\lambda}_2 - \hat{\lambda}_1)]. \end{aligned} \quad (3)$$

The elimination of c at the cost of the prefactor 2 on the right hand side of Eq. (3) is achieved by using the symmetries of the outcomes under the exchange $\hat{\lambda}_1 \longleftrightarrow -\hat{\lambda}_1$ and $\hat{\lambda}_2 \longleftrightarrow -\hat{\lambda}_2$ as outlined in Ref. [7]. Note that, although the correlation function $E(\theta)$ is nonlocal (Bob's output depends on c , which contains \hat{a}), after some recasting, despite the prefactor of 2 which accounts for nonlocality, it appears to be perfectly local, since it is the product of two sign functions containing merely \hat{a} and (separately) \hat{b} , respectively. (The two parameters $\hat{\lambda}_1, \hat{\lambda}_2$ are common shares.)

The $\hat{\lambda}_2$ integration can be performed by arranging \hat{b} along the positive y -axis $(0, 1)$ and by the parameterization $\hat{\lambda}_1 = (\sin t, \cos t)$ and $\hat{\lambda}_2 = (\sin \tau, \cos \tau)$. The positive contributions amount to $A_+ = 2 \int_0^t d\tau = 2t$; thus the negative contributions are $A_- = 2\pi - A_+$, and the entire integral is $A_+ - A_- = 2t/\pi - 1 = 2 \cos^{-1}(\hat{b} \cdot \hat{\lambda}_1)/\pi - 1$.

For the $\hat{\lambda}_1$ integration, \hat{a} is arranged along the positive y -axis $(0, 1)$, \hat{b} along $(\sin r, \cos r)$, and $\hat{\lambda}_1$ along $(\sin \tau, \cos \tau)$. Since $\cos^{-1}(\cos(x)) = |x|$ for $-1 \leq x \leq 1$, one obtains for the correlation value $(4/\pi^2) \int_0^\pi d\tau \text{sgn}(\cos \tau) |\tau - r|$. After evaluating all cases, the $\hat{\lambda}_1$ integration, for $0 \leq \theta =$

$\cos^{-1}(\hat{a} \cdot \hat{b}) \leq \pi$, yields

$$E(\theta) = \frac{4}{\pi^2} \left[\left(\theta^2 - \frac{\pi^2}{4} \right) - 2H \left(\theta - \frac{\pi}{2} \right) \left(\theta - \frac{\pi}{2} \right)^2 \right], \quad (4)$$

which is plotted in Fig. 4. An alternative derivation of Eq. (4) is via $(2/\pi) \int_0^{\pi/2} d\delta E(\theta, \delta)$ with

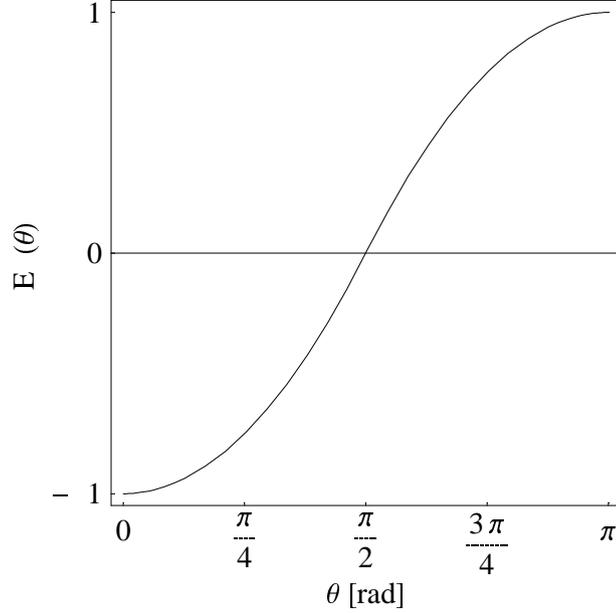


FIG. 4: The correlation $E(\theta)$ of Eq. (4) as a function of $\theta = \cos^{-1}(\hat{a} \cdot \hat{b})$ for the memoryless exchange of a single bit per experiment in the planar configuration. Note that, although the shape resembles the quantum cosine law, the function is piecewise quadratic.

$E(\theta, \delta)$ from Eq. (2).

Let us next enumerate a protocol requiring memory which, by the exchange of more than one bit, could give rise to maximal violations [9] of the CHSH inequality. The protocol is based on locating and communicating information about Alice's measurement direction \hat{a} to Bob, who then rotates his subsystem (or alternatively his measurement direction) so as to obtain the desired correlation function. It uses a single planar direction $\hat{\lambda}$ which could be interpreted as a parameter, and a binary search algorithm.

The first bit $c_1 = H[\text{sgn}(\hat{a} \cdot \hat{\lambda})]$ characterizes the location of \hat{a} within the halfspaces defined by the subspace orthogonal to $\hat{\lambda}$; e.g., whether $\text{sgn}(\hat{a} \cdot \hat{\lambda})$ is positive or negative. (H stands for the Heaviside unit step function.) The second bit $c_2 = H\left(\text{sgn}(\hat{a} \cdot \hat{\lambda})\text{sgn}[\hat{a} \cdot \hat{\Delta}(\pi/2)]\right)$ characterizes the location of \hat{a} within two opposite quadrants spanned by $\hat{\lambda}$ and the subspace orthogonal to $\hat{\lambda}$.

The third bit of information is about the location of \hat{a} with respect to the octants formed by dividing the quadrants used to determine the second bit, and so on. In general, the n th bit

$$c_n = H \left(\prod_{i=0}^{n-1} \text{sgn}[\hat{a} \cdot \hat{\Delta}((\pi/2)^i)] \right) \quad (5)$$

stands for an equipartition of the sphere into 2^n similar slices, which are associated alternatively with the two bit values. If the protocol stops after communicating the n th bit, then by convention \hat{a} is identified with the direction which is in the middle of the section sorted out by the previous segments.

The communicated information determines Alice's measurement direction \hat{a} for Bob with arbitrary precision. In order to obtain a stronger-than-classical correlation, Bob can use the information about the location \hat{a} by rotating his subsystem to compensate for the difference between his measurement direction \hat{b} and \hat{a} . In that way, either perfect correlation or perfect anticorrelation can be obtained, as Bob rotates his subsystem by either $\delta = \alpha - \beta$ or $\delta + \pi$, respectively. This yields an effective shift of $\hat{\lambda}$ by the opposite amount.

The number of required initial experiments increases proportional to any specified finite precision of the correlation function. Yet, any finite sequence of bits contributes little to the entire correlation function E if it is part of large sequence of $M + N$ outcomes in an experiment to determine E : The relative cost $M/(M + N)$ decreases as N increases.

Another, rather obvious, adaptive strategy requiring memory is to communicate Alice's outcome to Bob. The individual outcome is just an indication of whether Alice's measurement direction is located "above" or below the orthogonal subspace of the direction $\hat{\lambda}$, which is shared both by Alice and Bob. As more bits from different $\hat{\lambda}$ directions are communicated, the direction of \hat{a} can be inferred with increasing precision.

When compared to the protocol discussed by Toner and Bacon [7] or to the memoryless protocol introduced above, the adaptive protocols share some similarities. Both exchange very similar information, as can for instance be seen by a comparison between c_2 above and the bit $c = \text{sgn}(\hat{a} \cdot \hat{\lambda}_1) \text{sgn}(\hat{a} \cdot \hat{\lambda}_2)$ exchanged in the Toner and Bacon protocol. After the exchange of just a few bits, the adaptive protocols appear to be more efficient from the communication complexity point of view. However, these strategies require memory. Operationally, this presents no problem for Alice and Bob, but if one insists on nonadaptive, single particle strategies, these protocols must be excluded.

In summary we have found that, as long as adaptive protocols requiring memory at the re-

ceiver side are allowed, the CHSH inequality can be violated maximally. Furthermore, we have presented a type of memoryless, nonadaptive protocol giving rise to stronger-than-quantum correlations which does not yield maximal violations of the CHSH inequality but rather violates it by 3, as compared to the quantum Tsirelson bound $2\sqrt{2}$. We have thus solved a problem posed by Toner and Bacon, whether the exchange of a single classical bit offers an intriguing glimpse into the nature of correlations produced in quantum theory by enumerating protocols with stronger-than-quantum correlation functions at the cost of a single bit. The question still remains open whether memoryless single bit exchange protocols exist which violate the CHSH inequality maximally; i.e., by 4. Another open question is the effect of entangled quantum subsystems.

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- [1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
 - [2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, *Physical Review Letters* **70**, 1895 (1993), URL <http://dx.doi.org/10.1103/PhysRevLett.70.1895>.
 - [3] G. Brassard, R. Cleve, and A. Tapp, *Physical Review Letters* **83**, 1874 (1999), URL <http://dx.doi.org/10.1103/PhysRevLett.83.1874>.
 - [4] N. J. Cerf, N. Gisin, and S. Massar, *Physical Review Letters* **84**, 2521 (2000), URL <http://dx.doi.org/10.1103/PhysRevLett.84.2521>.
 - [5] N. J. Cerf, N. Gisin, S. Massar, and S. Popescu (2004), [quant-ph/0410027](http://arxiv.org/abs/quant-ph/0410027), URL <http://arxiv.org/abs/quant-ph/0410027>.
 - [6] N. Brunner, N. Gisin, and V. Scarani (2004), [quant-ph/0412109](http://arxiv.org/abs/quant-ph/0412109), URL <http://arxiv.org/abs/quant-ph/0412109>.
 - [7] B. F. Toner and D. Bacon, *Physical Review Letters* **91**, 187904 (2003), URL <http://dx.doi.org/10.1103/PhysRevLett.91.187904>.
 - [8] K. H. Schatten, *Physical Review A* **48**, 103 (1993), URL <http://dx.doi.org/10.1103/PhysRevA.48.103>.
 - [9] G. Krenn and K. Svozil, *Foundations of Physics* **28**, 971 (1998).