CDMTCS Research Report Series

Fast Generation of Fibonacci Permutations

A. Juarna, V. Vajnovszki

Université de Bourgone, Dijon, France

CDMTCS-242
July 2004

Centre for Discrete Mathematics and
Theoretical Computer Science

Fast Generation of Fibonacci Permutations

Asep JUARNA, Vincent VAJNOVSZKI
LE2I - UMR CNRS, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France

July 10, 2004

Abstract

In 1985, Simion and Schmidt showed that $|S_n(\tau_3)|$, the cardinality of the set of all length n permutations avoiding the patterns $\tau_3 = \{123, 213, 132\}$ is the Fibonacci numbers, f_{n+1}. They also developed a constructive bijection between the set of all binary strings with no *two* consecutive ones and $S_n(\tau_3)$. In May 2004, Egge and Mansour generalized this Simion-Schmidt counting result and showed that, $S_n(\tau_p)$, the set of permutations avoiding the patterns $\tau_p = \{12...p, 213, 132\}$ is counted by the $(p-1)$-generalized Fibonacci numbers, $f_{n+1}^{(p-1)}$.

The Simion-Schmidt's set of binary strings is $F_{n-1}^{(2)}$, the well known Fibonacci strings, which is a special case of $F_n^{(p)}$, p-generalized Fibonacci strings having no $p-1$ consecutive ones. In May 2001, Vajnovszki proposed a loopless algorithm for generating $\mathcal{F}_n^{(p)}$, a Gray code for $F_n^{(p)}$. This algorithm has a constant worst case time while the Hamming distance between any two consecutive strings in $\mathcal{F}_n^{(p)}$ is one.

In this paper we formalize and generalize the Simion-Schmidt bijection so that the new bijection now is between $F_{n-1}^{(p-1)}$ and $S_n(\tau_p)$, and we show that, $\mathcal{S}_n(\tau_p)$, the image of the ordered list $\mathcal{F}_n^{(p)}$ through this generalized bijection is a list for all length n permutations avoiding the patterns $\tau_p = \{12...p, 213, 132\}$ with the Hamming distance between any two consecutive permutations bounded by $(p-1)$, and so a Gray code for $S_n(\tau_p)$. We also propose a loopless algorithm, which is a modification of Vajnovszki's algorithm, and which generates $\mathcal{S}_n(\tau_p)$ also in constant worst case time.

Keywords: Pattern(s) avoiding permutations, Fibonacci numbers, generalized Fibonacci strings, Gray codes, continuous discrete bijections.

1 Introduction and Motivation

Let S_ℓ be the set of all permutations of $\{1, 2, ..., \ell\}$. Let $\pi \in S_n$ and $\tau \in S_k$ be two permutations, $k \leq n$. We say that π *contains* τ if there exists a subsequence $1 \leq i_1 < i_2 < ... < i_k \leq n$ such that $(\pi_{i_1}, ..., \pi_{i_k})$ is order-isomorphic to τ; in such context τ is usually called a *pattern*. We say that π *avoids* τ, or is τ-*avoiding*, if such a subsequence does not exist. The set of all τ-avoiding permutations in S_n is denoted $S_n(\tau)$ and $|S_n(\tau)|$ denotes its cardinality. For an arbitrary finite collection of patterns T, we say that π avoids T if π avoids any $\tau \in T$; the corresponding subset of S_n is denoted $S_n(T)$ and $|S_n(T)|$ denotes its cardinality.

The fundamental questions of the problems of pattern avoiding permutations are to determine $|S_n(T)|$, $T = \{\tau_1, \tau_2, ..., \tau_t\}$, viewed as a function of n for given T, to find an explicit bijection (a one-to-one correspondence) between $S_n(T)$ and $S_n(T')$ if $|S_n(T)| = |S_n(T')|$, and to find relations between $S_n(T)$ and other combinatorial structures. By determining $|S_n(T)|$ we mean finding explicit formula, or ordinary or exponential generating functions. From these researches, a number of enumerative results have been proved, new bijections found, and connections to other fields established.

The problems of pattern avoiding permutations were appeared since Knuth [1], in his text book, posed the problem of sorting with single stack. The problem actually was the 312-patterns avoiding permutations. In the other section of his book, he showed that the cardinality of all three lengths patterns avoiding permutations is the Catalan numbers. The investigations of the problems of pattern avoiding permutations then become wider to some set of patterns of three, four, five, and so on, some combinations of these patterns, generalized patterns, and permutations avoiding some patterns while in the same time containing exactly a numbers of other patterns.

Pattern avoiding permutations has proved to be useful language in a variety of seemingly unrelated problems, from theory of Kazhdan-Lusztig polynomials, to singularities of Schubert varieties, to Chebyshev polynomials, to rook polynomials for a rectangular board, to various sorting algorithms, sorting stacks and sortable permutations [2].

The first systematic study of patterns avoiding permutations was been undertaken in 1985 when Simion and Schmidt [3] solved the problem for every subset of S_3. Two of their propositions are:

1. For every $n \geq 1$, $|S_n(123, 213, 312)| = f_{n+1}$, where $\{f_n\}_{n\geq 1}$ is Fibonacci numbers, initialized by $f_0 = 1$, $f_1 = 1$.

2. For each $n \geq 1$, there is a constructive bijections between $S_n(123, 213,\ 312)$ and the set of binary strings of length $n - 1$ having no *two* consecutive ones.

In May 2004, Egge and Mansour [4] generalized these results and showed that for all integers n and $p \geq 2$, $|S_n(12...p, 132, 213)| = f^{(p-1)}_{n+1}$, where $\{f^{(p)}_n\}_{n\geq 1}$ is p-generalized Fibonacci numbers. The patterns $\{123, 213, 312\}$ is a special case of class of pattern $\{12...p, 213, 312\}$.

The set of binary strings that was meant by Simion-Schmidt is $F^{(2)}_{n-1}$, the well known Fibonacci strings. This set is a special case of $F^{(p)}_n$, *the set of n length binary strings with no p consecutive ones.* In May 2001 Vajnovszki [5], in his paper, proposed a loopless algorithm in generation $\mathcal{F}^{(p)}_n$, a Gray code for $F^{(p)}_n$ where the Hamming distance between two consecutive elements of this list is one.

In this paper we formalize and generalize the Simion-Schmidt bijection. With the formalization we mean that we construct new bijection between two lists:

$\mathcal{F}^{(2)}_{n-1}$ and $\mathcal{S}_n(123, 213, 312)$, that is, Gray codes for $F^{(2)}_{n-1}$ and $S_n(123, 213, 312)$, respectively.

The structure of this paper is follow. After this introduction section, Section 2 will review the Vajnovszki results about Fibonacci strings and its Gray code. Furthermore, Section 3 will present the construction of the list $\mathcal{S}_n(123, 213, 312)$ from list $\mathcal{F}^{(2)}_{n-1}$ and followed by Section 4 which will discuss about construction of recursive formulae of generation $\mathcal{S}_n(123, 213, 312)$ and computation of its Hamming distance. Section 5 then will show results concerning extension to general p.

2 Fibonacci Strings and Its Gray Code

The set $F_n^{(p)}$ of length n binary strings with no two consecutive ones is defined as follows [5]:

$$F_n^{(p)} = \begin{cases} \{\lambda\} & n = 0 \\ \{0,1\}^n & 1 \le n \le p \\ 0 \cdot F_{n-1}^{(p)} \cup 10 \cdot F_{n-2}^{(p)} \cup ... \cup 1^{p-1}0 \cdot F_{n-p}^{(p)} & n \ge p \end{cases} \quad (1)$$

where λ is the empty string, and for arbitrary string α and set of strings F we mean $\alpha \cdot F$ as concatenation of α to each string of F. It is easy to show that:

$$card(F_n^{(p)}) = f_{n+p}^{(p)} \quad (2)$$

where $f_n^{(p)}$ is the pth order n-th Fibonacci numbers defined by:

$$f_n^{(p)} = \begin{cases} 0 & 0 \le n \le p-1 \\ 1 & n = p-1 \\ \sum_{j=n-p}^{n-1} f_j^{(p)} & n \ge p \end{cases} \quad (3)$$

So $card(F_n^{(p)}) = f_{n+p}^{(p)}$ is p-right shifting of $f_n^{(p)}$. For $p = 2$ the relation (3) gives the well-known Fibonacci sequence $f_n^{(2)}$, which is usually written as f_n.

A Gray code for a combinatorial family is a listing of objects in the family so that successive objects differ in some pre-specified, usually small, way [6]. A Gray code list of Fibonacci strings defined by (1) is [5]:

$$\mathcal{F}_n^{(p)} = \begin{cases} \lambda & n = 0 \\ 0, 1 & n = 1 \\ 0 \cdot \overline{\mathcal{F}}_{n-1}^{(p)} \circ 10 \cdot \overline{\mathcal{F}}_{n-2}^{(p)} \circ ... \circ 1^{p-1}0 \cdot \overline{\mathcal{F}}_{n-p}^{(p)} & n > 1 \end{cases} \quad (4)$$

with $\circ$ is the operator of concatenation of two lists and $\overline{\mathcal{F}}$ is the list obtained by reversing $\mathcal{F}$. In [5] is shown that the Hamming distance between any two consecutive elements of its list equals to 1 for all n and for all p.

When $p = 2$ equation (4) is reduced to:

$$\mathcal{F}_n^{(2)} = \begin{cases} \lambda & n = 0 \\ 0, 1 & n = 1 \\ 0 \cdot \overline{\mathcal{F}}_{n-1}^{(2)} \circ 10 \cdot \overline{\mathcal{F}}_{n-2}^{(2)} & n > 1 \end{cases} \quad (5)$$

Example. Following are three examples of set of n length binary string with no *two* consecutive ones: $\mathcal{F}_3^{(2)} = \{010, 000, 001, 101, 100\}$, $\mathcal{F}_4^{(2)} = \{0100, 0101, 0001, 0000, 0010, 1010, 1000, 1001\}$, and $\mathcal{F}_5^{(2)} = \{01001, 01000, 01010, 00010\ 00000, 00001, 00101, 00100, 10100, 10101, 10001, 10000, 10010\}$

3 Construction of $\mathcal{S}_n(123, 132, 213)$

Simion and Schmidt showed that the cardinality set of all length n permutations avoiding the patterns $\tau_3 = \{123, 213, 132\}$ is the Fibonacci numbers,

f_{n+1}, where $\{f_n\}_{n \ge 1}$ is the sequence of Fibonacci numbers as defined by (3) for $p = 2$. Here is their nice proof. Let $\pi \in S_n$ be such a permutation and π^{-1} its inverse, that is $\pi^{-1}(\pi(i)) = i$.

Table 1: Generation $\mathcal{S}_6(123,132,213)$ from $\mathcal{F}_5^{(2)}$

rank	$\mathcal{F}_5^{(2)}$	$\mathcal{S}_6(123,132,213)$
1	01001	645312
2	01000	645321
3	01010	645231
4	00010	654231
5	00000	654321
6	00001	654312
7	00101	653412
8	00100	653421
9	10100	563421
10	10101	563412
11	10001	564321
12	10000	564321
13	10010	564231

It is trivial to verify the result for $n \leq 2$, that is $|S_1(123,132,213)| = 1$, namely permutation 1, and $|S_2(123,132,213)| = 2$, namely permutation 12 and 21. If $n \geq 3$, then $\pi^{-1}(n) \leq 2$, else either 123 or 213 could not be avoided. If $\pi(1) = n$, then $(\pi(2),...,\pi(n)) \in S_{n-1}$ with the same restrictions. If $\pi(2) = n$, then we must have $\pi(1) = n-1$, else 132 could not be avoided; thus$(\pi(3),...,\pi(n)) \in S_{n-2}$. Hence, for $n \geq 3$, $|S_n| = |S_{n-1}| + |S_{n-2}|$; this recurrence relation is satisfied by Fibonacci numbers of (3) for $p = 2$.

Simion and Schmidt also showed that for each $n \geq 1$, there exists a constructive bijection between the set of binary strings of length $n-1$ having no *two* consecutive ones and $S_n(123,132,213)$. Their proof is as follows. Let $s = s_1s_2...s_{n-1}$ be such a binary string. We construct its corresponding $\pi \in S_n(123,132,213)$ by determining $\pi(i)$, $1 \leq i < n$, as follows: if $X_i = \{1,2,...,n\} - \{\pi(1),...,\pi(i-1)\}$ then set

$$\pi(i) = \begin{cases} \text{largest element in } X_i \text{ if } s_i = 0 \\ \text{second largest element in } X_i \text{ if } s_i = 1 \end{cases} \tag{6}$$

Finally, $\pi(n)$ is the unique element in X_n.

The following table gives the list $\mathcal{S}_6(123,132,213)$ that are generated from $\mathcal{F}_5^{(2)}$ using (6), where $\mathcal{F}_5^{(2)}$ is recursively generated by (5).

4 Recursive Definition and Hamming Distance of $\mathcal{S}_n(123,213,132)$

Let $\tau_3 = \{123,\ 213,\ 312\}$. As we have seen in Section 3, $\mathcal{S}_n(\tau_3)$ is generated from $\mathcal{F}_{n-1}^{(2)}$, defined by (5). By considering of Simion-Schmidt bijection, the Gary code of Fibonacci strings $\mathcal{F}_n^{(2)}$ in (5) is transformed into the following Gray code for τ_3-avoiding permutations:

$$\mathcal{S}_n(\tau_3) = \begin{cases} \{1\} & n = 1 \\ \{21, 12\} & n = 2 \\ n \cdot \overline{\mathcal{S}}_{n-1}(\tau_3) \circ (n-1) \cdot n \cdot \overline{\mathcal{S}}_{n-2}(\tau_3) & n > 2 \end{cases} \tag{7}$$

where $\overline{\mathcal{S}}_n(\tau_3)$ is the list obtained by reversing

$\mathcal{S}_n(\tau_3)$. With equation (7), we can generate $\mathcal{S}_n(\tau_3)$ for

some n as follows: $\mathcal{S}_1(\tau_3)$ = {1}, $\mathcal{S}_2(\tau_3)$ = {21, 12}, $\mathcal{S}_3(\tau_3)$ = {312, 321, 231}, $\mathcal{S}_4(\tau_3)$ = {4231, 4321, 4312, 3412}, $\mathcal{S}_5(\tau_3)$ = {53421, 53412, 54312, 54321, 54231, 45231, 45321, 45312}, $\mathcal{S}_6(\tau_3)$ as is shown in the Table 1.

Proposition 1. *The Hamming distance between any two consecutive elements of $\mathcal{S}_n(\tau_3)$, which is constructed by Simion-Schmidt bijection, is* 2.

Proof: From (7), we can see that one of the basis of recursion is generation of permutations 21 and 12, the two elements having Hamming distance 2. Each two adjacent elements of $\overline{\mathcal{S}}_{n-2}$ (and also for $\overline{\mathcal{S}}_{n-1}$) have Hamming distance 2, provided Hamming distance between $last$ and $first$ of its subsets of $\overline{\mathcal{S}}_{n-2}$

(and also for $\overline{\mathcal{S}}_{n-1}$) have Hamming distance 2, where for a list $\mathcal{L} = x_1, x_2, ..., x_n$, $first(\mathcal{L}) = x_1$ and $last(\mathcal{L}) = x_n$. For $\mathcal{S}_n(\tau_3)$, here are the verification of Hamming distance between the $last(n \cdot \overline{\mathcal{S}}_{n-1}(\tau_3))$ and $first((n-1) \cdot n \cdot \overline{\mathcal{S}}_{n-2}(\tau_3))$. We omitted '$(\tau_3)$' in this proof.

$$\begin{aligned}
& last(n \cdot \overline{\mathcal{S}}_{n-1}) = n \cdot last(\overline{\mathcal{S}}_{n-1}) \\
& = n \cdot last(\overline{(n-1) \cdot \overline{\mathcal{S}}_{n-2} \circ (n-2) \cdot (n-1) \cdot \overline{\mathcal{S}}_{n-3}}) \\
& = n \cdot first((n-1) \cdot \overline{\mathcal{S}}_{n-2} \circ (n-2) \cdot (n-1) \cdot \overline{\mathcal{S}}_{n-3}) \\
& = n \cdot (n-1) \cdot first(\overline{\mathcal{S}}_{n-2})
\end{aligned}$$

$$first((n-1) \cdot n \cdot \overline{\mathcal{S}}_{n-2} = (n-1) \cdot n \cdot first(\overline{\mathcal{S}}_{n-2})$$

It is clear that Hamming distance between $last(n \cdot \overline{\mathcal{S}}_{n-1})$ and $first((n-1) \cdot n \cdot \overline{\mathcal{S}}_{n-2}$ is 2. $\diamondsuit$

5 Extension to General p

Refer to Egge-Mansour results, the set of patterns τ_3 = {123, 132, 213} is a special case of τ_p = {12...p, 213, 132} for $p = 3$. In the Sections 3 and 4 we have established the construction of $\mathcal{S}_n(\tau_3)$ from $\mathcal{F}^{(2)}_{n-1}$, together with its Hamming distance value, $p - 1 = 2$. In this section we will show that the construction procedure also holds for $p > 3$, but, by considering (4), we will show that the Hamming distance is bounded by $p - 1$, which is a constant for fixed p.

For pattern τ_p, the extended Simion-Schmidt bijection is:

$$\pi(i) = \begin{cases} \text{largest element in } X_i \text{ if } s_i = 0 \\ 2^{nd} \text{ largest element in } X_i \text{ if } s_i = 1 \text{ and} \\ \quad \text{(either } s_{i+1} = 0 \text{ or } i = n-1) \\ 3^{rd} \text{ largest element in } X_i \text{ if } s_i = s_{i+1} = 1 \text{ and} \\ \quad \text{(either } s_{i+2} = 0 \text{ or } i+1 = n-1) \\ \vdots \\ (p-2)^{th} \text{ largest element in } X_i \text{ if } s_i = s_{i+1} = ... = s_{i+p-4} \\ \quad = 1 \text{ and (either } s_{i+p-3} = 0 \text{ or } i+p-3 = n-1) \\ (p-1)^{th} \text{ largest element in } X_i \text{ if } s_i = s_{i+1} = ... = s_{i+p-3} \\ \quad = 1 \end{cases} \tag{8}$$

and $\pi(n)$ is the unique element in X_n. This bijection relates *the set of binary strings of length* $n-1$ *having no* $p-1$ *consecutive ones* with $S_n(\tau_4)$. Our formalized bijection then relates $\mathcal{F}_{n-1}^{(p-1)}$ to

$$\mathcal{S}_n(\tau_p) = \begin{cases} 1 & n = 1 \\ \{21, 12\} & n = 2 \\ n \cdot \overline{\mathcal{S}}_{n-1}(\tau_p) \circ (n-1) \cdot n \cdot \overline{\mathcal{S}}_{n-2}(\tau_p) \circ ... & \\ \circ (n-p+1) \cdot (n-p+2) \cdot ... \cdot n \cdot \overline{\mathcal{S}}_{n-p+1}(\tau_p) & n > 2 \end{cases} \tag{9}$$

Equation (9) lead us to the following lemma:

Lemma. *The Hamming distance between any two consecutive elements of $\mathcal{S}_n(\tau_p)$ that is constructed by Simion-Schmidt bijection is upper bounded to (p-1).*
Proof: The Hamming distance between $last$ and $first$ of the first two subsets of $\mathcal{S}_n(\tau_p)$ is 2 for $p = 3$, as we have shown in the Section 4. The upper bound of the distance for $p = 4$ also have been shown in Section 5. The Hamming distance between $last$ and $first$ of the last two subsets of $\mathcal{S}_n(\tau_p)$ is verified as follows (we omitted '(τ_p)'):

$$\begin{aligned} & last((n-p+1) \cdot (n-p+4) \cdot ... \cdot n \cdot \overline{\mathcal{S}}_{n-p+2}) \\ &= (n-p+1) \cdot (n-p+4) \cdot ... \cdot n \cdot last(\overline{\mathcal{S}}_{n+2}) \\ &= (n-p+1) \cdot (n-p+4) \cdot ... \cdot n \cdot \\ &\quad last(\overline{(n-p+2) \cdot \overline{\mathcal{S}}_{n-p+1} \circ (n-p+2) \cdot (n-p+2) \cdot \overline{\mathcal{S}}_{n-p}}) \\ &= (n-p+1) \cdot (n-p+4) \cdot ... \cdot n \cdot \\ &\quad first((n-p+2) \cdot \overline{\mathcal{S}}_{n-p+1} \circ (n-p+2) \cdot (n-p+2) \cdot \overline{\mathcal{S}}_{n-p}) \\ &= (n-p+1) \cdot (n-p+4) \cdot ... \cdot n \cdot first((n-p+2) \cdot \overline{\mathcal{S}}_{n-p+1}) \\ &= (n-p+1) \cdot (n-p+4) \cdot ... \cdot n \cdot (n-p+2) \cdot first(\overline{\mathcal{S}}_{n-p+1}) \end{aligned}$$

$$\begin{aligned} & first((n-p+2) \cdot (n-p+3) \cdot ... \cdot n \cdot \overline{\mathcal{S}}_{n-p+1}) \\ &= (n-p+2) \cdot (n-p+3) \cdot ... \cdot n \cdot first(\overline{\mathcal{S}}_{n-p+1}) \end{aligned}$$

Since the length of $(n-p+2) \cdot (n-p+3) \cdot ... \cdot n$ is $p-1$, the upper bound of Hamming distance between $last$ and $first$ of the two subset of $\mathcal{S}_n(\tau_p)$ is $p-1$. $\diamondsuit$

Table 2 shows $\mathcal{S}_5(\tau_4)$ and Figure 1 illustrates how the permutation $5674312 \in S_8^4$ is generated from the binary string $110001 \in F_7^3$.

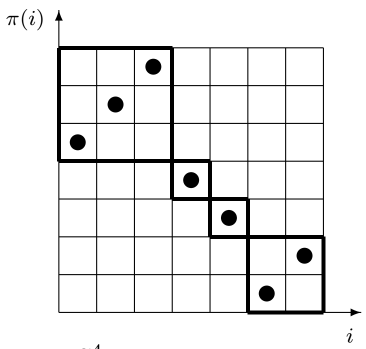

Figure 1: The permutation $5674312 \in S_8^4$ corresponding to the binary string $110001 \in F_7^3$.

Table 2: Generation $\mathcal{S}_5(1234, 132, 213)$ from $\mathcal{F}_4^{(3)}$

rank	$\mathcal{F}_4^{(3)}$	$\mathcal{S}_5(1234, 132, 213)$
1	0110	52341
2	0100	53421
3	0101	53412
4	0001	54312
5	0000	54321
6	0010	54231
7	0011	54123
8	1011	54123
9	1010	45231
10	1000	45321
11	1001	45312
12	1101	34512
13	1100	34521

6 Algorithmic Consideration

In Vajnovszki [5] is given a generating algorithm for the set $F_n^{(p)}$. We will use this algorithm for constructing $\mathcal{S}_n(\tau_p)$ from $\mathcal{F}_{n-1}^{(p-1)}$. Let $0 < s \leq i \leq t < n$, and let $x, x' \in \mathcal{F}_{n-1}^{(p-1)}$ are contiguous. x and x' differs in position i while the their others digits along position s to t are 1's. The Vajnovszki algorithm take a constant delay time for determining the position i.

The binary strings x and its successor x' however are

$$x = x_1 x_2 \dots x_{s-2} 0 \underbrace{1 \dots 1}_{i-s} x_i \underbrace{1 \dots 1}_{t-i-1} 0 x_{t+1} \dots x_n$$
$$x' = x_1 x_2 \dots x_{s-2} 0 \underbrace{1 \dots 1}_{i-s} x'_i \underbrace{1 \dots 1}_{t-i-1} 0 x_{t+1} \dots x_n$$

of course with $x'_i = 1 - x_i$ and $x_s \dots x_{i-1}$ and $x_{i+1} \dots x_{t-1}$ are, eventually empty, contiguous sequences of 1s. Obviously, the shape of π and π', the images of x and x' through the bijection in relation (11), are

$$\pi \quad = \pi(1)\pi(2) \dots \pi(s-1)\pi(s) \dots \pi(i) \dots \pi(t)\pi(t+1) \dots \pi(n)$$
$$\pi \quad = \pi(1)\pi(2) \dots \pi(s-1)\pi'(s) \dots \pi'(i) \dots \pi'(t)\pi(t+1) \dots \pi(n)$$

and only elements between position s and t are different in π and π'. In addition, $t - s \leq p$ since x and x' do not contain a sequence of

consecutive 1s of length equal or larger than p. Thus, we can find from i, the positions s and t and then update π between these positions in $O(p)$ time.

To resume, a slight modification of the algorithm [5] produces a generating algorithm for the set $S_n(\tau_p)$ which runs in $O(p)$ time between any two consecutive permutations.

7 Final Remark

This paper proves also, that the inverse of the bijection defined by relation (8) from $S_n(\tau_p)$ to $F_n^{(p)}$ is a continuous bijection, if $S_n(\tau_p)$ and $F_n^{(p)}$ are equipped with the discrete topology.

$S_n(\tau_p)$ is a restriction of the set, S_n, of all permutations of $\{1, 2, ..., n\}$, and $\mathcal{S}_n(\tau_p)$ is constructed from $\mathcal{F}_n^{(p)}$, a set of bitstrings that has Hamming distance 1 for all n and all p. From this facts, in the beginning of our investigation about this paper, we were hoped that $\mathcal{S}_n(\tau_p)$ can be generated with some variant of the Johnson-Trotter algorithm for S_n, which Hamming distance between any two consecutive elements is 2. If there exists any algorithm that can generate $S_n(\tau_p)$ such that the Hamming distance between any two consecutive elements is better than upper bounded to $(p-1)$ remains an open problem.

References

[1] Donald E. Knuth. *The Art of Programming, Vol. 1 Fundamental Algorithms, Section 2.2.1.*, Addison Wesley, Reading Massachusetts, 1973.

[2] Sergey Kitaev and Toufik Mansour. *A Survey on Certain Pattern Problems.* www.ms.uky.edu/~math/MAreport/survey.ps (2003)

[3] Rodica Simion and Frank W. Schmidt. *Restricted Permutations.* Europ. J. Combinatorics (1985) **6**, pp. 383-406.

[4] Eric S. Egge and Toufik Mansour. *Restricted Permutations, Fibonacci Numbers, and k-generalized Fibonacci Numbers.* www.gettysburg.edu/ ~eegge/kgenfib.pdf (2003).

[5] Vincent Vajnovszki. *A Loopless Generation of Bitstring without p Consecutive Ones.* Proceeding of 3-rd Discrete Mathematics and Theoretical Computer Science (2001). pp.227-239.

[6] J.T. Joichi, D.E. White, and S.G. Williamson. *Combinatorial Gray Codes.* SIAM Journal on Computing (1980) **9**, No. 1, pp. 130-141