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Single particle interferometric analogues of multipartite entanglement

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Based on research by Reck *et al.* [1] and Zukowski *et al.* [2], preparation and measurement configurations for the singlet states of two and three two- and three-state particles are enumerated in terms of multiport interferometers.

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I. UNIVERSAL QUANTUM NETWORKS

Any unitary operator in finite dimensional Hilbert space can be composed from a succession of two-parameter unitary transformations in two-dimensional subspaces and a multiplication of a single diagonal matrix with elements of modulus 1 in an algorithmic, constructive and tractable manner; this method is similar to Gaussian elimination and facilitates the parameterization of elements of the unitary group in arbitrary dimensions (e.g., [3, Chapter 2]). Reck, Zeilinger, Bernstein and Bertani have suggested to implement these group theoretic results by realizing interferometric analogues of any discrete unitary and hermitean operators in a unified and experimentally feasible way [1, 4]. Early on, one of the goals was to achieve experimentally realizable multiport analogues of multipartite correlation experiments; in particular for particle states in dimensions higher than two. The multiport analogues of many such experiments with higher than two-dimensional two-particle entangled states have been extensively discussed by Zukowski, Zeilinger and Horne [2]. This article contains also an extensive discussion of the novel features of multiport analogues of multipartite entangled systems; thus we shall just recall issues relevant for the considerations below.

Multiport analogues of multipartite configurations operate with *single* particles only. The output ports represent analogues of joint particle properties. Thus it makes not much sense to discuss *nonlocality* in this framework. The emphasis is on the issue of value definiteness of conceivable physical properties, such as on contextuality.

The observables of multiport interferometers are physical properties related to single particles passing the output ports. Particle detectors behind such output ports, one detector per output port, register the event of a particle passing through the detector. The observations indicating that the particle has passed through a particular output port are clicks in the detector associated with that port.

In what follows, lossless devices will be considered. There are many forms of suitable two-parameter unitary transformations corresponding to generalized two-dimensional “beam splitters” capable of being the factors of higher than two-dimensional unitary transformations (operating in the respective two-dimensional subspaces). Consider the matrix

$$\mathbf{T}(\omega, \phi) = \begin{pmatrix} \sin \omega & \cos \omega \\ e^{-i\phi} \cos \omega & -e^{-i\phi} \sin \omega \end{pmatrix}. \quad (1)$$

$\mathbf{T}(\omega, \phi)$ has physical realizations in terms of beam splitters and Mach-Zehnder interferometers equipped with an appropriate number of phase shifters. Two such realizations are depicted in Fig. 1. The elementary quantum interference device \mathbf{T}^{bs} in Fig. 1a) is a unit consisting of two phase shifters P_1 and P_2 in the input ports, followed by a beam splitter S , which is followed by a phase shifter P_3 in one of the output ports. The device can be quantum mechanically described by [5]

$$\begin{aligned} P_1 : |\mathbf{0}\rangle &\rightarrow |\mathbf{0}\rangle e^{i\alpha+\beta}, \\ P_2 : |\mathbf{1}\rangle &\rightarrow |\mathbf{1}\rangle e^{i\beta}, \\ S : |\mathbf{0}\rangle &\rightarrow \sqrt{T} |\mathbf{1}'\rangle + i\sqrt{R} |\mathbf{0}'\rangle, \\ S : |\mathbf{1}\rangle &\rightarrow \sqrt{T} |\mathbf{0}'\rangle + i\sqrt{R} |\mathbf{1}'\rangle, \\ P_3 : |\mathbf{0}'\rangle &\rightarrow |\mathbf{0}'\rangle e^{i\phi}. \end{aligned} \quad (2)$$

With $\sqrt{T(\omega)} = \cos \omega$ and $\sqrt{R(\omega)} = \sin \omega$, the corresponding unitary evolution matrix is given by

$$\mathbf{T}^{bs}(\omega, \alpha, \beta, \phi) = \begin{pmatrix} i e^{i(\alpha+\beta+\phi)} \sin \omega & e^{i(\beta+\phi)} \cos \omega \\ e^{i(\alpha+\beta)} \cos \omega & i e^{i\beta} \sin \omega \end{pmatrix}. \quad (3)$$

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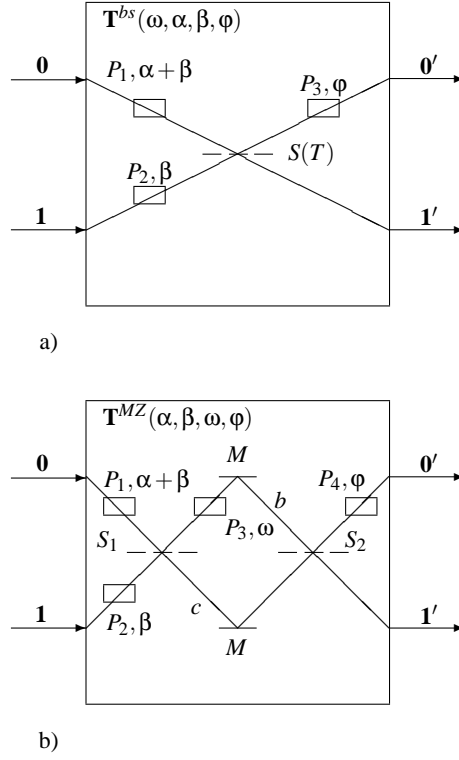


FIG. 1: A universal quantum interference device operating on a qubit can be realized by a 4-port interferometer with two input ports $\mathbf{0}, \mathbf{1}$ and two output ports $\mathbf{0}', \mathbf{1}'$; a) realization by a single beam splitter $S(T)$ with variable transmission T and three phase shifters P_1, P_2, P_3 ; b) realization by two 50:50 beam splitters S_1 and S_2 and four phase shifters P_1, P_2, P_3, P_4 .

Alternatively, the action of a lossless beam splitter may be described by the matrix [?]]

$$\begin{pmatrix} i\sqrt{R(\omega)} & \sqrt{T(\omega)} \\ \sqrt{T(\omega)} & i\sqrt{R(\omega)} \end{pmatrix} = \begin{pmatrix} i \sin \omega & \cos \omega \\ \cos \omega & i \sin \omega \end{pmatrix}.$$

A phase shifter in two-dimensional Hilbert space is represented by either $\text{diag}(e^{i\phi}, 1)$ or $\text{diag}(1, e^{i\phi})$. The action of the entire device consisting of such elements is calculated by multiplying the matrices in reverse order in which the quanta pass these elements [6, 7]; i.e.,

$$\mathbf{T}^{bs}(\omega, \alpha, \beta, \phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i \sin \omega & \cos \omega \\ \cos \omega & i \sin \omega \end{pmatrix} \begin{pmatrix} e^{i(\alpha+\beta)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix}. \quad (4)$$

The elementary quantum interference device \mathbf{T}^{MZ} depicted in Fig. 1b) is a Mach-Zehnder interferometer with *two* input and output ports and three phase shifters. The process can be quantum mechanically described by

$$\begin{aligned} P_1 : |\mathbf{0}\rangle &\rightarrow |\mathbf{0}\rangle e^{i(\alpha+\beta)}, \\ P_2 : |\mathbf{1}\rangle &\rightarrow |\mathbf{1}\rangle e^{i\beta}, \\ S_1 : |\mathbf{1}\rangle &\rightarrow (|b\rangle + i|c\rangle)/\sqrt{2}, \\ S_1 : |\mathbf{0}\rangle &\rightarrow (|c\rangle + i|b\rangle)/\sqrt{2}, \\ P_3 : |b\rangle &\rightarrow |b\rangle e^{i\omega}, \\ S_2 : |b\rangle &\rightarrow (|\mathbf{1}'\rangle + i|\mathbf{0}'\rangle)/\sqrt{2}, \\ S_2 : |c\rangle &\rightarrow (|\mathbf{0}'\rangle + i|\mathbf{1}'\rangle)/\sqrt{2}, \\ P_4 : |\mathbf{0}'\rangle &\rightarrow |\mathbf{0}'\rangle e^{i\phi}. \end{aligned} \quad (5)$$

The corresponding unitary evolution matrix is given by

$$\mathbf{T}^{MZ}(\alpha, \beta, \omega, \phi) = i e^{i(\beta+\frac{\omega}{2})} \begin{pmatrix} -e^{i(\alpha+\phi)} \sin \frac{\omega}{2} & e^{i\phi} \cos \frac{\omega}{2} \\ e^{i\alpha} \cos \frac{\omega}{2} & \sin \frac{\omega}{2} \end{pmatrix}. \quad (6)$$

Alternatively, \mathbf{T}^{MZ} can be computed by matrix multiplication; i.e.,

$$\mathbf{T}^{MZ}(\alpha, \beta, \omega, \phi) = i e^{i(\beta + \frac{\omega}{2})} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} e^{i\omega} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} e^{i(\alpha+\beta)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix}. \quad (7)$$

Both elementary quantum interference devices \mathbf{T}^{bs} and \mathbf{T}^{MZ} are universal in the sense that every unitary quantum evolution operator in two-dimensional Hilbert space can be brought into a one-to-one correspondence with \mathbf{T}^{bs} and \mathbf{T}^{MZ} . As the emphasis is on the realization of the elementary beam splitter \mathbf{T} in Eq. (1), which spans a subset of the set of all two-dimensional unitary transformations, the comparison of the parameters in $\mathbf{T}(\omega, \phi) = \mathbf{T}^{bs}(\omega', \beta', \alpha', \phi') = \mathbf{T}^{MZ}(\omega'', \beta'', \alpha'', \phi'')$ yields $\omega = \omega' = \omega''/2$, $\beta' = \pi/2 - \phi$, $\phi' = \phi - \pi/2$, $\alpha' = -\pi/2$, $\beta'' = \pi/2 - \omega - \phi$, $\phi'' = \phi - \pi$, $\alpha'' = \pi$, and thus

$$\mathbf{T}(\omega, \phi) = \mathbf{T}^{bs}(\omega, -\frac{\pi}{2}, \frac{\pi}{2} - \phi, \phi - \frac{\pi}{2}) = \mathbf{T}^{MZ}(2\omega, \pi, \frac{\pi}{2} - \omega - \phi, \phi - \pi). \quad (8)$$

Let us examine the realization of a few primitive logical “gates” corresponding to (unitary) unary operations on qbits. The “identity” element \mathbb{I}_2 is defined by $|\mathbf{0}\rangle \rightarrow |\mathbf{0}\rangle$, $|\mathbf{1}\rangle \rightarrow |\mathbf{1}\rangle$ and can be realized by

$$\mathbb{I}_2 = \mathbf{T}(\frac{\pi}{2}, \pi) = \mathbf{T}^{bs}(\frac{\pi}{2}, -\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}) = \mathbf{T}^{MZ}(\pi, \pi, -\pi, 0) = \text{diag}(1, 1) \quad . \quad (9)$$

The “not” gate is defined by $|\mathbf{0}\rangle \rightarrow |\mathbf{1}\rangle$, $|\mathbf{1}\rangle \rightarrow |\mathbf{0}\rangle$ and can be realized by

$$\text{not} = \mathbf{T}(0, 0) = \mathbf{T}^{bs}(0, -\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}) = \mathbf{T}^{MZ}(0, \pi, \frac{\pi}{2}, \pi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad . \quad (10)$$

The next gate, a modified “ $\sqrt{\mathbb{I}_2}$,” is a truly quantum mechanical, since it converts a classical bit into a coherent superposition; i.e., $|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle$. $\sqrt{\mathbb{I}_2}$ is defined by $|\mathbf{0}\rangle \rightarrow (1/\sqrt{2})(|\mathbf{0}\rangle + |\mathbf{1}\rangle)$, $|\mathbf{1}\rangle \rightarrow (1/\sqrt{2})(|\mathbf{0}\rangle - |\mathbf{1}\rangle)$ and can be realized by

$$\sqrt{\mathbb{I}_2} = \mathbf{T}(\frac{\pi}{4}, 0) = \mathbf{T}^{bs}(\frac{\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}) = \mathbf{T}^{MZ}(\frac{\pi}{2}, \pi, \frac{\pi}{4}, -\pi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad . \quad (11)$$

Note that $\sqrt{\mathbb{I}_2} \cdot \sqrt{\mathbb{I}_2} = \mathbb{I}_2$. However, the reduced parametrization of $\mathbf{T}(\omega, \phi)$ is insufficient to represent $\sqrt{\text{not}}$, such as

$$\sqrt{\text{not}} = \mathbf{T}^{bs}(\frac{\pi}{4}, -\pi, \frac{3\pi}{4}, -\pi) = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}, \quad (12)$$

with $\sqrt{\text{not}}\sqrt{\text{not}} = \text{not}$.

In $n > 2$ dimensions, the transformation \mathbf{T} in Eq. (1) can be expanded to operate in two-dimensional subspaces. It is possible to recursively diagonalize any n -dimensional unitary transformation $u(n)$ by a successive applications of such matrices \mathbf{T}' . The remaining diagonal entries of modulus 1 can be compensated by an inverse diagonal matrix \mathbf{D} ; such that $u(n)\mathbf{T}'\mathbf{T}'' \dots \mathbf{D} = \mathbb{I}_n$. Thus, the inverse of all these single partial transformations is equivalent to the original transformation; i.e., $u(n) = (\mathbf{T}'\mathbf{T}'' \dots \mathbf{D})^{-1}$. This technique is extensively reviewed in [3, Chapter 2] and in [1, 4]. Every single constituent and thus the whole transformation has a interferometric realization. We now turn to a discussion of multiport interferometric analogues of entangled multipartite states and their measurement.

II. TWO TWO-STATE PARTICLES ANALOGUE

A. States

Let us explicitly enumerate the case of two entangled two-state particles in one of the Bell basis states (e.g., [8]; the superscript T indicates transposition)

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(e_1 \otimes e_1 + e_2 \otimes e_2) \equiv \frac{1}{\sqrt{2}}(1, 0, 0, 1)^T, \quad (13)$$

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}(e_1 \otimes e_1 - e_2 \otimes e_2) \equiv \frac{1}{\sqrt{2}}(1, 0, 0, -1)^T, \quad (14)$$

$$|\Psi_3\rangle = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_1 \otimes e_2) \equiv \frac{1}{\sqrt{2}}(0, 1, 1, 0)^T, \quad (15)$$

$$|\Psi_4\rangle = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1) \equiv \frac{1}{\sqrt{2}}(0, 1, -1, 0)^T, \quad (16)$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ form the standard basis of the Hilbert space \mathbb{C}^2 of the individual particles. The state operators corresponding to (13)–(15) are the dyadic products of the normalized vectors with themselves; i.e.,

$$|\Psi_1\rangle \equiv \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad (17)$$

$$|\Psi_2\rangle \equiv \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad (18)$$

$$|\Psi_3\rangle \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (19)$$

$$|\Psi_4\rangle \equiv \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (20)$$

B. Observables

With the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (21)$$

two one-particle observables E, F can be defined by

$$E = \text{diag}(e_{11}, e_{22}), \quad (22)$$

$$F = R(-\frac{\pi}{4}) E R(\frac{\pi}{4}) = \frac{1}{2} \begin{pmatrix} e_{11} + e_{22} & e_{11} - e_{22} \\ e_{11} - e_{22} & e_{11} + e_{22} \end{pmatrix}. \quad (23)$$

Often, e_{11} and e_{22} are labelled by 0, 1 or +, −, respectively.

The corresponding single-sided observables for the two-particle case are

$$O_1 \equiv E \otimes \mathbb{I}_2 \equiv \text{diag}(e_{11}, e_{11}, e_{22}, e_{22}), \quad (24)$$

$$O_2 \equiv \mathbb{I}_2 \otimes F \equiv \frac{1}{2} \text{diag}(F, F) = \frac{1}{2} \begin{pmatrix} e_{11} + e_{22} & e_{11} - e_{22} & 0 & 0 \\ e_{11} - e_{22} & e_{11} + e_{22} & 0 & 0 \\ 0 & 0 & e_{11} + e_{22} & e_{11} - e_{22} \\ 0 & 0 & e_{11} - e_{22} & e_{11} + e_{22} \end{pmatrix}. \quad (25)$$

Here, $\text{diag}(A, B)$ stands for the matrix with diagonal blocks A, B ; all other components are zero. \mathbb{I}_2 stands for the unit matrix in two dimensions.

As the commutator $[A \otimes \mathbb{I}, \mathbb{I} \otimes B] = (A \otimes \mathbb{I}) \cdot (\mathbb{I} \otimes B) - (\mathbb{I} \otimes B) \cdot (A \otimes \mathbb{I}) \equiv A_{ij} \delta_{lm} \delta_{jk} B_{ms} - \delta_{ij} B_{lm} A_{jk} \delta_{ms} = A_{ik} B_{ls} - B_{ls} A_{ik} = 0$ vanishes for arbitrary matrices A, B , also $[O_1, O_2] = 0$ vanishes, and the two corresponding observables are commensurable. Hence the two measurements of O_1 and O_2 can be performed successively without disturbing each other.

In order to represent O_1 and O_2 by beam splitters, we note that their eigenvectors form the bases

$$\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}, \quad \text{and} \\ \{(1/\sqrt{2})(0, 0, -1, 1), (1/\sqrt{2})(0, 0, 1, 1), (1/\sqrt{2})(-1, 1, 0, 0), (1/\sqrt{2})(1, 1, 0, 0)\} \quad (26)$$

with eigenvalues $\{e_{11}, e_{11}, e_{22}, e_{22}\}$ and $\{e_{22}, e_{11}, e_{22}, e_{11}\}$, respectively. By identifying those eigenvectors as rows of a unitary matrix and stacking them in numerical order, one obtains the unitary operators “sorting” the incoming amplitudes into four output ports, corresponding to the eigenvalues of O_1 and O_2 , respectively. (Any other arrangement would also do, but would

change the port identifications.) That is,

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (27)$$

$$U_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \quad (28)$$

The operator

$$\begin{aligned} O_{12} &= (E \otimes \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes F) = E \otimes F \\ &= \frac{1}{2} \text{diag}(e_{11}F, e_{22}F) = \frac{1}{2} \begin{pmatrix} e_{11}(e_{11}+e_{22}) & e_{11}(e_{11}-e_{22}) & 0 & 0 \\ e_{11}(e_{11}-e_{22}) & e_{11}(e_{11}+e_{22}) & 0 & 0 \\ 0 & 0 & e_{22}(e_{11}+e_{22}) & e_{22}(e_{11}-e_{22}) \\ 0 & 0 & e_{22}(e_{11}-e_{22}) & e_{22}(e_{11}+e_{22}) \end{pmatrix} \end{aligned} \quad (29)$$

combines both O_1 and O_2 . The interferometric realization of O_{12} in terms of a unitary transformation is the same as for O_2 , since they share a common set of eigenstates with different eigenvalues $\{e_{22}^2, e_{11}e_{22}, e_{11}e_{22}, e_{11}^2\}$. Thus, $U_{12} = U_2$.

C. Preparation

The interferometric setup can be decomposed into two phases. In the first phase, the state is prepared. In the second phase, the state is analyzed by successive applications of U_1 and U_2 , or just $U_{12} = U_2$, and by observing the output ports.

Suppose the interferometric input and output ports are labelled by $1, \dots, 4$; and let the corresponding states be represented by $|1\rangle \equiv (1, 0, 0, 0)^T$, $|2\rangle \equiv (0, 1, 0, 0)^T$, $|3\rangle \equiv (0, 0, 1, 0)^T$ and $|4\rangle \equiv (0, 0, 0, 1)^T$. The initial state can be prepared by unitary transformations. For instance, the unitary transformation U_p transforming the state of a particle entering the first port $|1\rangle$ into the singlet state (16) is

$$U_p = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \quad (30)$$

D. Predictions

To check the validity of the calculations, consider a measurement of the singlet state $|\Psi_4\rangle$ in (16) with parallel directions. Thus, instead of F in (23), the second operator is the same as E in (22). As a result, $O'_{12} \equiv E \otimes E \equiv \text{diag}(e_1^2, e_1e_2, e_1e_2, e_2^2)$. Since the eigenvectors of O'_{12} are just the elements of the standard basis of the Hilbert space \mathbb{C}^4 , $U'_{12} = U_1$ has only unit entries in its counterdiagonal. Hence, $U'_{12}|\Psi_4\rangle \equiv (1/\sqrt{2})(0, -1, 1, 0)^T$, and since $|\langle n|U'_{12}|\Psi_4\rangle|^2 = 0$ for $n = 1, 4$ and $|\langle n|U'_{12}|\Psi_4\rangle|^2 = 1/2$ for $n = 2, 3$, there is a 50:50 chance to find the particle in port 2 and 3, respectively. The particle will never be measured in detectors behind the output ports 1 or 4.

These events could be interpreted in the following way: The first and the forth detectors stand for the property that both “single-particle” observables are the same; the second and the third detectors stand for the property that both “single-particle” observables are different. Since the input state was chosen to be a singlet state (16), only the latter case can occur. Similar considerations hold for the other states of the bell basis defined in (13)–(15). In particular, for Ψ_1 and Ψ_2 , the detectors behind output ports 1 or 4 will record events, and the detectors behind ports 2 and 3 will not.

The singlet state (16), when processed through U_{12} in Eq. (29), yields equal chances of output through any one of the four output ports of the interferometer; i.e., $U_{12}|\Psi_4\rangle \equiv (1/2)(1, -1, 1, 1)^T$, and thus $|\langle n|U_{12}|\Psi_4\rangle|^2 = 1/4$, $n = 1, \dots, 4$. This result could be expected, as in (22) and (23) the relative distance between E and F has been chosen to be $\pi/4$.

A more general computation for arbitrary $0 \leq \theta \leq \pi$ yields the set

$$\{(\cos \theta, \sin \theta, 0, 0), (-\sin \theta, \cos \theta, 0, 0), (0, 0, \cos \theta, \sin \theta), (0, 0, -\sin \theta, \cos \theta)\}$$

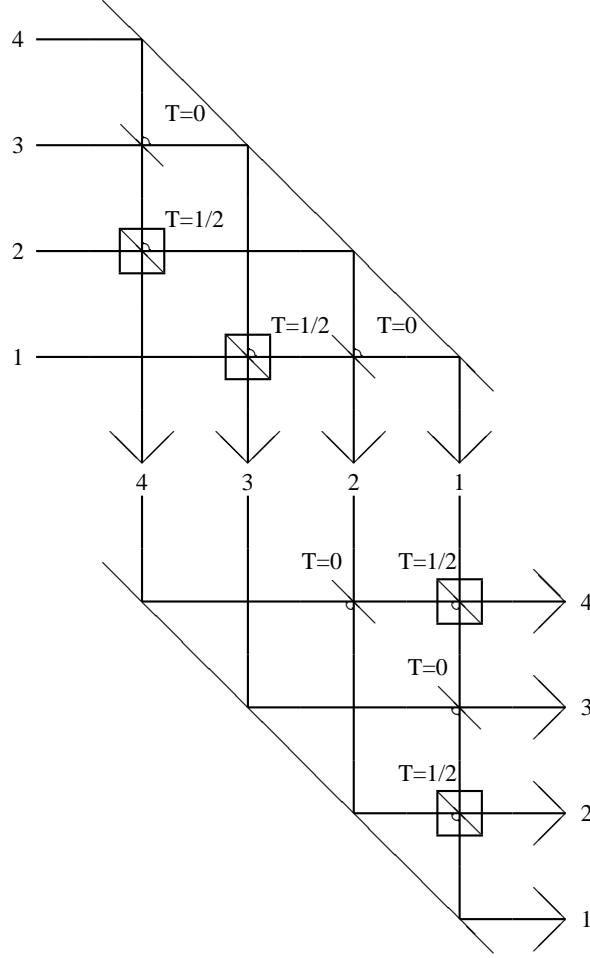


FIG. 2: Preparation and measurement setup of an interferometric analogue of a two two-state particles setup in the singlet state. A single particle enters the upper port number 1 and leaves one of the lower ports 2 or 3.

of normalized eigenvectors for $O_{12}(\theta)$. As a result, the corresponding unitary operator is given by

$$U_{12}(\theta) = \text{diag}(R(\theta), R(\theta)) = \begin{pmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{pmatrix}. \quad (31)$$

Thus, $U_{12}(\theta)|\Psi_4\rangle \equiv (1/\sqrt{2})(\sin \theta, \cos \theta, -\cos \theta, \sin \theta)^T$, and $|\langle 1|U_{12}(\theta)|\Psi_4\rangle|^2 = |\langle 4|U_{12}(\theta)|\Psi_4\rangle|^2 = \frac{1}{2} \sin^2 \theta$, $|\langle 2|U_{12}(\theta)|\Psi_4\rangle|^2 = |\langle 3|U_{12}(\theta)|\Psi_4\rangle|^2 = \frac{1}{2} \cos^2 \theta$.

E. Interferometric setup

Based on the decomposition of an arbitrary unitary transformation in four dimensions into unitary transformations of two-dimensional subspaces [3], Reck *et al.* [1] have developed an algorithm [9] for the experimental realization of any discrete unitary operator. When applied to the preparation and analyzing stages corresponding to (30) and (27)–(28) respectively, the arrangement is depicted in Fig. 2.

III. TWO THREE-STATE PARTICLES ANALOGUE

A. Singlet state preparation

A group theoretic argument shows that in the case of two three-state particles, there is just one singlet state [10–12]

$$|\Phi\rangle = \frac{1}{\sqrt{3}}(e_1 \otimes e_3 - e_2 \otimes e_2 + e_3 \otimes e_1) \equiv \frac{1}{\sqrt{3}}(0, 0, 1, 0, -1, 0, 1, 0, 0)^T, \quad (32)$$

where again $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ refer to elements of the standard basis of Hilbert space \mathbb{C}^3 of the individual particles.

A unitary transformation rendering the singlet state (32) from a particle in the first port $|1\rangle$ is

$$U_p = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (33)$$

B. Observables

For the sake of the argument toward a limited quantum noncontextuality [13], rotations in the $e_1 - e_2$ plane along e_3 are considered; the corresponding matrix being

$$R_{12}(\theta) = \text{diag}(R(\theta), e_{33}) = \begin{pmatrix} \cos\theta & \sin\theta & & \\ -\sin\theta & \cos\theta & & \\ & & 0 & 1 \end{pmatrix} \quad (34)$$

Two one-particle observables E, F can be defined by

$$E = \text{diag}(e_{11}, e_{22}, e_{33}), \quad (35)$$

$$F = R_{12}\left(-\frac{\pi}{4}\right) E R_{12}\left(\frac{\pi}{4}\right) = \frac{1}{2} \begin{pmatrix} e_{11} + e_{22} & e_{11} - e_{22} & & \\ e_{11} - e_{22} & e_{11} + e_{22} & & \\ 0 & 0 & 2e_{33} & \end{pmatrix}. \quad (36)$$

Often, e_{11} and e_{22} are labelled by $-1, 0, 1$ or $-, 0, +$, respectively.

The corresponding single-sided observables for the two-particle case are

$$O_1 \equiv E \otimes \mathbb{I}_3 \equiv \text{diag}(e_{11}, e_{11}, e_{11}, e_{22}, e_{22}, e_{22}, e_{33}, e_{33}, e_{33}), \quad (37)$$

$$O_2 \equiv \mathbb{I}_3 \otimes F \equiv \frac{1}{2} \text{diag}(F, F, F) = \frac{1}{2} \begin{pmatrix} F & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}. \quad (38)$$

\mathbb{I}_3 stands for the unit matrix in three dimensions.

Let the matrix $[v_i^T v_i]$ stand for the dyadic product of the vector v_i with itself, and let $P_1 = [e_1^T e_1] = \text{diag}(1, 0, 0)$, $P_2 = [e_2^T e_2] = \text{diag}(0, 1, 0)$, $P_3 = [e_3^T e_3] = \text{diag}(0, 0, 1)$ be the projections onto the axes of the standard basis. Then, the following observables can be defined: $x_1 = P_1 F = \text{diag}(e_{11}, 0, 0)$, $x_2 = P_2 F = \text{diag}(0, e_{22}, 0)$, $x_3 = P_3 F = \text{diag}(0, 0, e_{33})$. Likewise, x'_1 , x'_2 and x'_3 can be defined by rotated projections. The configuration of the observables is depicted in Fig. 3a), together with its representation in a Greechie (orthogonality) diagram [14] in Fig. 3b), which represents orthogonal tripods by points symbolizing individual legs that are connected by smooth curves. In the dual Tkadlec diagram depicted in Fig. 3c), the role of points and smooth curves are interchanged: points represent complete tripods and maximal smooth curves represent single legs.

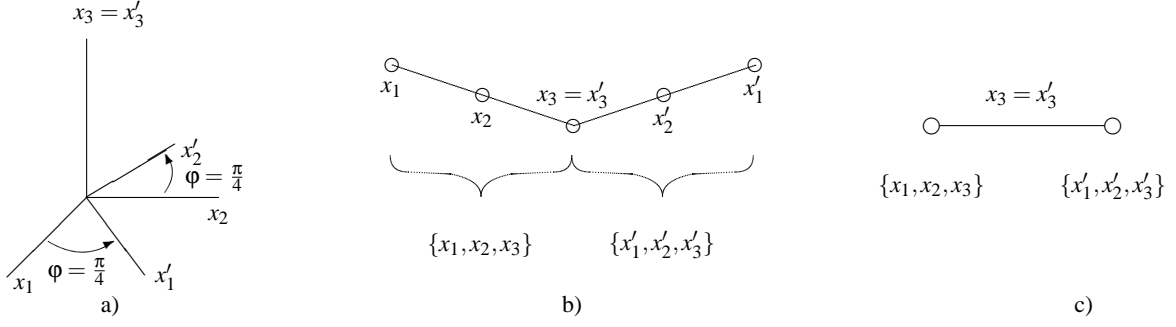


FIG. 3: Three equivalent representation of the same geometric configuration: a) Two tripods with a common leg; b) Greechie (orthogonality) diagram: points stand for individual basis vectors, and orthogonal tripods are drawn as smooth curves; c) Tkadlec diagram: points represent complete tripods and smooth curves represent single legs interconnecting them.

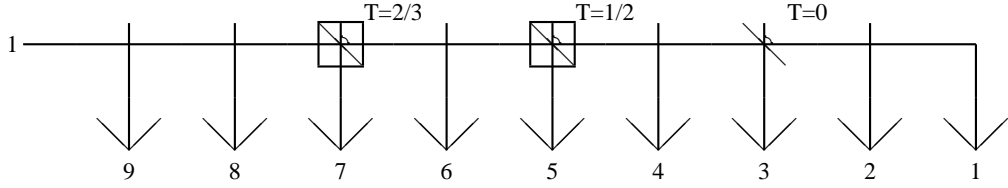


FIG. 4: Preparation stage of a two three-state particles singlet state setup derived from the unitary operator U_p in Eq. (33). Only the bottom part of the element pyramid is drawn.

C. Interferometric implementation

A multiport implementation of U_p in Eq. (33) is depicted in Fig. 4. The entire matrix corresponds to a pyramid of beam splitters and phase shifters, but only the bottom row contributes toward the transformation $|1\rangle \rightarrow |\Phi\rangle$.

The unitary matrices needed for the interferometric implementation of O_1 and O_2 are again just the ordered eigenvectors of O_1 and O_2 ; i.e., U_1 is a matrix with unit entries in the counterdiagonal and zeroes otherwise, and

$$U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (39)$$

The interferometric implementation of U_2 is drawn in Fig. 5.

D. Predictions

The probabilities to find the particle in the input ports can be computed by $U_2|\Phi\rangle = (0, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}, 0, 0)$, and finally $\langle n|U_2|\Phi\rangle$, $n = 1, \dots, 9$. It is $1/3$ for port number 7, $1/6$ for ports number 2, 3, 5, 6 and 0 for ports number 1, 4, 8, 9, respectively. This result can be interpreted as follows. Port number 7 corresponds to the occurrence of observable corresponding to $x_3 \wedge x'_3$, where \wedge stands for the logical “or.” By convention, the single particle state vectors e_1, e_2, e_3 and their rotated counterparts $e'_1, e'_2, e'_3 = e_3$ can be referred to by the labels “+,” “−,” “0,” respectively; thus port number 7 can be referred to as the “00 case.” The ports number 2, 3, 5, 6 correspond to the four equal-weighted possibilities $x_1 \wedge x'_1, x_2 \wedge x'_1, x_1 \wedge x'_2, x_2 \wedge x'_2$, which are also

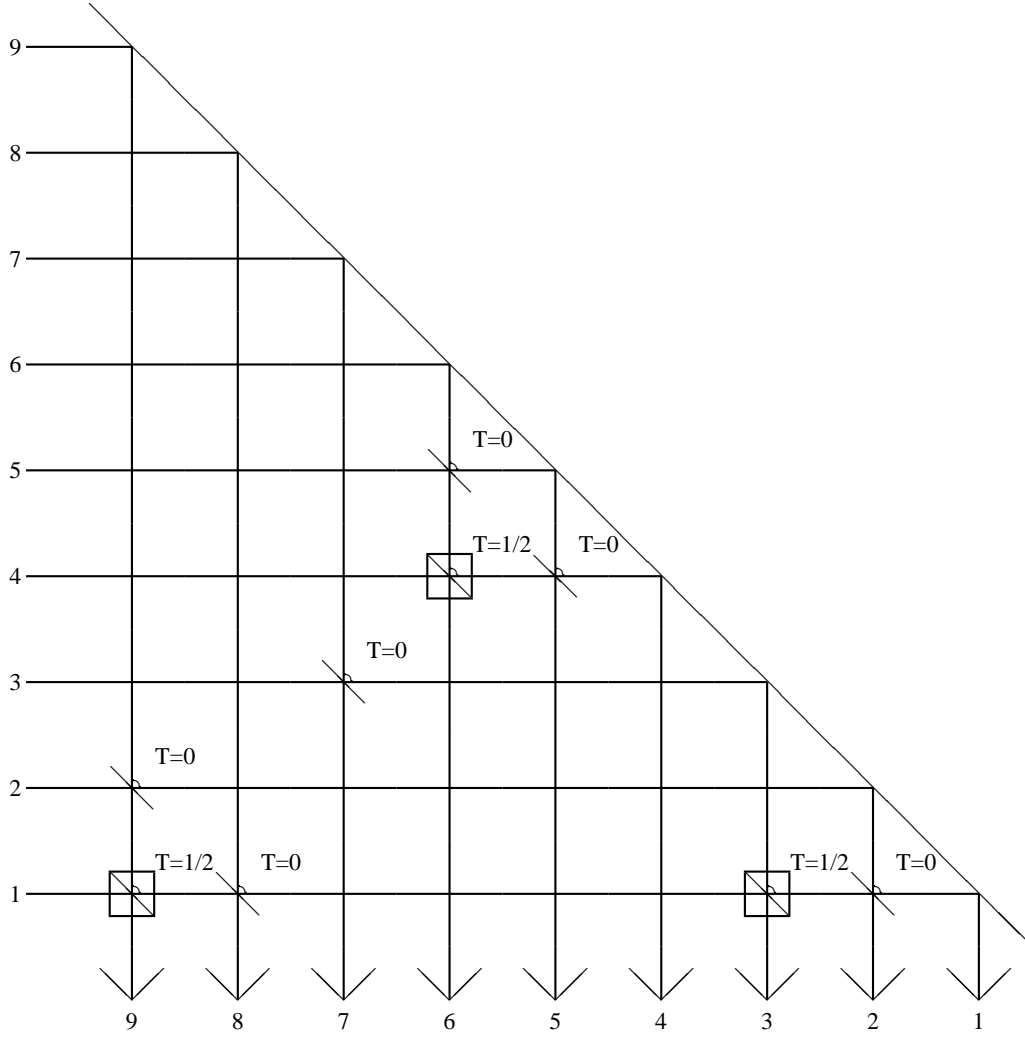


FIG. 5: Measurement setup of an interferometric analogue of a measurement of O_2 in Eq. (39).

known as $++$, $--$, $+-$, $-+$ cases. The ports number 1, 4, 8, 9 correspond to the four $x_1 \wedge x'_3$, $x_2 \wedge x'_3$, $x_3 \wedge x'_1$, $x_3 \wedge x'_2$, which are also known as $+0$, -0 , $0+$, $0-$ cases, which cannot occur, since the particle enters the analysing part of the interferometer in the singlet state in which it was prepared for.

IV. THREE THREE-STATE PARTICLES ANALOGUE

We shall briefly sketch the considerations yielding to an interferometric realization which is analogous to a configuration of three three-state particles in a singlet state, measured along three particular directions, such that the context structure is $x''_3 - x''_2 - x''_1 = x_1 - x_2 - x_3 = x'_3 - x'_1 - x'_2$.

Group theoretic considerations [12, 15] show that the only singlet state for three three-state particles is

$$|\Delta\rangle = \frac{1}{\sqrt{6}}(|-+0\rangle - |-0+\rangle + |+0-\rangle - |+ -0\rangle + |0-+\rangle - |0+-\rangle). \quad (40)$$

If the labels “+,” “-,” “0” are again identified with the single particle state vectors e_1, e_2, e_3 forming a standard basis of \mathbb{C}^2 , Eq. (40) can be represented by

$$\begin{aligned} |\Delta\rangle &\equiv \frac{1}{\sqrt{6}}(e_2 \otimes e_1 \otimes e_3 - e_2 \otimes e_3 \otimes e_1 + e_1 \otimes e_3 \otimes e_2 - e_1 \otimes e_2 \otimes e_3 + e_3 \otimes e_2 \otimes e_1 - e_3 \otimes e_1 \otimes e_2) \\ &\equiv \frac{1}{\sqrt{6}}(0, 0, 0, 0, 0, -1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, -1, 0, 0, 0, -1, 0, 1, 0, 0, 0, 0, 0). \end{aligned} \quad (41)$$

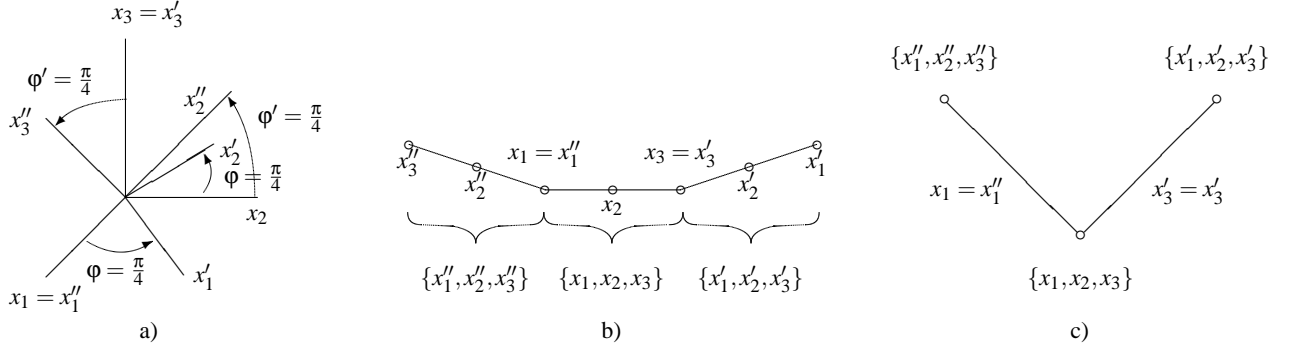


FIG. 6: Three equivalent representation of the same geometric configuration: a) Three tripods interconnected at two common legs; b) Greechie diagram of a); c) Tkadlec diagram of a).

We shall study rotations in the $e_1 - e_2$ plane around e_3 , as well as in the $e_2 - e_3$ plane around e_1 ; the corresponding matrix being $R_{23}(\theta) = \text{diag}(e_{11}, R(\theta))$. With the rotation angles $\pi/4$, three one-particle observables E, F, G can be defined by

$$E = \text{diag}(e_{11}, e_{22}, e_{33}), \quad (42)$$

$$F = R_{12}\left(-\frac{\pi}{4}\right) E R_{12}\left(\frac{\pi}{4}\right), \quad (43)$$

$$G = R_{23}\left(-\frac{\pi}{4}\right) E R_{23}\left(\frac{\pi}{4}\right). \quad (44)$$

The corresponding single-sided observables for the two-particle case are

$$O_1 \equiv E \otimes \mathbb{I}_3 \otimes \mathbb{I}_3, \quad (45)$$

$$O_2 \equiv \mathbb{I}_3 \otimes F \otimes \mathbb{I}_3, \quad (46)$$

$$O_3 \equiv \mathbb{I}_3 \otimes \mathbb{I}_3 \otimes G. \quad (47)$$

O_1, O_2, O_3 are commensurable, as they represent analogues of the observables which are measured at the separate particles of the singlet triple. The joint observable

$$O_{123} \equiv E \otimes F \otimes G \quad (48)$$

has normalized eigenvectors which form a unitary basis, whose elements are the rows of the unitary equivalent U_{123} of O_{123} . An interferometric implementation of this operator is depicted in Fig. 7.

V. DISCUSSION

With the apparent lack of entanglement resources for higher than two-dimensional subsystems, multipart analogues of multipartite entanglement have been developed with limited quantum noncontextuality in mind [13]. Although there is no principal limit to the number of entangled particles involved, certain tasks, such as the encoding of “explosion views” of Kochen-Specker configurations appear as insurmountable challenge. Such “explosion views” of Kochen-Specker type configurations of observables can be imagined in the following way. Let N be the number of contexts, or “blocks,” or inter-rotated triples of observables occurring in the Kochen-Specker proof. First, a singlet state of a “large” number N of three-state particles has to be realized. $N = 118$ in the original Kochen-Specker argument [16], and $N = 40$ in Peres’ [17, 18] proof. Any such state should be invariant with respect to unitary transformations $u(n^N) = \bigotimes_{i=1}^N u_i(n)$ composed of identical unitary transformations $u_i(n)$ in n dimensions. Then, every one of the N particle would be measured along the N contexts or blocks, one particle per context, respectively. All steps, in particular the construction and formation of n -partite singlet states by group theoretic methods, as well as the interferometric implementation of these states and of all observables in the many different contexts required by the Kochen-Specker proof, are constructive and computationally tractable. Yet, the configurations would require an astronomical number (of the order of 3^{80} in the Peres’ case of the proof) of beam splitters. Thus, a physical realization of the full context structure of properties appearing in an “explosion view” of a Kochen-Specker type proof is remains a *Gedankenexperiment* at the moment. Even weaker forms of nonclassicality such as structures with a nonseparating set of states—the Γ_3 in Kochen and Specker’s original work [16] would require $N = 16$ (corresponding to sixteen particles) and would still be too complex to realize.

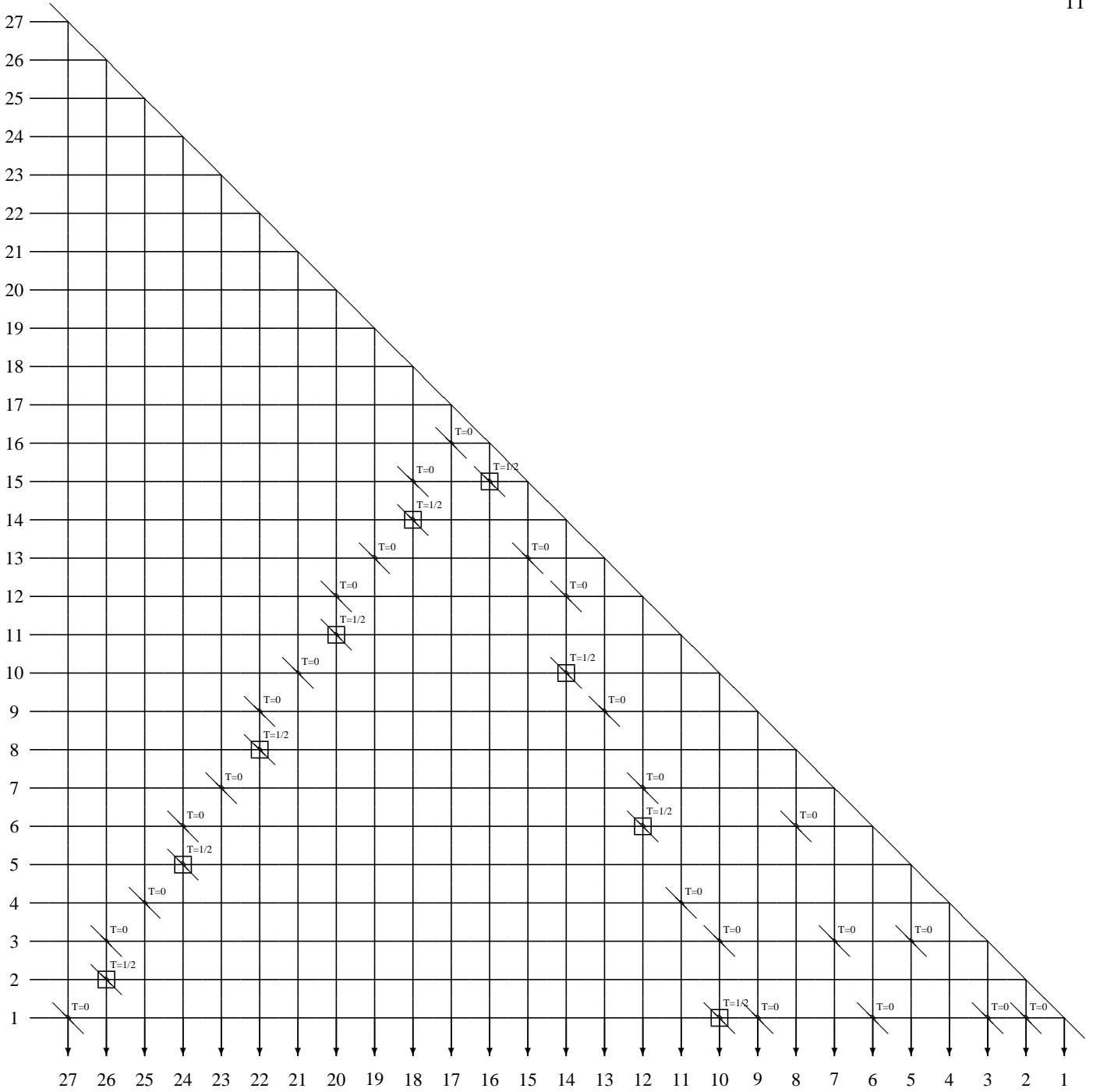


FIG. 7: Measurement setup of an interferometric analogue of a measurement of the three-particle operator O_{123} in Eq. (48).

However, interferometric analogues of two- and three-particle configurations are realizable with today's techniques. Such configurations have been explicitly enumerated in this article.

It may also be worthwhile to search not only for purely optical implementations of the necessary elementary interferometric cells realizing two-dimensional unitary transformations. Solid state elements and purely electronic devices may be efficient models of multiport interferometric analogues of multipartite entangled states.

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- [] The standard labelling of the input and output ports are interchanged, therefore sine and cosine are exchanged in the transition matrix.