CDMTCS
Research
Report
Series

Lowness Properties of Reals and Randomness

André Nies
Department of Computer Science,
University of Auckland

CDMTCS-201
December 2002

Centre for Discrete Mathematics and
Theoretical Computer Science

Lowness properties of reals and randomness

André Nies

December 2, 2002

1 Introduction

We investigate three properties of sets of natural numbers which have been discovered independently by different researchers, and show they are all the same. Sets of natural numbers are identified with infinite strings over $\{0,1\}$, and will be called *reals*.

- Chaitin [5] and Solovay [19] introduced the class $\mathcal{K}$ of K-trivial reals (using different notation). The real A is K-trivial if the prefix complexity of each initial segment of A is minimal.
- Van Lambalgen [21] and Zambella studied the class $\mathcal{LR}$ of low for random reals, a property which says the real is computationally weak when used as an oracle, since each random real is already A-random.
- Andrei Muchnik (1998) worked on the class $\mathcal{SK}$ of strongly K-trivial reals, reals which are of no use as an oracle for compressing a string. By an easy argument, $\mathcal{SK}$ is included in both $\mathcal{K}$ and $\mathcal{LR}$, and all reals in $\mathcal{SK}$ are low_1.

We investigate properties of those classes, in particular showing that $\mathcal{K}$ is closed downward under Turing reducibility. This eventually leads to a proof that all three classes coincide. At the same time, they represent very different aspects of the same notion. The class $\mathcal{LR}$ expresses that the real is computationally weak, while $\mathcal{K}$ states that the real is far from random. The part $\mathcal{K} = \mathcal{SK}$ is joint with D. Hirschfeldt, and can be proved by modifying the argument that $\mathcal{K}$ is closed downward.
Our results contribute to a recent line of research joining two areas of computability theory which had previously been pursued separately: the complexity of reals, and their randomness properties. To classify reals by their absolute complexity, one introduces a hierarchy of classes: computable, recursively enumerable, Δ^0_2 etc. $\mathcal{K}$ lies in between computable and Δ^0_2. The complexity of reals is compared via reducibilities, for example Turing reducibility $\leq_T$. To study the classes in this hierarchy and the degree structures arising from these reducibilities, increasingly difficult forcing arguments and priority constructions are needed.

The most commonly accepted notion of algorithmic randomness is the one introduced by Martin-Löf [10]. A *Martin-Löf test* is a uniformly r.e. sequence (U_n) of open sets in Cantor Space 2^ω such that $\mu(U_n) \leq 2^{-n}$, where μ is the usual Lebesgue measure on 2^ω. A real X is *Martin-Löf random* if it passes each test in the sense that $X \notin \bigcap_n U_n$. Schnorr [15] proved that a real X is random in this sense if and only if the algorithmic prefix complexity K of all its initial segments is large, namely $\forall n\ K(X \upharpoonright n) \geq n - \mathcal{O}(1)$. The methods used to study algorithmic randomness have been quite different from the ones mentioned above - they were effective measure theoretic or, when dealing with K-complexity, combinatorial.
The class $\mathcal{K}$ induces a Σ^0_3 ideal in the r.e. Turing degrees, which generates the whole of $\mathcal{K}$ under Turing downward closure. As in computational complexity theory, such closure properties can be taken as further evidence that this common class $\mathcal{K}$ is a very natural one. $\mathcal{K}$ is the first known example of an intermediate Σ^0_3-ideal defined by a property not directly related to Turing reducibility. $\mathcal{K}$ also is the first Σ^0_3-ideal not obtained by a direct construction. Moreover, $\mathcal{K}$ is *degree invariant*, namely, for Turing equivalent reals X, Y, the relativized classes $\mathcal{K}^X$ and $\mathcal{K}^Y$ coincide. This relates to Sacks' question whether there is a degree invariant solution to Post's Problem [14]. A degree invariant ideal which is also principal would give such a solution (at least as a Borel operator). However, we also prove that $\mathcal{K}^X$ is never a principal ideal in the r.e. degrees relative to X.
In the following we discuss the relevant classes and concepts in an informal way, deferring the formal definitions to Section 2. A *lowness property* of a real A says that, in some sense, A has low computational power when used as an oracle. We require that such a property be downward closed under $\leq_T$. An example of a (rather weak) lowness property is the usual lowness, $A' \equiv_T \emptyset'$. The lowness property $\mathcal{LR}$ is itself based on relative randomness: A is *low for random* if each random real X is already random relative to A, i.e. X passes all A-r.e. tests. Terwijn and Kucera [9, submitted 1997] proved the existence of a non-computable r.e. low for random real.
The class $\mathcal{K}$ of K-trivial reals embodies being far from random. While random reals have high initial segment complexity, for K-trivial reals this complexity is as low as possible, namely $\forall n\ K(X \upharpoonright n) \leq K(n) + \mathcal{O}(1)$. Clearly each computable real is K–trivial. Chaitin [4] proved that $\mathcal{K} \subseteq \Delta^0_2$. Solovay (1975), in an unpublished but widely circulated manuscript [19], gave the first, rather complicated construction of a non-computable real in $\mathcal{K}$. Kummer (unpublished) and Downey (see [7]) independently built an r.e. non-computable real in $\mathcal{K}$ via similar, very short and elegant constructions. (See [7] for more on the history of this.)
Let $K^A(y)$ be the prefix complexity of y relative to the oracle A. We call a real A *strongly K-trivial* if $\forall y\ K(y) \leq K^A(y) + \mathcal{O}(1)$. In other words, the oracle A cannot be used to further compress the string y. The class of such reals is denoted $\mathcal{SK}$. Andrei Muchnik (unpublished, 1998) constructed a non-computable r.e. real in this class.

The constructions of an r.e. set in those apparently very different classes are quite similar (this was a first indicator that the classes might in fact be the same). We describe a common framework for those constructions, called the cost function method. A *cost function* is a computable function $c : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{Q}_0^+$ such that $\lim_x \lim_s c(x, s) = 0$. Suppose we are building a Δ_2^0-set A, via a Δ_2^0–approximation (A_r). At stage s, if x is least such that $A_s(x)$ changes (for instance to meet a requirement in a list of requirements ensuring that A is non-computable), the cost of this change is $c(x, s)$. There is a global restraining requirement that the sum of the costs over all stages be finite.

It remains to define a cost function which is appropriate for constructing a real in the relevant class. For $\mathcal{K}$, one uses $c(x, s) = \mathcal{O}(1) \sum_{x<y\leq s} 2^{-K_s(y)}$, where $K_s(y)$ denote the prefix complexity of y by stage s (the particular choice of the constant is irrelevant). This method is interesting because it has no injury to requirements, thereby giving a new injury free solution to Post's problem.

The method seemed to be the only way to construct K-trivial reals. We will see that this uniqueness of the construction was unavoidable: If $A \in \mathcal{K}$, then this Δ_2^0 set can be viewed as being built via the cost function method for $\mathcal{K}$. This characterization can be used to derive further information about $\mathcal{K}$, for instance, each real $A \in \mathcal{K}$ is truth-table below an r.e. set in $\mathcal{K}$.

We sketch the recent history of these results. Kucera and Terwijn asked for a low for random real not in Δ_2^0. (This is also Problem 4.4. in Ambos-Spies and Kucera [2]). The work of Terwijn and Zambella on Schnorr low sets [20] suggested the existence of such a set, since there are continuum many Schnorr low sets, and they are necessarily outside Δ_2^0. Stephan and Nies showed that $\{e : W_e \in \mathcal{LR}\}$ is Σ_3^0. To do so they gave a characterization of $\mathcal{LR}$ [13]. Using a modified form of this characterization, the author proved $\mathcal{LR} = \mathcal{SK}$, which implies $\mathcal{LR} \subseteq \Delta_2^0$. Hirschfeldt made an important step towards showing $\mathcal{K}$ is a lowness property, showing each $A \in \mathcal{K}$ is Turing incomplete (see [7, Thm 4.1]). The author showed the stronger result that $\mathcal{K}$ is closed downwards under $\leq_T$, and gave the characterization of $\mathcal{K}$. Hirschfeldt conjectured that $\mathcal{K} = \mathcal{SK}$, and together they developped the modification of the proof that $\mathcal{K}$ is closed downwards which suffices.

Few examples of natural ideals are known in the r.e. degrees: the non-cuppable degrees, the non-promptly simple degrees (which coincide with the cappable degrees by [1]) and the almost deep degrees ($\mathbf{a}$ is almost deep if $\mathbf{a} \vee \mathbf{b}$ is low for each low r.e. degree $\mathbf{b}$ [6]). The latter two classes are interesting since, as is the case for $\mathcal{K}$, their defining property is not directly related to Turing reducibility. However, unlike for $\mathcal{K}$, this defining property is not Σ_3^0. It is known that to be of promptly simple degree is Σ_4^0–complete.

An example of a lowness property from the theory of inductive inference which is analogous to $\mathcal{LR}$ is the class of reals of trivial EX-degree, i.e. the reals A such that $\mathrm{EX}[A] = \mathrm{EX}$, where $\mathrm{EX}[A]$ is the class of sets of computable functions which can be learned with an oracle A. Slaman and Solovay [17] proved that the nonrecursive reals in this class coincide with the reals Turing equivalent to a 1-generic set in Δ_2^0. Thus, none of them is r.e.

The plan of the paper is as follows. In Section 2 we define the classes and discuss their basic properties. In Section 3 we discuss an important tool, the Kraft-Chaitin Theorem, and we review a construction of a non-computable r.e. set in $\mathcal{K}$, based on [4]. In Section 4 we prove that low for random equals strongly K-trivial. Section 5 contains the Theorems that $\mathcal{K}$ is downward closed, that $\mathcal{K} = \mathcal{SK}$ and that the construction from Section 3 actually provides a characterization of $\mathcal{K}$. In a final Section, we discuss relativizations of $\mathcal{K}$ and reducibilities related to $\mathcal{LR}$ and $\mathcal{SK}$.

Notation. We identify a string σ in $2^{<\omega}$ with the natural number n such that the binary representation of $n+1$ is 1σ.

For each real A, we want to define $K^A(y)$, the length of a shortest prefix description of y using oracle A. An *oracle machine* is a partial recursive functional $M : 2^\omega \times 2^{<\omega} \mapsto 2^{<\omega}$. We write $M^A(x)$ for $M(A, x)$. M is an *oracle prefix machine* if the domain of M^A is an antichain under inclusion of strings, for each A. Let $(M_d)_{d\in\mathbb{N}^+}$ be an effective listing of all oracle prefix machines. The universal oracle prefix machine U is given by

$$U^A(0^d1\sigma) = M_d^A(\sigma).$$

If $U^A(\sigma) = y$, we say that σ is a U^A-*description* of y. Let $\Omega^A = \mu(\text{dom } U^A)$, and

$$K^A(y) = \min\{|\sigma| : U^A(\sigma) = y\}.$$

When fixing the oracle to $\emptyset$, we obtain the notions of prefix machine and universal prefix machine. We simply write Ω and $K(y)$. Note that $K(y) = \lim_s K_s(y)$, where $K_s(y) = \min\{|\sigma| : U_s(\sigma) = y\}$. For a string y, $K(y)$ is not far greater than $|y|$, since a prefix code $\widehat{y}$ for y can serve as a description of y. Since there is such a code of length $|y| + 2\log|y|$ [3, Example 2.4], a computable upper bound is $K(y) \le |y| + 2\log|y| + c^*$ for a certain constant c^* (which will be used below). A Δ^0_2*–approximation* $(A_r)_{r\in\mathbb{N}}$ of a real A is an effective sequence of finite sets such that $A(x) = \lim_r A_r(x)$. Note that $A \le_{tt} \emptyset'$ iff $A \le_{wtt} \emptyset'$ iff the number of changes in such an approximation is recursively bounded. Reals with that property are called ω-r.e.

2 The classes and their basic properties

2.1 Far from random: the class $\mathcal{K}$.

Note that $K(|y|) \le K(y) + \mathcal{O}(1)$, since one can compute $|y|$ from y. Thus, the following expresses that the K-complexity of initial segments is as small as possible.

Definition 2.1 (Chaitin, [4]) *A real A is K–trivial via a constant b if*

$$\forall n\ K(A \upharpoonright n) \le K(n) + b.$$

Let $\mathcal{K}$ denote the class of K–trivial reals.

The intuitive meaning is "far from random", since, by Schnorr [15], A is Martin-Löf-random if, for some c, $\forall n\ K(A \upharpoonright n) \geq n - c$. Thus, A is random if for each n, $K(A \upharpoonright n)$ is close to its upper bound, and A is K–trivial if $K(A \upharpoonright n)$ is within $\mathcal{O}(1)$ of its lower bound $K(n)$. We discuss properties of K-trivial reals.

Theorem 2.2 (Chaitin, [4]) $\mathcal{K} \subseteq \Delta^0_2$.

The proof in [7] uses trees of bounded width. The Δ^0_2 tree $T_b = \{\sigma : \forall \rho \subseteq \sigma\ K(\rho) \leq K(|\rho|) + b\}$ has width at most $O(2^b)$. If A is K–trivial via the constant b, then A is a path on T_b. All paths on T_b are isolated, so $A \in \Delta^0_2$.
For reals A, B, let $A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$.

Theorem 2.3 ([7], Thm 6.2) *If $A, B \in \mathcal{K}$, then $A \oplus B \in \mathcal{K}$.*

Let $(\Theta_e)_{e \in \mathbb{N}}$ be an effective listing of all *tt*-reduction procedures. The following is easily checked.

Fact 2.4 $\{e : \Theta_e$ *total* $\&\ \Theta_e(\emptyset') \in \mathcal{K}\} \in \Sigma^0_3$.

As a consequence, there is a u.r.e. listing of all the r.e. K-trivial reals, and this class has a Σ^0_3 index set. Then, in fact, the index set is Σ^0_3-complete, since it is easy to show that any nontrivial Σ^0_3 class of r.e. sets which is closed under finite differences and contains the computable sets has a Σ^0_3-complete index set.

2.2 Low computational power: the class $\mathcal{LR}$.

Note that if $B \leq_T A$, then $K^A(y) \leq K^B(y) + \mathcal{O}(1)$. In particular, $K^A(y) \leq K(y) + \mathcal{O}(1)$. A real X is A-*random* if, for some c, $\forall n\ K^A(X \upharpoonright n) \geq n - c$. Let $\mathrm{RAND}(A)$ denote this class of reals, and let RAND denote $\mathrm{RAND}(\emptyset)$. Then $\mathrm{RAND}(A) \subseteq \mathrm{RAND}$.

Definition 2.5 (Kucera and Terwijn,[9]) *A real A is* low for random *if* $\mathrm{RAND}(A) = \mathrm{RAND}$. *In other words,* $\mathrm{RAND}(A)$ *is as large as possible. Let $\mathcal{LR}$ denote the class of low for random reals.*

Note that this is a Π^1_1 definition, and that $\mathcal{LR}$ is closed downward under $\leq_T$. Recall that A is generalized low$_1$ (in brief, GL_1) if $A' \leq_T A \oplus \emptyset'$. A result of Kucera [8, Thm. 2] implies that each low for random A is GL_1.

2.3 Both: the class $\mathcal{SK}$.

We next consider the reals which, when used as an oracle, do not decrease K.

Definition 2.6 *A is* strongly K–trivial *if* $\forall y\ K(y) \leq K^A(y) + \mathcal{O}(1)$.
Let $\mathcal{SK}$ denote this class of reals.

Note that $\mathcal{SK} \subseteq \mathcal{LR}$, since RAND$(X)$ is defined in terms of K^X. Moreover, $\mathcal{SK} \subseteq \mathcal{K}$, since $\forall n\ K^A(A \upharpoonright n) \leq K^A(n) + \mathcal{O}(1)$, and we can replace K^A by K if $A \in \mathcal{SK}$. Trivially, $\mathcal{SK}$ is closed downward under $\leq_T$.
We show that the reals A in $\mathcal{SK}$ satisfy a lowness property saying that $U^A(\sigma)$ has few possible values. A related property, being recursively traceable, was used in [20] to characterize the Schnorr low sets. Given $T \subseteq \mathbb{N}$, let $T^{[x]} = \{y : \langle y, x\rangle \in T\}$.

Definition 2.7 *(i) A r.e. set $T \subseteq \mathbb{N}$ is a* trace *if for some computable h, $\forall x\ |T^{[x]}| \leq h(x)$. We say that h is a* bound *for the trace T.*

*(ii) The real A is U-*traceable *if there is a r.e. trace T such that*

$$\forall \sigma\ (U^A(\sigma) \downarrow \Rightarrow U^A(\sigma) \in T^{[|\sigma|]}).$$

Equivalently, one can require that there is a trace S such that $\{e\}^A(e)$ is in $S^{[e]}$ in case $\{e\}^A(e)$ defined. It is not hard to show that U-traceable sets are in GL_1 (see [12]).

Proposition 2.8 *Each strongly K-trivial real is U-traceable, and low.*

Proof. For U-traceablity, suppose $A \in \mathcal{SK}$ via a constant b. Clearly, if $U^A(\sigma)$ is defined then $K^A(U^A(\sigma)) \leq K^A(\sigma) + \mathcal{O}(1)$. Since $A \in \mathcal{SK}$, this implies $\forall \sigma\ K(U^A(\sigma)) \leq K(\sigma) + \mathcal{O}(1)$. Now $K(\sigma) \leq |\sigma| + 2\log_2(|\sigma|) + \mathcal{O}(1)$, so it is sufficient to let $T^{[n]} = \{y : K(y) \leq n + 2\log_2(n) + d\}$, for an appropriate constant d (which can in fact be determined effectively from b). T is a trace because $|T^{[n]}| = \mathcal{O}(2^n n^2)$.
Since A is Δ^0_2 and GL_1, A is low. $\diamondsuit$

We summarize the properties of our classes we have seen *so far*.

	$\mathcal{K}$	$\mathcal{LR}$	$\mathcal{SK}$
Closed under $\oplus$	yes	?	?
$\leq_T$- downward closure	?	yes	yes
Index set of r.e. members	Σ^0_3-complete	?	?
Superclasses	Δ^0_2	GL_1	Low, U–traceable

3 Constructing a K-trivial real

An important tool will be the Kraft-Chaitin Theorem.

Definition 3.1 *An r.e. set $W \subseteq \mathbb{N} \times 2^{<\omega}$ is a* Kraft-Chaitin set (KC set) *if*

$$\sum_{\langle r,y\rangle \in W} 2^{-r} \leq 1.$$

Given W, for any $E \subseteq W$, let the weight of E be $wt(E) = \sum\{2^{-r} : \langle r, n\rangle \in E\}$. If $X \subseteq \mathbb{N}$, the weight (in the context of W) is

$$wt(X) = \sum_{n\in X} \sum\{2^{-r} : \langle r, n\rangle \in W\}.$$

The pairs enumerated into such a set W are called *axioms*.

Theorem 3.2 (Chaitin, [4], Thm 3.2) *From a Kraft-Chaitin set W one can effectively obtain a prefix machine M such that*

$$\forall \langle r, y\rangle \in W \exists w\ (|w| = r\ \&\ M(w) = y)$$

We say that M is a prefix machine for W*.*

For later reference, we give a quick review of the proof (based on [4, Thm 3.2]).
Proof. Let $\langle r_n, y_n\rangle_{n\in\mathbb{N}}$ be an effective enumeration of W. At stage n, we will find a string w_n of length r_n, and we set $M(w_n) = y_n$. At each stage we have a finite set D_{n-1} of unused strings. We let $D_{-1} = \{\lambda\}$.
It is useful to think of a string x as the half-open subinterval $I(x) \subseteq [0, 1)$ of real numbers whose binary representation extends x. Let z_n be the longest string in D_n of length $\leq r_n$. Choose w_n so that $I(w_n)$ is the leftmost subinterval of $I(z_n)$ of length 2^{-r_n}, i.e., let $w_n = z_n 0^{r_n - |z_n|}$. To obtain D_n, first remove z_n from D_{n-1}. If $w_n \neq z_n$ then also add the strings $z_n 0^i 1$, $0 \leq i < r_n - |z_n|$.
One checks inductively that for each $n \geq 0$ the following hold:

(a) z_n exists

(b) all the strings in D_n have different lengths

(c) $\{I(z) : z \in D_n\} \cup \{I(w_i) : i \leq n\}$ is a partition of $[0, 1)$

We prove (a) for $n \geq 0$, assuming (b) and (c) for $n - 1$ (these are trivial statements for $n = 0$). If z_n fails to exist, then r_n is less than the length of each string in D_{n-1}, so that $2^{-r_n} > \sum\{2^{-|x|} : x \in D_{n-1}\}$ by (b) for $n - 1$. Then $\sum_{i=0}^{n} 2^{-r_i} > 1$ by (c) for $n - 1$. This contradicts the assumption that W is a KC-set.
Next, (b) for n holds if $w_n = z_n$. Otherwise $|z_n| < |w_n|$ but also $|w_n|$ is less than the next shortest string in D_{n-1}, so (b) holds by the definition of D_n. (c) is satisfied by the definition of D_n. $\Diamond$

Suppose $A(x) = \lim_r A_r(x)$ for a Δ^0_2–approximation (A_r). We will give a sufficient condition on (A_r) for the K–triviality of A (based on [7]). Then we meet this condition in order to construct an non-computable K-trivial r.e. set.
To show A is K-trivial, we wish to enumerate a KC set W such that, for each $y \in \mathbb{N}$, $\langle K(y) + 1, A \upharpoonright y\rangle \in W$. Since neither $K(y)$ nor $A \upharpoonright y$ are known, we have to work with approximations at stages r. Firstly, if $y = r$, or $y < r$ and $K_r(y) < K_{r-1}(y)$, then we put an axiom $\langle K_r(y), A_r \upharpoonright y\rangle$ into W. This contributes at most weight $1/2$. Secondly, when $x < r$ is minimal such that $A_{r-1}(x) \neq A_r(x)$, then we put an axiom $\langle K_r(y) + 1, A \upharpoonright y\rangle$ into W for each y, $x < y \leq r$. In this case, the axioms for descriptions of $A_{r-1} \upharpoonright y$ we enumerated previously are "wasted". Thus, each A-change carries a cost, the weight wasted

on descriptions of previous strings $A_{r-1} \upharpoonright y$. When we enumerated the axiom $\langle K_s(y)+1, A_s \upharpoonright y\rangle \in W$ at stage $s < r$, we spent $2^{-(K_s(y)+1)}$. Since $2^{-K_s(y)} \leq 2^{-K_r(y)}$, an upper bound for the cost of changing $A(x)$ is

$$c(x,r) = 1/2 \sum_{x<y\leq r} 2^{-K_r(y)}.$$

Note that $c(x,r)$ is nondecreasing in r, $\lim_r c(x,r) \leq 1/2$ for each x, and $\lim_x \lim_r c(x,r) = 0$. Our sufficient condition for K-triviality implies that the sum of the costs of all changes is at most $1/2$.

Proposition 3.3 *Suppose that $A(x) = lim_r A_r(x)$ for a Δ^0_2–approximation (A_r) such that*

$$S = \sum \{c(x,r) : r > 0 \;\&\; x \textit{ is minimal s.t. } A_{r-1}(x) \neq A_r(x)\} \leq 1/2. \qquad (1)$$

Then A is K-trivial.

Proof. We enumerate a KC set W in stages s:
Put the axiom $\langle K_s(w)+1, A_s \upharpoonright w\rangle$ into W in case

(a) $s = w$, or

(b) $s > w$ and $K_s(w) < K_{s-1}(w)$, or

(c) $A_{s-1} \upharpoonright w \neq A_s \upharpoonright w$.

To show W is a KC set, suppose an axiom $\langle K_s(w)+1, A_s \upharpoonright w\rangle$ is put into W at stage s.
Stable case. $\forall r > s\; A_s \upharpoonright w = A_r \upharpoonright w$. The contribution of such axioms is at most $\Omega/2$.
Change case. $\exists r > s\; A_s \upharpoonright w \neq A_r \upharpoonright w$, where r is chosen minimal. Since $2^{-K_s(w)} \leq 2^{-K_r(w)}$, the contribution of such axioms for a single r is at most $c(x,r)$, where x is minimal such that $A_{r-1}(x) \neq A_r(x)$ (so that $x < w$). Since $S \leq 1/2$, the total contribution is at most $1/2$.
Let M_e be the prefix machine for W. Then, for each w, $K(A \upharpoonright w) \leq K(w)+e+1$. For, given w, let $s > 0$ be greatest such that $A_{s-1} \upharpoonright w \neq A_s \upharpoonright w$. If s does not exist or $s \leq w$ then the axioms in (a) at stage w cause this inequality. Otherwise, (c) causes $K_s(A_s \upharpoonright w) \leq K_s(w)+e+1$. If $K_t(w) < K_{t-1}(w)$ for some $t > s$, then we maintain the inequality via (b). $\diamondsuit$

Recall that an r.e. set A is *promptly simple* if A is co-infinite and, for each e,

$$|W_e| = \infty \Rightarrow \exists s \exists x\, [x \in W_{e,s} - W_{e,s-1} \;\&\; x \in A_s - A_{s-1}].$$

Theorem 3.4 ([7]) *There is a promptly simple K-trivial set A.*

Proof. Define an enumeration (A_r) as follows. Let $A_0 = \emptyset$. At stage $s > 0$, for each $e < s$, if $A_{s-1} \cap W_{e,s} = \emptyset$ and there is $x \ \geq 2e$ such that $x \in W_{e,s}$ and $c(x,s) \leq 2^{-(e+2)}$, then put x into A_s.
The condition (1) is satisfied since we need to make at most one change for each e. We verify that A is promptly simple. If W_e is infinite, there is an $x \geq 2e$ in W_e such that $c(x,s) \leq 2^{-(e+2)}$ for all $s > x$. Thus there is a unique x in $A \cap W_e$. Since $c(x,s)$ is nondecreasing in s, we enumerate x into A at the stage where x appears in W_e. Thus A is promptly simple. ♢

One can combine this technique with the Robinson guessing method for low sets (see [18]) to obtain the following.

Theorem 3.5 ([12]) *For each low r.e. set B, there is an r.e. $A \in \mathcal{K}$ such that $A \not\leq_T B$.*

The condition (1) is very restrictive. For instance,

Fact 3.6 *A Δ^0_2–approximation $A_r(y)$ satisfying (1) changes at most $O(y^2)$ many times.*

For, given $y < r$, when $A_{r-1}(y) \neq A_r(y)$, then S increases by at least $2^{-K_r(y)}$. Since $K_r(y) \leq 2\log_2(y) + \mathcal{O}(1)$, $2^{-K_r(y)} \geq \mathcal{O}(1)y^{-2}$. Since $S \leq 1/2$, the required bound on the number of changes follows.
A modification of the proof of Theorem 3.4 yields the existence of a promptly simple set in $\mathcal{SK}$: we work with the cost function

$$c(x,r) = 1/2 \sum \{2^{-|\sigma|} : \ U^A(\sigma)\downarrow [r-1] \ \& \ x < \ \textit{the use of this computation}\}.$$

Running the construction in the proof of Theorem 3.4 with this new cost function, we obtain an r.e. set A in $\mathcal{SK}$, via the KC set W defined as follows: when a new computation $U^A(\sigma) = y$ appears, then enumerate $\langle |\sigma| + 1, y\rangle$ into W. To see that W is a KC set, note that the computations with a stable A contribute a weight of at most $\Omega^A/2$, while the others contribute at most $1/2$. Our enumeration into W causes $K(y) \leq K^A(y) + \mathcal{O}(1)$ for each y.
The cost function method in itself does not provide an injury free construction. For instance, one can define a cost function encoding the restraint of the usual lowness requirements $\exists^\infty s \ \{e\}^A(e)\downarrow [s\text{-}1] \ \Rightarrow \ \{e\}^A(e)\downarrow$ in the canonical construction of a low simple set [18, Thm. VII.1.1]. If $\{e\}^A(e)$ converges at stage $s-1$, then one defines $c(x,s) = \max\{c(x,s-1), 2^{-(e+2)}\}$ for each x below the use of $\{e\}^A(e)$. Then this computation can only be destroyed by the finitely many simplicity requirements which are allowed to spend $2^{-(e+2)}$.
The construction in the proof of Theorem 3.4 can be considered injury free because $c(x,s)$ is defined in advance, not depending on A_{s-1}.

4 Low for random reals are strongly K–trivial

Our first main result is $\mathcal{LR} \subseteq \mathcal{SK}$. We apply the usual topological notions for Cantor space 2^ω. For a string y, $[y]$ denotes the clopen set $\{X : y \subseteq X\}$ (so that $\mu[y] = 2^{-|y|}$). An open set V is identified with the set of strings y such that $[y] \subseteq V$. A set $R \subseteq 2^\omega$ is Σ^0_1 if $R = \bigcup_{y \in W}[y]$ for some r.e. set of strings W (in particular, R is open).
We first provide two preliminary results: an oracle version of the Kraft-Chaitin Theorem, and a characterization of the low for random sets.

Definition 4.1 *Consider an r.e. set $L \subseteq \mathbb{N} \times 2^{<\omega} \times 2^{<\omega}$. The elements $\langle r, z, \gamma\rangle$ of L (also called axioms) will be written in the form $\langle r, z\rangle^\gamma$. L is called an oracle Kraft–Chaitin (oracle KC) set if, for all $\rho \in 2^{<\omega}$,*

$$L^\rho = \{\langle r, y\rangle : \exists \gamma \subseteq \rho \ \langle r, z\rangle^\gamma \in L\} \tag{2}$$

is a Kraft–Chaitin set.

Proposition 4.2 *From an index for an oracle KC set L, one can effectively obtain an index d for an oracle prefix machine M^X_d such that*

$$\forall X \subseteq \mathbb{N}\ \forall \langle r, z\rangle^\gamma \in L\ [\gamma \subseteq X \ \Rightarrow\ \exists w(|w| = r \ \&\ M^X_d(w) = z)].$$

Proof. For each real X, $L^X = \{\langle r, y\rangle : \exists \gamma \subseteq X\ \langle r, z\rangle^\gamma \in L\}$ is a KC set relative to X. Applying the construction in the proof of Theorem 3.2, we obtain an index d (which only depends on an r.e. index for L) such that M^X_d is an oracle prefix machine as desired. $\diamondsuit$

Theorem 4.3 (with F. Stephan, also see [13]) *A is low for random $\Leftrightarrow$ $\exists b \in \mathbb{N}\ \exists R \subseteq 2^\omega$ ($R \in \Sigma^0_1$ & $\mu R < 1$ &*

$$\forall z \in 2^{<\omega}[K^A(z) \le |z| - b \ \Rightarrow\ [z] \subseteq R]). \tag{3}$$

Proof. To gain insight we reformulate the condition in Proposition 4.3. For each $X \subseteq \mathbb{N}$ and $b \in \mathbb{N}$, let $R^X_b = \{z : \exists w \subseteq z\ K^X(w) \le |w| - b\}$, so that $(R^X_b)_{b\in\mathbb{N}}$ is a universal Martin-Löf test relative to X (namely, $\bigcap_b R^X_b = 2^\omega - \mathrm{RAND}(X)$). Then A is low for random iff $\bigcap_b R^A_b \subseteq \bigcap_n S_n$ for some (unrelativized) Martin Löf test (S_n). By a method of Kucera (see e.g. [9, Lemma 1.5]), this is equivalent to $\bigcap_b R^A_b \subseteq R$ for some Σ^0_1 set $R \subseteq 2^{<\omega}$ such that $\mu R < 1$ (to obtain (S_n), one "iterates" R). The condition in Proposition 4.3 states that already for some b, R^A_b is contained in such a set R. The direction from right to left follows.
To prove the converse direction, suppose $A \in \mathcal{LR}$, and fix a Σ^0_1 set $R_0 \subset 2^\omega$ of measure less that 1 containing all the nonrandom sets (say R_0 is the first component of a universal Martin Löf test). We claim that, for some string x such that $[x] \not\subseteq R_0$ and some $m \in \mathbb{N}$,

$$\forall y \supseteq x\ (K^A(y) \le |y| - m \ \Rightarrow\ [y] \subseteq R_0). \tag{4}$$

Suppose otherwise. Define a sequence of strings (y_m) as follows: let y_0 be the empty string, and let y_{m+1} be some proper extension of y_m such that $K^A(y) \leq |y| - m$ but $[y_{m+1}] \not\subseteq R_0$. Then $Y = \bigcup_m y_m$ is not A–random, but Y is random since $Y \notin R_0$.
Now fix x, m such that (4) holds, and let $R = \{z : xz \in R_0\}$. Then $R \neq 2^\omega$. Since the Π^0_1 set R^c is contained in the random sets and no random set can be in a Π^0_1 set of measure 0, $\mu R < 1$.
Since $K^A(xz) \leq K^A(x) + K^A(z) + \mathcal{O}(1)$, letting $b = K^A(x) + m + \mathcal{O}(1)$, we obtain, for each y,

$$K^A(z) \leq |z| - b \;\Rightarrow\; K^A(xz) \leq |xz| - m \;\Rightarrow\; [xz] \subseteq R_0 \;\Rightarrow\; [z] \subseteq R.$$

$\diamondsuit$

Theorem 4.4 *Any low for random real is strongly K-trivial.*

Proof. Suppose A is low for random. The proof will be uniform: a constant for the strong K–triviality of A is obtained effectively in b, R, q, where b, R are as in (3) and $q > 0$ is a rational such that $\mu R \leq 1 - q$. We define an effective sequence $(T_s)_{s\in\mathbb{N}}$ of finite subtrees of $2^{<\omega}$ (viewed as characteristic functions) such that the limit tree T given by $T(\gamma) = \lim_s T_s(\gamma)$ exists. The real A is a path of T, and each path of T is K-trivial, via a constant which can be determined effectively from b, R and q. To ensure this, we enumerate a KC set W such that, for some constant c determined below, if $\gamma \in T$ and $K^\gamma(y) = p$, then $\langle p + c, y\rangle \in W$ (so that $K^\gamma(y) \leq p + \mathcal{O}(1)$ by the Kraft-Chaitin Theorem). Of course, the condition "$\gamma \in T$ and $K^\gamma(y) = p$" is only Δ^0_2, so we need to work with approximations. At stage t, if $\gamma \in T_t$, $K^\gamma_t(y) = p$ and some further conditions hold, then we plan to enumerate $\langle p + c, y\rangle$ into W. While defining (T_s) we enumerate an auxiliary oracle KC set L, which ensures that we do not make too many errors in this enumeration of W (putting axioms for strings $\gamma \notin T$), so that W is indeed a KC set. Our enumeration of L at stage t exploits (3) in a way which makes it harder for a string $\gamma \in T_t$ to reappear on T_s at a later stage s.
Preliminaries and the general framework. We may assume that an index d for the oracle prefix machine M_d corresponding to L is given (d can even be obtained effectively in the parameters b, q and a Σ^0_1–index for R). The reason is that, for any index of an r.e. set $Q \subseteq \mathbb{N} \times 2^{<\omega} \times 2^{<\omega}$, we can effectively obtain an index for an oracle KC set $\widetilde{Q}$ such that $\widetilde{Q} = Q$ in case Q already is an oracle KC set. Let d be an index for the oracle prefix machine effectively obtained from $\widetilde{Q}$ via Proposition 4.2. Our construction will effectively produce an oracle KC set L from d (for any $d \in \mathbb{N}$). By the Recursion Theorem with parameters, we can assume that $Q = L$. Thus Q is an oracle KC set, and M_d is a machine for L.
Let $c \in \mathbb{N}$ be least such that $c \geq b + d$ and $2^{-c} \leq q/2$. We define T_t and L_t by recursion on t. For strings $\gamma \in T_t$ we will enumerate finitely many axioms $\langle r, z\rangle^\gamma$, $r = |z| - c$, into L at stages t. Such an enumeration will cause $z \in R_s$ in case $\gamma \in T_s$ at a later stage s.

Let T_0 contain only the empty string and let $L_0 = \emptyset$. Suppose $t > 0$ and T_s and L_s have been determined for $s < t$. We define T_t by a subrecursion on the length of strings. We begin by putting the empty string into T_t. Suppose currently the string γ, $|\gamma| < t$, is a leaf of T_t. For $i = 0, 1$, let $s_i < t$ be the greatest s such that $\gamma\widehat{\ }i \in T_s$ or $s_i = 0$ if there is no such stage. For $i = 0$ or $i = 1$, if

$$\forall z[\langle |z| - c, z\rangle^{\gamma\widehat{\ }i} \in L_{s_i} \Rightarrow [z] \subseteq R_t]],$$

then put $\gamma\widehat{\ }i$ into T_t.
It remains to define L_t, by enumerating finitely many axioms at stage t. We first show that, no matter how we do this, as long as $L = \bigcup_t L_t$ is an oracle KC set (and $Q = L$), A will be a path of T. In variants of the construction (Theorem 6.3, (i) below), the limit tree may fail to exists, but we can as well work with the Π^0_2-tree $T = \{\gamma : \exists^\infty t\ \gamma \in T_t\}$.

Lemma 4.5 *Suppose L is an oracle KC set and M_d is an oracle machine for L in the sense of 4.2. Then each real A satisfying* (3) *is a path of T.*

Proof. If $\langle |z| - c, z\rangle^\gamma \in L$ then, since M_d is a machine for L, for each set X extending γ, $M_d^X(w) = z$ for some w of length $|z| - c$. Hence $U^X(0^{d-1}1w) = z$ and $K^X(z) \le |z| - b$ (recall that $c = b + d$).
Suppose A satisfies (3). We show by induction on m that $A \upharpoonright m$ is on T for each m. We may suppose that $m > 0$. By inductive hypothesis, there are infinitely many s such that $A \upharpoonright m - 1 \in T_s$. Suppose for a contradiction that t is greatest such that $\gamma = A \upharpoonright m \in T_t$. Then, by the above remarks (for $\gamma = A \upharpoonright m$ and $X = A$), there is $v > t$ such that $[z] \subseteq R_v$ for each of the finitely many z such that $\langle |z| - c, z\rangle^\gamma \in L_t$. Then at a stage $s \ge v$ such that $A \upharpoonright m - 1 \in T_s$, we put γ into T_s, contrary to the choice of t. $\diamondsuit$

For each γ, if $g = \sum\{2^{-r} : \langle r, z\rangle^\gamma$ *enters L at s*$\}$, then we say we *put measure g on γ at s*. We view this as a cost, as it conflicts with our goal to make L an oracle KC set, which requires that, for each ρ, the total measure put on substrings of ρ be at most 1.
Some more intuition. Recall that if $\gamma \in T_t$ and $K_t^\gamma(y) = p$, then we want to enumerate $\langle p+c, y\rangle$ into W. A *strategy* α is a triple $\langle \sigma, y, \gamma\rangle$, where $\sigma, y, \gamma \in 2^{<\omega}$, $|y| < |\gamma|$ and $|\sigma| \le |y| + 2\log|y| + c^*$ (σ will be a U^γ-description of y). We start α at a stage t which is least such that $\gamma \in T_t$ & $U_t^\gamma(\sigma) = y$, and γ is the shortest among such strings at t.
Let $p = |\sigma|$. Simplifying, the idea is to choose a clopen set $C = C(\alpha)$, $\mu C = 2^{-(p+c)}$, which is disjoint from R and the sets chosen by other strategies. The strategy α puts an axiom $\langle |z| - c, z\rangle^\gamma$ into L for each string $z \in C$ of minimal length. If at a stage $s > t$, once again $\gamma \in T_s$, then $C \subseteq R_s$. At this stage, we put $\langle p+c, y\rangle$ into W. Using that $\mu R \le 1$ and that the sets belonging to different strategies are disjoint, we want to argue that W is a KC set. Moreover, L is an oracle KC set, since the measure put on any substring γ of a string ρ is a sum

of quantities $2^{-|\sigma|}$, where $U^\gamma(\sigma) = y$ for some y. Then each set L^ρ in (2) is a KC-set.

The problem is to make the sets C chosen by strategies at different stages disjoint. Suppose $\beta \neq \alpha$ is a strategy which chose its set $C(\beta)$ at a stage before stage t. If β, or rather, its last component, has reappeared on the tree, then $C(\beta) \subseteq R_t$, so there is no problem since α chooses its set disjoint from R. However, if β has not reappeared (and it possibly never will), then β keeps away its set from assignment to other strategies. The solution is to build up the set $C(\alpha)$ in small pieces D_α, whose measure is a fixed fraction of $2^{-(p+c)}$. Recall $\alpha = \langle \sigma, y, \gamma \rangle$ and $p = |\sigma|$. If α always reappears after assigning such a set, then eventually $C(\alpha)$ reaches the required measure $2^{-(p+c)}$, in which case we are allowed to enumerate the axiom $\langle p + c, y \rangle$ into W. Otherwise, α only keeps away one single set D_α, whose measure is so small that the union (over all strategies) of sets kept away is at most $q/2$. Thus there is always a clopen set of measure $\geq 1 - \mu R - q/2 \geq q/2$ available for other strategies.

For a strategy $\alpha = \langle \sigma, y, \gamma \rangle$, let $n_\alpha \geq |\sigma|$ be a natural number assigned to α in some effective one-one way.

Inductive definition of L_t and of the sets $C_t(\alpha)$.

Let $L_0 = \emptyset$ and $C_0(\alpha) = \emptyset$ for each strategy α. Suppose $t > 0$, and T_s $(s \leq t)$ and L_s $(s < t)$ have been defined.

1. For each $\gamma \in T_t$, if $\alpha = \langle \sigma, y, \gamma \rangle$ is a strategy, $V_t^\gamma(\sigma) = y$, $V_{t-1}^\gamma(\sigma)$ is undefined and, for σ, y, the string γ is the shortest such string, then start the strategy $\langle \sigma, y, \gamma \rangle$.

2. For each strategy $\alpha = \langle \sigma, y, \gamma \rangle$ which is now running, if $\gamma \in T_t$, then do the following. If $\mu C_{t-1}(\alpha) = 2^{-(|\sigma|+c)}$, then let α end. For the remaining such strategies α, pick pairwise disjoint clopen sets D_α such that $\mu D_\alpha = 2^{-(n_\alpha+c)}$, and

$$D_\alpha \cap R_t = \emptyset \ \& \ \forall \beta \neq \alpha \ [D_\alpha \cap C_{t-1}(\beta) = \emptyset]$$

(we will verify that this is possible). Put D_α into $C(\alpha)$ and, for each string $z \in D_\alpha$ which is minimal under inclusion of strings, enumerate an axiom $\langle |z| - c, z \rangle^\gamma$ into L (this puts measure 2^{-n_α} on γ). We say α *acts via* D_α. This completes the definition of L_t.

Verification. Note that, by definition of T_t, for each $\alpha = \langle \sigma, y, \gamma \rangle$, if $\gamma \in T_t$, then $C_{t-1}(\alpha) \subseteq R_t$. Thus for each strategy β, $\mu(C_{t-1}(\beta) - R_t) \leq 2^{-(n_\beta+c)}$. Then the union S of all such sets, which represents the strings outside R being kept away for assignment for other strategies, has measure at most $q/2$ (recall that $2^{-c} \leq q/2$). Thus we always have a clopen set of measure at least $q/2$ at our disposal at a stage t, which suffices for the strategies α which want to choose sets D_α at stage t.

Let $C(\alpha) = \bigcup_t C_t(\alpha)$. Clearly $\alpha \neq \beta$ implies $C(\alpha) \cap C(\beta) = \emptyset$.

To see that L is an oracle KC set, fix ρ. We need to show that, for each ρ, $\sum_{\langle r,z \rangle \in L^\rho} 2^{-r} \leq 1$, where L^ρ is defined in (2). For each $\gamma \subseteq \rho$, a strategy $\alpha = \langle \sigma, y, \gamma \rangle$ puts measure at most $2^{-|\sigma|}$ on γ, since the maximum measure $C(\alpha)$ can reach is $2^{-(|\sigma|+c)}$. Then, the total put on all substrings of ρ is bounded by

$\mu(\mathrm{dom}(U^\rho)) \leq 1$. (Note that we did not assume M_d is an oracle machine for L, as required. Such an assumption is only needed in the proof of Fact 4.5.)
Defining a KC set W which shows that each path of T is K-trivial. We first verify that $\lim_s T_s(\gamma)$ exists. There are only finitely many strategies $\alpha = \langle \sigma, y, \gamma \rangle$. Each time such a strategy acts and then γ reappears on the tree, we increased $\mu C(\alpha)$ by at least $2^{-(n_\alpha + c)}$. So eventually the strategy ends, and the limit exists.
Define W as follows. For each $\alpha = \langle \sigma, y, \gamma \rangle$, if α ends at t, then put $\langle |\sigma| + c, y \rangle$ into W. To verify that W is a KC set, we note that

$$\sum_t \sum \{2^{-(|\sigma|+c)} : \langle |\sigma| + c, y \rangle \text{ is put into } W \text{ via } \langle \sigma, y, \gamma \rangle \text{ at stage } t\} \leq \mu R.$$

For, when α ends at t then $\mu C_{t-1}(\alpha) = 2^{-(|\sigma|+c)}$ and $C_{t-1}(\alpha) \subseteq R$. Since the sets $C(\alpha)$ are pairwise disjoint, the required inequality holds.
Let M_e be a prefix machine for W according to the Kraft-Chaitin Theorem 3.2. We claim that, for each path X of T and each string y, $K(y) \leq K^X(y) + c + e$. For choose a shortest U^X-description σ of y, and choose $\gamma \subseteq X$ shortest such that $|\gamma| > y$ and $U^\gamma(\sigma) = y$. Then at some stage t, we start the strategy $\langle \sigma, y, \gamma \rangle$. Since $\gamma \in T$, the strategy ends and we put $\langle |\sigma| + c, y \rangle$ into W, causing $K(y) \leq K^X(y) + c + e$.
We obtained the constant $c + e$ effectively from the parameters b, R and q, since we used the Recursion Theorem with parameters in the proof. $\diamondsuit$

Theorem 2.2 also answers [2, Problem 4.4], first asked in [9, p.1400].

Corollary 4.6 *Any low for random set is low, and hence Δ^0_2.*

5 K-trivial reals

We prove that the class $\mathcal{K}$ is closed downward under Turing reducibility, and give the modifications needed to prove that actually $\mathcal{K} = \mathcal{SK}$. The first version of the proof also shows that Proposition 3.3 in fact provides a characterization of the K-trivial sets. This yields some corollaries which further restrict our common lowness property.

Theorem 5.1 *If A is K-trivial and $B \leq_T A$, then B is K-trivial.*

As noted in [7], the corresponding fact is easily verified for weak truth table reducibility: Suppose $B \leq_T A$ via a Turing reduction Γ such that the use of Γ is bounded by a recursive function g. Then, up to constants,

$$K(B \upharpoonright n) \leq K(A \upharpoonright g(n)) \leq K(g(n)) = K(n).$$

Hirschfeldt and Nies modified the proof of Theorem 5.1 and obtained a stronger result. However, the first version of the proof is also needed for the characterization of $\mathcal{K}$.

Theorem 5.2 (with Hirschfeldt) *Each K-trivial real A is strongly K-trivial.*

We note the modifications needed to obtain a proof of Theorem 5.2 in brackets [...].
Proof. Suppose A is K-trivial via a constant b. For Theorem 5.1, let $B = \Gamma^A$, where Γ is a Turing functional whose use is nondecreasing in the input. Let $(A_r)_{r\in\mathbb{N}}$ be a Δ^0_2-approximation of A. For each s, one can effectively determine an $f(s) > s$ such that $\forall n < s\ K(A \upharpoonright n) \le K(n) + b\ [f(s)]$, i.e., the inequality holds at stage $f(s)$. Let $s_0 = 0$ and $s_{i+1} = f(s_i)$. The construction is restricted to *stages* in $\{s_i : i \in \mathbb{N}\}$. We use italics to emphazise this. In the following, s, t, u always will denote such *stages*. We may modify the approximation (A_r) so that that $A_r(x) = A_{s_i}(x)$ for all r, $s_i \le r \le s_{i+1} - 1$. We say that $A(x)$ *changes at s* if $A_{s-1}(x) \ne A_s(x)$.
We will determine a KC set W in order to show that B is K-trivial [A is strongly K-trivial]. We also enumerate an auxiliary KC set L to exploit the hypothesis that A is K-trivial. For each n, at most one axiom $\langle r_n, n\rangle$ will be enumerated into L. Recall that, for $E \subseteq \mathbb{N}$, $wt(E) = \sum_{n\in E} 2^{-r_n}$.
As in the proof of Theorem 4.4, we may assume that an index d for a machine M_d is given, and we can think of M_d as being a prefix machine for L: For any index for an r.e. set $Q \subseteq \mathbb{N} \times 2^{<\omega}$, we can effectively obtain an index for a KC set $\widetilde{Q}$ such that $\widetilde{Q} = Q$ in case Q already is a KC set. Let M_d be the machine effectively obtained from $\widetilde{Q}$ via the Kraft-Chaitin Theorem. Our construction effectively produces a KC set L from d. Thus, if $Q = L$, then Q is a KC set and M_d is a machine for L. Of course, first we have to show that L is a KC-set, in the absence of any assumption on d.
Let $c = b + d$ and $k = 2^{c+1}$. (Then, putting $\langle r, n\rangle$ into L causes $K(n) \le r + d$ and hence $K(A \upharpoonright n \le r + c$, assuming M_d is a machine for L.)
To gain some intuition, we first give a direct proof that no K–trivial set A is satisfies $\emptyset' \le_{wtt} A$ (which also follows from the downward closure of $\mathcal{K}$ under $\le_{wtt}$ and the fact that the wtt-complete set Ω is not K–trivial). Suppose $\emptyset' \le_{wtt} A$. Now we *build* an r.e. set B, and by the Recursion Theorem we can assume we are given a total wtt-reduction Γ such that $B = \Gamma^A$, whose use is bounded by a computable function g. We wait till $\Gamma^A(k)$ converges, let $n = g(k)$ and put the single axiom $\langle r, n\rangle$ into L, where $r = 1$. Our total investment is 1/2. Each time the opponent has a U-description of $A \upharpoonright n$ of length $\le r + c$ we force $A \upharpoonright n$ to change, by putting into B the largest number $\le k$ which is not yet in B. If we reach $k + 1$ such changes, then his total investment is $(k + 1)2^{-(r+c)} > 1$, contradiction.
In the proof of Hirschfeldt's more general result that K-trivial reals are T-incomplete (see [7, Thm 4.1]), there is no recursive bound on the use of $\Gamma^A(k)$. The problem now is that the opponent might, before giving a description of $A_s \upharpoonright n$, move this use beyond n, thereby depriving us of the possibility to cause further changes of $A \upharpoonright n$. The solution is to carry out many attempts in parallel, based on different computations $\Gamma^A(m)$. Each time the use of such a computation changes, the attempt is cancelled. What we placed in L for this attempt now becomes "garbage". We have to ensure that the weight of the

garbage does not build up too much, otherwise L is not a KC set.
Ingredients. Our proof builds on three main ideas. The essence of the first one, and some elements of the third, first appeared in the proof of Hirschfeldt's result. Roughly speaking, for an axiom $\langle r, n\rangle \in L$, either n reaches a k–set (as defined below) or n is garbage. The weight of numbers of either type is at most 1/2. The first idea is present even in the proof of *wtt*–incompleteness above. The third is a way to deal with the garbage. Both together ensure that L is a KC set.
The second idea is needed to identify W. We use a tree of runs of procedures, where the branchings are determined by U–descriptions [U^A-descriptions]. Each branching nodes emulates the construction of a [strongly] K–trivial real. That is, $B \in \mathcal{K}$ [$A \in \mathcal{SK}$] for the same reason as in the proof of Theorem 3.4 [in the construction near the end of Section 3]. We discuss the ideas in detail.
1. The concept of a j–set. For $1 \leq j \leq k$, we say that a finite set $E \subseteq \mathbb{N}$ is a *j-set* at *stage* t if, for all $n \in E$, at some *stage* $u < t$ an axiom $\langle r_n, n\rangle$ went into L and now there are j distinct strings z of length n such that $K_t(z) \leq r_n + c$. (Intuitively speaking, the opponent provides j times as many descriptions as we, though his ones may be by up to c longer.) A r.e. set with an enumeration $E = \bigcup E_t$ is a j-set if E_t is a j-set at each *stage* t. In our construction, the strings z will have the form $A_s \upharpoonright n$ at certain *stages* s, $u \leq s \leq t$.

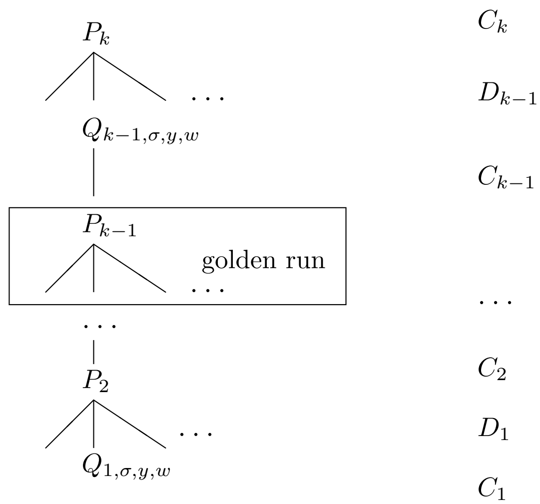
Fact 5.3 *If the r.e. set E is a k–set, then $wt(E) \leq 1/2$.*

Proof. For all $n \in E$, there is an axiom $\langle r_n, n\rangle$ in L and there are k distinct strings z of length n such that $K(z) \leq r_n + c$. Since for each $n \in E$, we have descriptions of k distinct strings of length n,

$$1 \geq \mu(\mathrm{dom}(V)) \geq k \textstyle\sum_{n \in E} 2^{-(r_n+c)} = k2^{-c}wt(E).$$

Because $k = 2^{c+1}$, this implies $wt(E) \leq 1/2$. Note that we did not assume here that M_d is a machine for L. $\diamondsuit$

2. The golden run and indexing procedures by descriptions
As in the proof of *wtt*-incompleteness, we attempt to enumerate a k-set C_k of weight 1. Now we use a tree of runs of procedures. The successor relation is given by recursive calls. Each run of a procedure enumerates a set and has a goal, the weight this set has to reach so that the run can end. Runs may also be cancelled by runs of procedures which are above this run on the tree. The root procedure is P_k, which has goal 1. It calls several procedures of type Q_{k-1}. These call a single procedure P_{k-1} and so on till we reach the bottom level, consisting of procedures of type Q_1. All procedures have further indices or parameters, discussed below. The failure of P_k to reach $wt(C_k) = 1$ implies that there is a level i and a run of a procedure of type P_i which does not return, though all its subprocedures (of type Q_{i-1}) return unless they are cancelled. Using this "golden run" we are able to define a KC set W as desired. However, to be "golden" is merely a Π^0_2 property of runs.



To reach C_k, a number has to pass through j-sets C_j ($1 \leq j < k$) and j-sets D_j ($1 \leq j < k$), $C_1 \supseteq D_1 \supseteq \cdots \supseteq D_{k-1} \supseteq C_k$. The procedures of type P_i ($1 < i \leq k$) enumerate numbers n from D_{i-1} into C_i (thereby adding a further string z of length n as in the definition of j-sets), and the procedures of type Q_j ($1 \leq j < k$) enumerate C_1 for $j = 1$, and move numbers from C_j to D_j. Thus C_1 is just the right domain of L.

A main idea is to index the procedures of type Q_j by descriptions σ, and also by the object y being described and a certain A–use w. Each procedure P_i may call procedures $Q_{i-1,\sigma,y,w}$ for all σ such that $U(\sigma) = y$ [$U^A(\sigma) = y$]. Ultimately we want to show $K(B \upharpoonright y) \leq |\sigma| + \mathcal{O}(1)$ [$K(y) \leq |\sigma| + \mathcal{O}(1)$], provided the run of P_i is a golden one, since this would make B K-trivial [it would make A strongly K-trivial]. We prove the K–triviality of B by emulating the construction of a K–trivial set. The failure of P_i to reach its goal means that there are few A-changes, hence the weight of axioms placed in W for which the change case in Proposition 3.3 applies is small.

To give an outline of the procedures, let us pretend that $k = 2$. Now the single run of the root procedure P_2 attempts to enumerate a 2-set C_2 of weight 1, but never completes this task. It proceeds as follows. Each string σ is *available* in the beginning. At a *stage* s, for each available σ, if $U(\sigma) = y$ and $\Gamma^A(y')$ converges for each $y' < y$ [if $U^A(\sigma) = y$], then declare σ *unavailable.* Let $w = \gamma^A(y-1)$. Start a procedure $Q_{1,\sigma,y,w}$ attempting to obtain a 1–set D, $w \leq \min(D)$, of weight 2^{-r}, where $r = |\sigma|$. In this simplified outline, D is a singleton. The procedure $Q_{1,\sigma,y,w}$ picks a large number $n > w$ and puts the axiom $\langle r, n\rangle$ into L. Then at some later *stage* s, $D = \{n\}$ is a 1-set (i.e., we see a description of $A_s \upharpoonright n$ of length $\leq r + c$). If $A \upharpoonright w$ has not changed by stage s, then $Q_{1,\sigma,y,w}$ returns the set D. Now P_2 waits for an $A \upharpoonright w$ change, since this would make D a 2-set. If it obtains the change, then it puts D into C_2 and declares σ available again. If $A \upharpoonright w$ changes before we see such a description, we cancel the run of $Q_{1,\sigma,y,w}$ and declare σ available.

The KC set W is defines as follows. When a run $Q_{1,\sigma,y,w}$ returns at stage s, then put the axiom $\langle|\sigma|+1, B_s \upharpoonright y\rangle$ into W [put $\langle|\sigma|+1, y\rangle$ into W]. Note that $\Gamma^A \upharpoonright y$ did not change, hence still $w = \gamma^A(y-1)$. We have the same two cases as in the construction of a K-trivial set in Proposition 3.3.
Stable case. $A \upharpoonright w$ is stable from s on. Then $B \upharpoonright y$ is stable [the computation $U^A(\sigma) = y$ is stable]. So the axiom is as desired, assuming that σ is a shortest description. For each σ, this case can occur at most once, so the total contribution to W in this case is $\leq \Omega/2$ [$\leq \Omega^A/2$].
Change case. $A \upharpoonright w$ changes after s. Then $B \upharpoonright y$ may change [$U^A(\sigma) = y$ may be destroyed], in which case the axiom we placed into W is wasted. However, its weight is added to C_2, so that in the construction, P_2 makes progress towards reaching its goal. Assuming that $wt(C_2)$ never exceeds $1/2$, the contribution of those axioms is bounded by $1/2$.
We now discuss the general case where $k = 2^{c+1}$. At each stage we have a finite tree with 2k levels of runs of procedures. The leaves are the runs of procedures of type Q_1, which act in the way indicated above. Each n enumerated by such a procedure into C_1 at stage t corresponds to a unique run of a procedure at each level at stage t (we say n *belongs* to that run). Since n is chosen large, it is bigger than the parameter w of any run of a Q-type procedure n belongs to. Thus $A \upharpoonright w$-changes contribute to the aim that n reaches the k–set C_k.
A procedure P_i has a parameter p, its *goal*, which is the weight it wants to move from D_{i-1} to C_i. Similarly, a procedure Q_j has goal q, the weight it wants to move from C_j to D_j. P_i calls several procedures $Q_{i-1,\sigma,y,w}$ which enumerate $i-1$–sets $D \subseteq D_{i-1}$ where $\min D \geq w$. Eventually such a procedure may reach its goal and return its set D. In this case P_i waits for an $A \upharpoonright w$ change, and then puts D into C_i. Note that D is now an i-set. If $A \upharpoonright w$ changes before $Q_{i-1,\sigma,y,w}$ returns, then this very change turns the current set D into an i-set, so P_i is entitled to put D into C_i. However, P_i also has to cancel the run of $Q_{i-1,\sigma,y,w}$.
Identifying strings with numbers, we may view the tree at stage s as a subtree of $\{\gamma \in \omega^{<\omega} : |\gamma| \leq 2k-1 \ \& \ \forall i < k \ \gamma(2i+1) = 0\}$.
3. Waste management. A number $n \in C_j$ may not be promoted to D_j if the run $Q_{j,\sigma,y,w}$ during which it was placed into C_j is cancelled. Similarly, a number from D_{i-1} may fail to go into C_i if the required $A \upharpoonright w$–change does not occur. These 'garbage numbers' jeopardize the requirement that L be a KC set. To avert this, each run of a procedure is equipped with a *garbage quota*, assigned in an effective (if somewhat arbitrary) way during the construction. A procedure P_i has as a further parameter a garbage quota α, the amount it is allowed to waste by leaving it in $D_{i-1} - C_i$. Similarly, $Q_{j,\sigma,y,w}$ has garbage quota β, the amount it may leave in $C_j - D_j$. All goals and garbage quotas are of the form 2^{-l}, $l \in \mathbb{N}$. We denote runs of P_i-procedures with such parameters by $P_i(p,\alpha)$, and runs of Q_j-procedures by $Q_{j,\sigma,y,w}(q,\beta)$. The goal parameter of a run must be chosen small in order to meet the garbage quota of the run immediately above on the tree which called it.
The strategies proceed as follows, making sure not to exceed their garbage quotas.

$Q_{j,\sigma,y,w}(q,\beta)$: If $j = 1$, the procedure chooses n large, puts an axiom $\langle r, n\rangle$ into L, where $2^{-r} = \beta$, and waits for $K_t(n) \le r + d$ at a later stage t, at which point n is put into D_1. This is repeated till the goal has been reached. If $j > 1$, while the goal q has not been reached, the run $Q_{j,\sigma,y,w}(q,\beta)$ continues to call a single procedure $P_j(\beta,\alpha)$ for varying values of α, and waits till it returns a set C', at which time C' is put into D_j. Thus the amount of garbage left in $C_j - D_j$ is produced during a single run of a procedure P_j, which does not reach its goal β. So it is bounded by β.

$P_i(p,\alpha)$: this procedure calls procedures $Q_{j,\sigma,y,w}(2^{-|\sigma|}\alpha,\beta)$ for an appropriate value of β. Then the weight left in $D_{i-1} - C_i$ by all the returned runs of Q_{i-1}-procedures which never receive an A-change adds up to at most $\Omega\alpha$ $[\Omega^A\alpha]$, since this is a one-time event for each σ. The runs of procedures Q_{i-1} which are cancelled and have enumerated D so far do not contribute to the garbage of P_i, since D goes into C_i upon cancellation.

To assign the garbage quotas, at any substage of *stage* s, let

$$\alpha_i^* = 2^{-(2i+3+n_{P,i})},$$

where $n_{P,i}$ is the number of runs of P_i-procedures started so far. Let

$$\beta_j^* = 2^{-(2j+2+n_{Q,j})},$$

where $n_{Q,j}$ is the number of runs of Q_j-procedures started prior to this substage of *stage* s. When P_i is called at a substage of *stage* s, its parameter α will be at most α_i^*. Similarly, Q_j's parameter β will be at most β_j^*. This ensures $wt(C_1 - C_k) \le 1/2$. Since $wt(C_k) \le 1/2$ by Fact 5.3, L is a KC set.

It is instructive to compare the proofs of Theorem 4.4 and of Theorem 5.2. In both cases we prove that a real A is strongly K-trivial. In the first case we do not have levels of procedures, which is why the proof is uniform. The strategy $\alpha = \langle\sigma, y, \gamma\rangle$ closely corresponds to a strategy $Q_{j,\sigma,y,w}$ at a fixed level j. Both are based on a description of y, $U^\gamma(\sigma) = y$ in the first case, and $U_s^A(\sigma) = y$ in the second. Both are stopped when their guess about A turns out wrong. Both carry out their actions in small bits to avert too much damage in case this happens. Reserving only a small set D_α of measure $2^{-(n_\alpha+c)}$ at a time corresponds to calling a procedure P_j with a small goal β. A strategy waiting to reappear on a tree T_s corresponds to P_{j+1}'s waiting for an $A \upharpoonright w$ change after $Q_{j,\sigma,y,w}$ returned.

We give the formal description of the procedures and the construction.

The procedure $P_i(p,\alpha)$ ($1 < i \le k$, $p = 2^{-l}, \alpha = 2^{-r}$ for some $r \ge l$).
It enumerates a set C. Begin with $C = \emptyset$.
At *stage* s, declare each σ, $|\sigma| = s$, available (availability is a local notion for each run of a procedure). For each σ, $|\sigma| \le s$, do the following.

($\mathrm{P1}_\sigma$) If σ is available, and $U(\sigma) = y$ for some y, $y < s$, $\Gamma^A(y') \downarrow$ for each $y' < y$ $[U^A(\sigma) = y$ for some $y < s]$ let $w = \gamma^A(y-1)$ [let w be use of this computation] and call the procedure $Q_{i-1,\sigma,y,w}(2^{-|\sigma|}\alpha,\beta)$, where $\beta = \min(2^{-|\sigma|}\alpha, \beta_i^*)$. Declare σ unavailable.

(P2$_\sigma$) If σ is unavailable due to a run $Q_{i-1,\sigma,y,w}(q,\beta)$, and $A_s \upharpoonright w \neq A_{s-1} \upharpoonright w$, declare σ available.

(a) Say the run is *released*. If $wt(C \cup D_{i-1,\sigma}) < p$, then put $D_{i-1,\sigma}$ into C and go on to (b). Otherwise, choose a subset $\widetilde{D}$ of $D_{i-1,\sigma}$ such that $wt(C \cup \widetilde{D}) = p$, and put $\widetilde{D}$ into C. *Return the set* C, cancel all runs of subprocedures and end this run of P_i. ($\widetilde{D}$ exists since $p = 2^{-l}$ for some l, and $r_n > l$ for each $n \in D$ – now order the numbers r_n in a nondecreasing way.) Also, if we inductively assume that $D_{i-1,\sigma}$ was an $i-1$-set already at the last *stage*, then C is an i-set, since $\min(D_{i-1,\sigma}) > w$.

(b) If the run $Q_{i-1,\sigma}$ has not returned yet, cancel this run and all the runs of subprocedures it has called.

The procedure $Q_{j,\sigma,y,w}(q,\beta)$ $(0 < j < k,\ \beta = 2^{-r}, q = 2^{-l}$ for some $r \geq l)$.
It enumerates a set $D = D_{j,\sigma}$. Begin with $D = \emptyset$.

(Q1)

Case $j = 1$. Pick a large number n. Put n into C_1, and put $\langle r_n, n\rangle$ into L, where $2^{-r_n} = \beta$. Wait for a *stage* t such that $K_t(n) \leq r_n + d$, and go to (Q2). (If M_d is a machine for L, then t exists.)

Case $j > 1$. Call $P_j(\beta,\alpha)$, where $\alpha = \min(\beta, \alpha^*_j)$, and goto (Q2).

(Q2)

Case $j = 1$. Put n into D (D remains a 1-set).

Case $j > 1$. Wait till $P_j(\beta,\alpha)$ returns a set C'. Put C' into D.

In any case, if $wt(D) < q$ then goto (Q1). Else *return the set* D. (Note that in this case, necessarily $wt(D) = q$. Also, D is a j-set, assuming inductively that the sets C' are j–sets if $j > 1$.)

At *stage* 0, we begin the construction by calling $P_{k,0}(1,\alpha^*_k)$. At each *stage*, we descend through the levels of procedures of type P_k, $Q_{k-1} \ldots P_2, Q_1$. At each level we start or continue finitely many runs of procedures. This is done in some effective order, say from left to right on that level of the tree of runs of procedures, so that the values α^*_i and β^*_j are defined at each substage. Since we descend through the levels, a possible termination of a procedure in (P2$_\sigma$) (b) occurs before the procedure can act.
Verification. In the beginning, we do not assume that M_d is a machine for L. C_1, the right domain of L, is enumerated in (Q1). For $1 \leq j < k$, let $D_{j,t}$ be the union of sets $D_{j,\sigma}$ enumerated by runs of a procedure $Q_{j,\sigma}$ up to the end of stage t. Let $C_{i,t}$ be the union of sets C_t enumerated by runs of a procedure P_i $(1 < i \leq k)$ by the end of stage t.

Lemma 5.4 *The r.e. sets C_i are i–sets*

Proof. By the comments in ($\mathrm{P2}_\sigma$) and (Q2) above, D_1 is a 1-set. For $2 \le i \le k$, assume inductively that D_{i-1} is an $i-1$-set. Then C_i (and hence D_i for $i < k$) is an i-set. $\diamondsuit$

We next verify that L is a KC set. First we show that no procedure exceeds its garbage quota.

Lemma 5.5 *(a) Let $1 \le j < k$. The weight of all numbers in $C_j - D_j$ which belong to a run $Q_{j,\sigma,y,w}(q,\beta)$ is at most β.*
(b) Let $1 < i \le k$. The weight of all numbers in $D_{i-1} - C_i$ which belong to a run $P_i(p,\alpha)$ is at most α.

Proof. We actually prove that the bounds hold at any stage of the run. This suffices for the lemma, even if the run gets cancelled.
a) For $j = 1$ the bound holds since the run has at most one number n in $C_1 - D_1$ at any given stage. So if the run gets stuck waiting at (Q1), it has left weight β in $C_1 - D_1$. If $j > 1$, all numbers as in (a) belong to a single run of a procedure $P_j(\beta,\alpha)$ called by $Q_{j,\sigma,y,w}(q,\beta)$, because, once such a run returns a set C', this set is put into D_j. Since the run of P_j does not return, it does not reach its goal β. Thus the weight of such numbers is $\le \beta$ at any stage of the run of $Q_{j,\sigma,y,w}$.
b) Suppose n belongs to a run $P_i(p,\alpha)$ and $n \in D_{i-1,t}$ at *stage* t. Then n was put there during a run of a procedure $Q_{i-1,\sigma,y,w}(2^{-|\sigma|}\alpha,\beta)$ called by P_i. We claim that, *if n does not reach C_i, then no further procedure $Q_{i-1,\sigma,y',w'}$ is called after stage t during the run of P_i.* Firstly assume that $A_s \restriction w \ne A_{s-1} \restriction w$ for some *stage* $s > t$. The only possible reason that n does not reach C_i is that the run of P_i did not need n to reach its goal in ($\mathrm{P2}_\sigma$) (i.e., $n \notin \widetilde{D}$), in which case the run of P_i ends at s. Secondly, assume there is no such s. Then the run of P_i, as far as it is concerned with σ, keeps waiting at $(P2_\sigma)$, and σ does not become available again. This proves the claim.
The claim implies that, for each σ there is at most one run $Q_{i-1,\sigma,y,w}(2^{-|\sigma|}\alpha,\beta)$ called by $P_i(p,\alpha)$ which leaves numbers in $D_{i-1} - C_i$. The sum of the weights of such numbers over all such σ is at most $\Omega\alpha$. [For Theorem 5.2, we distinguish two cases. If the run of P_i returns at *stage* s, then the sum of the weights is bounded by the value of Ω^A at the last *stage* before s. Otherwise the sum is bounded by $\Omega^A\alpha$.] $\diamondsuit$

By the previous lemma and the definitions of the values α_i^*, β_j^* at substages,

$$wt(C_1 - C_k) \le \textstyle\sum_{j=1}^{k-1} wt(C_j - D_j) + \sum_{i=2}^{k} wt(D_{i-1} - C_i) \le 1/2.$$

By Fact 5.3, $wt(C_k) \le 1/2$. We conclude that $wt(C_1) \le 1$, and L is a KC-set. From now on we may assume that M_d is a machine for L, using the Recursion Theorem as explained above.

Lemma 5.6 *There is a run of a procedure P_i, called a* golden run*, such that* *(i) the run is not cancelled*

(ii) each run of a procedure $Q_{i-1,\sigma,y,w}$ started by P_i returns unless cancelled

(iii) the run of P_i does not return.

Proof. Suppose no such run exists. We claim that *each run of a procedure returns unless cancelled*. This yields a contradiction, since we call P_k with goal 1, this run is never cancelled, but if it returns, it has enumerated weight 1 into C_k, contrary to Fact 5.3.
To prove the claim we use induction on levels of procedures of type Q_1, P_2, Q_2, ..., Q_{k-1}, P_k. Suppose the run of a procedure is not cancelled.
$Q_{j,\sigma,y,w}(q, \beta)$: In case $j = 1$, by the hypothesis we always reach (Q2) after putting n into C_1, because the run is not cancelled and M_d is a machine for L. In case $j > 1$, inductively each run of a procedure P_j called by $Q_{j,\sigma,y,w}$ returns, as it is not cancelled. In any case, each time the run is at (Q2), the weight of D increases by β. Therefore $Q_{j,\sigma,y,w}$ reaches its goal and returns.
$P_i(p, \alpha)$: The run satisfies (i) by hypothesis, and (ii) by inductive hypothesis. Thus, (iii) fails, i.e., the run returns. $\diamondsuit$

Lemma 5.7 *B is K-trivial. [A is strongly K-trivial].*

Proof. Choose a golden run of a procedure $P_i(p, \alpha)$ as in Lemma 5.6. We enumerate a KC set W. Note that $p/\alpha = 2^g$ for some $g \in \mathbb{N}$. At *stage* s, when a run $Q_{j,\sigma,y,w}(2^{-|\sigma|}\alpha, \beta)$ returns, then put $\langle |\sigma| + g + 1, B_s \restriction y\rangle$ into W [put $\langle |\sigma| + g + 1, y\rangle$ into W] . We prove that W is a KC-set, namely,

$$S_W = \sum_s \sum\{2^{-r} : \langle r, z\rangle \in W_s - W_{s-1}\} \leq 1.$$

Suppose $\langle r, z\rangle$ enters W at *stage* s due to a run $Q_{i-1,\sigma,y,w}(2^{-|\sigma|}\alpha, \beta)$ which returns.
Stable case. The contribution to S_W of those axioms $\langle r, z\rangle$ where $A \restriction w$ is stable from s on is bounded by $2^{-(g+1)}\Omega$ [$2^{-(g+1)}\Omega^A$], since for each σ such that $U(\sigma)$ is defined [$U^A(\sigma)$ is defined] this can only happen once.
Change case. Now suppose that $A \restriction w$ changes after *stage* s. Then the set D returned by $Q_{i-1,\sigma,y,w}$, whose weight is $2^{-|\sigma|}\alpha$, went into C_i. Since the run of P_i does not return, $\sum_s \sum\{2^{-|\sigma|}$: $Q_{i-1,\sigma,y,w}$ *returns at* s *and* $A \restriction w$ *changes at some* $t > s$ $\} < 2^g$, otherwise the run of P_i reaches its goal α. Thus the contribution of the corresponding axioms to S_W is less than 1/2.
Let M_e be the machine for W according to the Kraft-Chaitin Theorem. We claim that, for all y,

$$K(B \restriction y) \leq K(y) + g + e + 1$$

[$K(y) \leq K^A(y) + g + e + 1$]. Suppose that s is the minimal *stage* such that $U(\sigma) = y$, $\Gamma^A \restriction y \downarrow$ and $A \restriction \gamma(y - 1)$ is stable [a stable computation $U^A(\sigma) = y$ appears], where σ is a shortest description of y. Let w be as in (P1$_\sigma$), namely,

$w = \gamma^A(y-1)$ [let w be the use of this computation]. Then σ is available at s: otherwise some run $Q_{i-1,\sigma,y',w'}$ is waiting to be released at (P2$_\sigma$). In that case, $A \upharpoonright w'$ has not changed since that run was started. Then $w = w'$ and $y = y'$, contrary to the minimality of s. So we call $Q_{i-1,\sigma,y,w}$. Since $A \upharpoonright w$ is stable and the run of P_i is not cancelled, this run is not cancelled, so it returns by (ii) of Lemma 5.6. At this *stage* we put $\langle |\sigma| + g + 1, B_s \upharpoonright y\rangle$ into W [we put $\langle |\sigma| + g + 1, y\rangle$ into W], causing the required inequality. $\diamondsuit$

In the following we show that Proposition 3.3 actually provides a characterization of K-trivial sets. We extract some additional information from the proof of Theorem 5.1.

Theorem 5.8 *The following are equivalent.*

(i) A is K-trivial

(ii) There is a Δ^0_2–approximation $(\widetilde{A}_r)$ of A such that

$$S = \sum\{c(x,r) : x \textit{ is minimal s.t. } \widetilde{A}_{r-1}(x) \neq \widetilde{A}_r(x)\} < 1/2, \qquad (5)$$

where $c(x,r) = 1/2 \sum_{x<y\leq r} 2^{-K_r(y)}$.

By (ii) and Fact 3.6, any $A \in \mathcal{K}$ is ω–r.e.

Proof. (ii) $\Rightarrow$ (i) is Proposition 3.3, with $(\widetilde{A}_r)$ instead of (A_r).

(i)$\Rightarrow$ (ii). Let $P(m) = 2^{-K(m)}$, and $P_t(m) = 2^{-K_t(m)}$. Let (A_s) be the modified Δ^0_2–approximation from the proof of Theorem 5.1. We first prove that there a constant $g \in \mathbb{N}$ and a recursive sequence $q(0) < q(1) < \ldots$ such that

$$\widehat{S} = \sum\{\widehat{c}(x,r) : x \text{ is minimal s.t. } A_{q(r+1)}(x) \neq A_{q(r+2)}(x)\} < 2^g \qquad (6)$$

where $\widehat{c}(z,r) = \sum_{z<y\leq q(r)} P_{q(r+1)}(y)$.

We apply Lemma 5.6, for the special case that $B = A$ and Γ is the identity functional, where $\gamma(y)$ is defined to be $y + 1$. Choose a golden run $P_i(p,\alpha)$ which enumerates a set C. We claim that, for each *stage* s, there is a *stage* $t > s$ such that, for all $y < s$, *if σ is a shortest description of y at t, then a run $Q_{i-1,\sigma,y,y+1}$ has returned by t and is not released yet, that is, P_i waits at (P2$_\sigma$).* Such a t exists because, for each y, there are only finitely many possible σ's. Once $A \upharpoonright y+1$ has settled, a run of a procedure $Q_{i-1,\sigma,y,y+1}$ is not cancelled, therefore it returns by property (ii) of golden runs.

Note that the least such t can be determined effectively. Let $q(0) = 0$. If $s = q(r)$ has been defined, let $q(r+1)$ be the least t satisfying this condition for s.

Again, let $g \in \mathbb{N}$ is the number such that $p/\alpha = 2^g$. We show that $\widehat{S} < 2^g$. Suppose x is minimal such that $A_{q(r+1)}(x) \neq A_{q(r+2)}(x)$. Then $A_{s-1}(x) \neq A_s(x)$ for some *stage* s, $q(r+1) < s \leq q(r+2)$. No later that s, the runs of procedures $Q_{i-1,\sigma,y,y+1}$, $x \leq y < p(r)$ which are still waiting at (P2$_\sigma$) are

released. This adds a weight of at least $\widehat{c}(x,r)$ to C_i. Thus $\widehat{S} < 2^g$, otherwise the run of P_i reaches its goal.
We obtain the required Δ^0_2-approximation $\widetilde{A}_r(x)$ after some manipulations. First let $A^*_r(x) = A_{q(r+2)}(x)$. Note that, for $z < r$, $c(z,r) = 1/2 \sum_{z<y\leq r} P_r(y) \leq \widehat{c}(z,r)$, so that $\sum\{c(x,r) : x$ is minimal s.t. $A_{r-1}(x) \neq A_r(x)\} \leq \widehat{S} < 2^g$. Now choose r_0 so large that the sum over all $r \geq r_0$ is at most 1/2. Let $\widetilde{A}_r(x) = A^*_{r_0}(x)$ for $r \leq r_0$, and $\widetilde{A}_r(x) = A^*_r(x)$ else. This shows (i)⇒ (ii). ♢

Theorem 5.9 *For each K-trivial set A, there is an r.e. K-trivial set D such that $A \leq_{tt} D$, via a polynomial time tt-reduction.*

Proof. We may assume that $\widetilde{A}_0(y) = 0$. Recall that, by the remark after Proposition 3.3, $\widetilde{A}_r(y)$ can only change $O(y^2)$ many times. Choose a constant c such that $\widetilde{A}_r(y)$ changes at most cy^2 times, and let $f(x) = c\sum_{0\leq z<x} z^2$. Define the r.e. set D as follows: when $\widetilde{A}_r(x) \neq \widetilde{A}_{r+1}(x)$ for the $i+1$st time, then enumerate $f(x)+i$ into D. Then $A \leq_{tt} D$, by a polynomial time tt-reduction (where numbers are identified with strings) since y
$niiA$ iff the greatest $i < cy^2$ such that $f(y)+i \in D$ is odd or there is no such i, and $\widetilde{A}_0(y) = 1$ otherwise.
To see D that is K-trivial, note that for each r and each $x < r$,

$$D_{r-1} \restriction x \neq D_r \restriction x \Rightarrow \widetilde{A}_{r-1} \restriction x \neq \widetilde{A}_r \restriction x.$$

Thus the sum in (5) for (D_r) is no greater that the sum for $(\widetilde{A}_r)$. ♢

Definition 5.10 *The real A is super-low if $A' \leq_{tt} \emptyset'$.*

Of course, super-low sets A are ω-r.e., that is, $A \leq_{tt} \emptyset'$. In Nies [12] it is proved that super-lowness and U-traceability coincide on the r.e. sets, but no inclusion holds between the classes on the ω-r.e. sets.
The following could be proved directly via a modification of the proof of Theorem 5.1. However, we prefer to use Theorem 5.2 and Proposition 2.8.

Theorem 5.11 *Each K-trivial real A is super-low.*

Proof. It suffices to show that the r.e. set D obtained via Theorem 5.9 is super-low. D is strongly K-trivial by Theorem 5.2, hence U-traceable by Proposition 2.8. Thus D is super-low by [12]. ♢

It is not hard to show that there are super-low r.e. sets A,B such that $A \oplus B$ is Turing complete [12]. Thus not all super-low r.e. sets are K–trivial.
Recall that a total function f is fixed-point free if $\forall x\ W_{f(x)} \neq W_x$. As an immediate consequence of Theorems 5.9, 5.11 and the Arslanov completeness criterion [18, Thm V.5.1], we obtain

Corollary 5.12 *No real $A \in \mathcal{K}$ computes a fixed-point free function.*

In particular, not every Π^0_1-class has a K-trivial member.
Each K-trivial real has a Δ^0_2-approximation which changes as little as desired (we thank Frank Stephan for pointing this out).

Corollary 5.13 *Let $A \in \mathcal{K}$. Given a nondecreasing recursive h such that $lim_n h(n) = \infty$, there is a Δ^0_2-approximation (A_r) of A such that $A_r(y)$ changes at most $h(y)$ times.*

Proof. By Theorem 5.9, there is an r.e. $D \in \mathcal{K}$ such that $A = \Phi^D$ for a *tt*-reduction Φ with recursive use φ. D is U-traceable by Theorem 5.11 and [12]. By the method of [20, Fact 1], there is an r.e. trace with bound h for the total D-recursive function $p(y) = \mu s\ D_s \upharpoonright \varphi(y) = D \upharpoonright \varphi(y)$, that is, $\forall y\ p(y) \in T^{[y]}$. Now let $A_r(y) = 1$ if $\Phi^{D_v}(y) = 1$ where $v = \max T^{[y]}_r$, and let $A_r(y) = 0$ otherwise. $\diamondsuit$

Theorem 5.14 *The K-trivial reals form a Σ^0_3 ideal in the ω–r.e. T-degrees, which is generated by its r.e. members. Moreover, this ideal is nonprincipal.*

Proof. By Theorems 5.1, 2.3 and 5.9 the K-trivial reals induce an ideal generated by the r.e. members. This ideal is Σ^0_3 by Fact 2.4. Suppose the ideal equals $[\mathbf{0}, \mathbf{b}]$ for some degree $\mathbf{b}$, then $\mathbf{b}$ is r.e. and low by Theorem 5.11. This contradicts Theorem 3.5. $\diamondsuit$

Corollary 5.15 *There is an r.e. low_2 set E such that $A \leq_T E$ for each K-trivial real A.*

Proof. It suffices to give such a bound E for the r.e. K-trivial reals. By [11], any Σ^0_3 ideal in the r.e. degrees has a low_2 upper bound. $\diamondsuit$

By Theorem 3.5, no such E is low_1.

6 Relativizations, operators, and reducibilities

We review some extensions and related notions.
Operators. Let $\mathcal{K}(X)$ be the class of reals K–trivial relative to X, that is, $\mathcal{K}(X) = \{A : \forall n\ K^X(A \upharpoonright n) \leq K^X(n) + \mathcal{O}(1)\}$. The relativization of the class of strongly K-trivial reals is $\mathcal{SK}(X) = \{A : \forall y\ K^X(y) \leq K^{A\oplus X}(y) + \mathcal{O}(1)$.
We show that $\mathcal{K}$ is an operator with good closure properties and a very simple representation. Firstly, $\mathcal{K}$ is degree invariant as an operator, since

$$X \equiv_T Y \ \Rightarrow\ \forall z\ |K^X(z) - K^Y(z)| \leq \mathcal{O}(1) \ \Rightarrow\ \mathcal{K}(X) = \mathcal{K}(Y).$$

All the results on $\mathcal{K}$ we have discussed relativize.

Theorem 6.1 *(i) $\mathcal{K}(X)$ is closed under $\oplus$ and closed downward under $\leq_T$.*

(ii) There is an r.e. index e such that, for each X, $W_e^X \in \mathcal{K}(X)$ and $X <_T W_e^X$.

(iii) $\mathcal{SK}(X) = \mathcal{K}(X)$

(iv) $A \in \mathcal{K}(X) \;\Rightarrow\;$ A is tt–below some $D \in \mathcal{K}(X)$ which is r.e. in X, via a polynomial time tt–reduction as in Theorem 5.9.

(v) $A \in \mathcal{K}(X) \;\Rightarrow\; A' \leq_{tt} X'$.

Proof. One obtains (i)-(iv) by examining the proofs of Theorems 2.3, 5.1, 3.4 and 5.9. For (v), suppose that $A \in \mathcal{K}(X)$. By (iv) we can suppose A is r.e. in X. Since $A \oplus X \in \mathcal{K}(X)$, $A \oplus X \in \mathcal{SK}(X)$ by (iii). Relativizing Proposition 2.8, $A \oplus X$ is jump traceable relative to X. Then, relativizing the fact from [12] that each U traceable set is super low, $A' \leq_{tt} X'$. $\diamondsuit$

Theorem 6.2 *There is an effective listing (Γ_e) of tt-reduction procedures such that, for each X, $\mathcal{K}(X) = \{\Gamma_e(X') : \; e \in \mathbb{N}\}$.*

Proof. Since $\{e : W_e^X \in \mathcal{K}(X)$ is $\Sigma_3^0(X)$ via a fixed index, there is an effective listing (V_j) of oracle enumeration procedures such that for each X, $\{V_j^X : j \in \mathbb{N}\}$ equals the set of reals in $\mathcal{K}(X)$ which are r.e. in X. Let (Φ_i) be an effective listing of the tt-reduction procedures needed in Theorem 5.9. For each pair i, j we can effectively determine a tt reduction Γ_e, $e = \langle i, j \rangle$ such that $\Gamma_e(X') = \Phi_i(V_j^X)$. $\diamondsuit$

Slaman [16] studied Borel operators $\mathcal{M} : \mathcal{P}(\mathbb{N}) \mapsto \mathcal{P}(\mathcal{P}(\mathbb{N}))$ such that, for each X, Y, $\mathcal{M}(X)$ is an ideal in the Turing degrees containing X, but not all sets, and $\mathcal{M}$ is *monotone*, that is, for each X, Y, $X \leq_T Y \;\Rightarrow\; \mathcal{M}(X) \subseteq \mathcal{M}(Y)$, a property stronger than degree invariance. Slaman proves that, on an upper cone in the Turing degrees, any such operator is given by (possibly transfinite) iterates of the jump. For instance, possibilities for $\mathcal{M}(X)$ are $\{Y : Y \leq_T X\}$, $\{Y : Y \leq_T X'\}$, or $\{Y : \exists n \in \mathbb{N}\; Y \leq_T X^{(n)}\}$.
Since the operator $\mathcal{K}$ is not given by such iterates, it cannot be monotone. An explicit example of non-monotonicity was pointed out by R. Shore: By Theorem 3.4, let A be a promptly simple set in $\mathcal{K}(\emptyset) = \mathcal{K}$. Then A is low cuppable, i.e. there is a low r.e. G such that $K \leq_T A \oplus G$. Hence $A \in \mathcal{K}(\emptyset) - \mathcal{K}(G)$, otherwise $A \oplus G \in \mathcal{K}^G$ and hence $(A \oplus G)' \leq_T G'$ by Theorem 6.1 (v), contradiction.

Reducibilities. For reals A, B, let $A \leq_{\mathcal{LR}} B \;\Leftrightarrow\; \mathrm{RAND}(B) \subseteq \mathrm{RAND}(A)$, and $A \leq_{\mathcal{SK}} B \;\Leftrightarrow\; \forall y\; K^B(y) \leq K^A(y) + \mathcal{O}(1)$.
Clearly, $\leq_T$ implies $\leq_{\mathcal{SK}}$, which in turn implies $\leq_{\mathcal{LR}}$. In [13] we prove

Theorem 6.3 ([13]) *(i) For r.e. A, B, $A \leq_{\mathcal{LR}} B$ implies $A' \leq_{tt} B'$*

(ii) There is an r.e. A which is T-incomplete but $\mathcal{SK}$–complete.

Let $\mathcal{L}(X) = \{A : A \leq_{\mathcal{LR}} X\}$. Like $\mathcal{K}$, $\mathcal{L}$ is a Σ^0_3 operator, but unlike $\mathcal{K}$, $\mathcal{L}$ is monotone in the sense of Slaman. Since $\mathcal{L}(X)$ is downward closed under $\leq_T$, by Slaman's result, $\mathcal{L}(X)$ cannot be $\oplus$–closed. The explicit counterexample we used for $\mathcal{K}$ can be used again: Note that $A \in \mathcal{L}(\emptyset)$. Thus $A \leq_{\mathcal{LR}} G$, and trivially $G \leq_{\mathcal{LR}} G$, but $K \equiv_T A \oplus G \not\leq_{\mathcal{LR}} G$ by Theorem 6.3 (i), since G is low. In particular, $\oplus$ does not determine a supremum in the r.e. $\leq_{\mathcal{LR}}$–degrees.
Note that the result $\mathcal{LR} = \mathcal{K}$ relativizes, as follows: $A \oplus X \leq_{\mathcal{LR}} X \Leftrightarrow A \in \mathcal{K}(X)$. Thus $\mathcal{K}(X)$ is a subclass of $\mathcal{L}(X)$. By relativizing our counterexample, we see that for each X there is $G \geq_T X$ such that $\mathcal{K}(G)$ is a proper subclass of $\mathcal{L}(G)$. Then, since $\mathcal{K}$ and $\mathcal{L}$ are degree invariant, by arithmetic determinacy, this holds on an upper cone of Turing degrees.
Using Theorem 6.1 (v), $A \equiv_{\mathcal{SK}} B$ implies $A' \equiv_T B'$ for all reals A, B. We do not know if this holds for $\equiv_{\mathcal{LR}}$ in place of $\equiv_{\mathcal{SK}}$.
Many other questions remain. For instance, is $\mathcal{K}$ definable in the (r.e.) Turing degrees?

References

[1] K. Ambos-Spies, Carl G. Jockusch, Jr., Richard A. Shore, and Robert I. Soare. An algebraic decomposition of the recursively enumerable degrees and the coincidence of several degree classes with the promptly simple degrees. *Trans. Amer. Math. Soc.*, 281:109–128, 1984.

[2] K. Ambos-Spies and A. Kucera. Randomness in computability theory. In Peter Cholak, Steffen Lempp, Manny Lerman, and Richard Shore, editors, *Computability Theory and Its Applications: Current Trends and Open Problems*. American Mathematical Society, 2000.

[3] Cristian Calude. *Information and randomness*. Monographs in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 1994. An algorithmic perspective, With forewords by Gregory J. Chaitin and Arto Salomaa.

[4] G. Chaitin. A theory of program size formally identical to information theory. *J. Assoc. Comput. Mach.*, 22:329–340, 1975.

[5] G. Chaitin. Information-theoretical characterizations of recursive infinite strings. *Theoretical Computer Science*, 2:45–48, 1976.

[6] P. Cholak, M. Groszek, and T. Slaman. An almost deep degree. *J. Symbolic Logic*, 66(2):881–901, 2001.

[7] R. Downey, D. Hirschfeldt, A. Nies, and F. Stephan. Trivial reals. In *Electronic Notes in Theoretical Computer Science (ENTCS)*, 2002.

[8] A. Kucera. On relative randomness. *Ann. Pure Appl. Logic*, 63:61–67, 1993.

[9] A. Kucera and S. Terwijn. Lowness for the class of random sets. *J. Symbolic Logic*, 64:1396–1402, 1999.

[10] P. Martin-Löf. The definition of random sequences. *Inform. and Control*, 9:602–619, 1966.

[11] A. Nies. Ideals in the recursively enumerable degrees. To appear.

[12] A. Nies. Reals which compute little. To appear.

[13] A. Nies and F. Stephan. Variations on low for random and $\mathcal{K}$-trivial. To appear.

[14] Gerald E. Sacks. *Degrees of Unsolvability*, volume 55 of *Annals of Mathematical Studies*. Princeton University Press, 1963.

[15] Claus-Peter Schnorr. *Zufälligkeit und Wahrscheinlichkeit. Eine algorithmische Begründung der Wahrscheinlichkeitstheorie*. Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 218.

[16] T. Slaman. Aspects of the Turing jump. To appear, 2001.

[17] T. Slaman and R. Solovay. When oracles do not help. In *Fourth Annual Conference on Computational Learning Theory*, pages 379–383. Morgan Kaufman, Los Altos, CA, 1991.

[18] R. Soare. *Recursively Enumerable Sets and Degrees.* Perspectives in Mathematical Logic, Omega Series. Springer–Verlag, Heidelberg, 1987.

[19] R. Solovay. Draft of a paper (or series of papers) on chaitin's work. IBM Thomas J. Watson Research Center, Yorktown Heights, NY, 215 pages, 1975.

[20] S. Terwijn and D. Zambella. Algorithmic randomness and lowness. *J. Symbolic Logic*, 66:1199–1205, 2001.

[21] M. van Lambalgen. *Random Sequences.* University of Amsterdam, 1987.