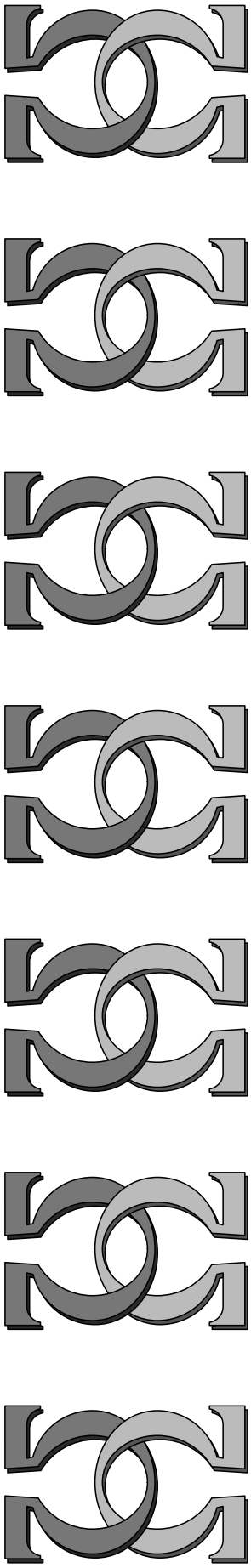

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A Quantum Scattering Approach to Undecidable Problems: Preliminary Version

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A Quantum Scattering Approach to Undecidable Problems: Preliminary Version*

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Abstract

In [12] a probabilistic solution to the Infinite Merchant's Problem, an undecidable problem equivalent to the Halting Problem, was proposed. The solution uses a real Hilbert space and is based on the estimation of the exponential growth of an unbounded semigroup. The aim of this paper is to offer an alternative solution in terms of scattering processes on quantum dots. We will reduce the problem to a special scattering problem, observe the results of multiple acts of scattering and, finally, estimate the deviation of the scattered data from the input data.

1 Introduction

Recently various proposals to break Turing' Barrier (see [13, 43, 8, 10]) have been discussed in a series of papers [18, 12, 25]. The common denominator of all approaches is the essential use of some physical theory, relativity theory in [18], quantum theory [12, 25], an attitude advocated by Landauer [32, 33] (*information is inevitably physical*) and Deutsch [16, 17] (*the reason why we find it possible to construct, say, electronic calculators, and indeed why we can perform mental arithmetic ... is that the laws of physics "happen" to permit the existence of physical models for the operations of arithmetic*).

The aim of the present paper is to revisit the solution offered in [12] and to produce an alternative solution in terms of scattering processes on quantum dots. We will formulate our results for the Merchant Problems. Recall that in the *finite* version of the Merchant's

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Problem we have N stacks of coins and we know that *at most one stack may contain false coins*. We are allowed to take just one coin from each stack and have a single "weighting". Can we determine whether there is a stack containing false coins, and in the affirmative, which? The solution reduces the problem to the weighting of a special combination of coins : one coin from the first stack, two coins from the second stack, ... N coins from the N-th stack. Assume that true coins weight 1; a false coins weights $1 + \gamma > 1$. If there are false coins in the i-th stack, then the weight of the combination is $N(N+1)/2 + i\gamma$; otherwise, it is $N(N+1)/2$.[1] For the Infinite Merchant's Problem we assume that *we have countable many stacks, all of them, except at most one, containing true coins only*. We are allowed to take just one coin from each stack and we want to determine whether all coins are true or there is a stack of false coins. In the first problem we can check classically whether there is a false coin, but we cannot find its position. The second problem is equivalent to the Halting Problem (decide whether an arbitrary program, Turing machine, probabilistic Turing machine, Java program, etc., eventually halts), so classically undecidable. In both cases some missing information prevents a classical solution. However, the information *is* in the data specifying each of the problems and the challenge is to make use of it.

In fact there are solutions for these problems, see [12]. The solutions we are going to discuss in this paper are also *probabilistic*. As in [12], the discussion is purely mathematical; the proposed model is still very far from any technical or even experimental implementation, which may require joint efforts of physicists and engineers.

The structure of our paper is the following. In the next section we will exemplify our main scattering technique by showing that the quantum truth-table of the $C_{\rm NOT}$ gate can be realized as a scattering matrix of a resonance quantum gate manipulated by optical signals.

In Section 3 we briefly present the probabilistic solutions from [12]. In Section 4 we develop the mathematical solution for the Finite Merchant's Problem based on the observation of a series of independent acts of scattering. As in [12], this solution cannot be extended to a solution of the Infinite Merchant's Problem and we offer a reason for this statement. In Section 5 we present a resonance method of amplification of the signal which permits the presentation of the results of the general scattering experiment in a similar form as in Section 4. The solution of the Infinite Merchant's Problem discussed in Section 6 follows the pattern in [12]; it requires a probabilistic structure on the input space of all trajectories of some Markov process. This purely quantum approach reveals also the important, explicit role played by the characteristics of resonances.

We postpone till the end of the paper the discussion of abstract algebraic structures of solutions of scattering problems involving quantum dots. Explicit formulae for scattering matrices of simple solvable models of quantum dots constructed via a symplectic operator-extension technique will be presented in the Appendix.

For quantum computing see [24, 45, 11]. For scattering theory see for example [38]; [44] contains a detailed discussion of some applications of scattering theory to quantum computing.

[1] This problem was widely spread in allies armies during the Second World War (evidence given to BP by Dr. A.N. Ignatov). Probably the elegant solution described above was the very first solution of a computational problem bearing typical features of quantum computing.

2 C_{NOT} as a Scattering System

In [1] the idea to consider a single act of quantum computation as a scattering process was suggested. The possibility of interpreting every unitary transformation as a single act of a quantum computation was also discussed during the First International Conference on "Unconventional Models of Computation" held in Auckland, 1998, see [9].

Quantum computing exploits the principles of quantum mechanics to perform operations, in particular to perform many operations simultaneously as a result of the applications of operators to superpositions of classical states.

In this section, which plays only an illustrative role, we will show a simple quantum scattering system realizing the quantum C_{NOT} (controlled-NOT) gate, i.e., a quantum gate satisfying exactly to the same truth-table as the classical controlled-NOT gate. The C_{NOT} device has two input and output channels. Each channel can be only in two different states, say $|0\rangle, |1\rangle$. The *in* and *out* states of the control-channel are the same, $|I_{in}\rangle = |I_{out}\rangle$ but *in* and *out* states of the current-channel may be different, $|J_{in}\rangle \neq |J_{out}\rangle$, depending on the state $|J_{in}\rangle$ and the control-channel state. The classical controlled-NOT gate has the following truth table:

I_{in}	J_{in}	I_{out}	J_{out}
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

which describes the effect of the device on the above *in*-states,

$\lvert I_{in}\ J_{in}\rangle$	$\longrightarrow$	$\lvert I_{out}\ J_{out}\rangle$
$\lvert 0\ 0\rangle$	$\longrightarrow$	$\lvert 0\ 0\rangle$
$\lvert 0\ 1\rangle$	$\longrightarrow$	$\lvert 0\ 1\rangle$
$\lvert 1\ 0\rangle$	$\longrightarrow$	$\lvert 1\ 1\rangle$
$\lvert 1\ 1\rangle$	$\longrightarrow$	$\lvert 1\ 0\rangle$

The quantum C_{NOT} gate operates not only on "classical" states $|0\rangle$ and $|1\rangle$, ($C_{\mathrm{NOT}}|ij\rangle = |ik\rangle$, where $i, j \in \{0, 1\}$ $k = i \oplus j \pmod 2$), but also on all their linear combinations,

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \longrightarrow \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|11\rangle + \alpha_{11}|10\rangle.$$

This quantum transformation can be presented via the unitary matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1}$$

with respect to the basis $(e_1, e_2, e_3, e_4) = (|00\rangle, |01\rangle, |10\rangle, |11\rangle)$. More importantly, $C_{\rm NOT}$ is *universal*[2] and truly quantum as *it cannot be written as a product of two operators.*

We claim that the matrix U in (1) can be realized as a *scattering matrix* of a special *quantum dot*. First, here is the motivation. Consider Figure 1 in which two isolated quantum wires are placed in proximity and there is a window region in which the two wires are coupled. An electron moving in the window region oscillates between the two quantum wires and the probability of the electron exiting a specific quantum wire depends on the length of the window. We can arrange the setup in such a way that, under normal conditions, the electron exits from the same wire it enters, but switches to the other wire when an extra potential is applied, a realization of the $C_{\rm NOT}$ gate.

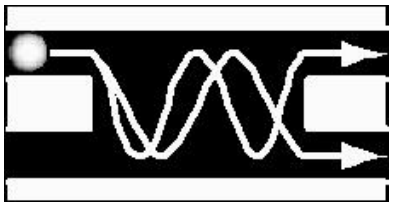

Figure 1: $C_{\rm NOT}$ as quantum dot

One could associate to the above quantum system a product space $H_e \times H_d$ constituted by the states of the electron and the states of the dot, and then consider the evolution of the system generated by the total Hamiltonian $\mathcal{H}_e + \mathcal{H}_d + \mathcal{H}_{int}$ with a proper interaction term. This would be a quite sophisticated problem of quantum mechanics, similar to three-body problem (see for instance [37]). We assume now that the state of the dot is selected independently and thus reduce the above problem to the corresponding one-body problem for an electron scattered in the quantum wire depending on the state of the dot. The corresponding device might be called *quantum relay* rather than *quantum dot*; it can be transformed into a quantum dot if the state of the dot is considered as a quantum state (with finite life-time). This model is acceptable if the life-time of the state of the dot is long enough for the scattering experiment.

We continue with the mathematical model and assume that a quantum dot is inserted between the wires and electrons are redirected according to the state of the dot. We assume also that the *inner* Hamiltonian of the quantum dot is presented by a finite diagonal matrix which is either $A_1 = \text{diag}\,\{\alpha_1^2, \alpha_3^2, \ldots, \alpha_N^2\}$ or $A_2 = \text{diag}\,\{\alpha_2^2, \alpha_3^2, \ldots, \alpha_N^2\}$, with positive diagonal elements, $0 < \alpha_1^2 < \alpha_2^2 < \alpha_3^2 < \ldots < \alpha_N^2$. We assume that the quantum dot is inserted in an one-dimensional quantum wire $-\infty < x < \infty$ at the origin and a proper boundary condition is satisfied (31) (see the Appendix) for connecting it with the Schrödinger operator on the wire defined in the space of square-integrable vector-functions $L_2(\mathbf{R}, \mathcal{E})$

$$l = -\frac{d^2}{dx^2}. \tag{2}$$

[2]Every classical computable function can be computed by a small universal set of gates like {OR, NOT} or {NAND}. A set of quantum gates $\mathcal{S}$ is called universal if any unitary operation can be approximated with an arbitrary accuracy by a quantum circuit involving gates in $\mathcal{S}$; see more in [20, 21, 24, 11].

The corresponding general "zero-range" quantum Hamiltonian (solvable model) is described in the Appendix as a self-adjoint extension $\mathbf{A}_\beta$ of the orthogonal sum $A_0 \oplus l_0$ restricted to $A_0 \oplus l_0$ in $L_2(\mathbf{R}, \mathcal{E}) \oplus E$ a onto proper domain; here $\mathcal{E}$ is the input space and E is the inner space (dim $(E) \geq 2$). The positive part of the *spectrum* σ_β of the operator $\mathbf{A}_\beta$ is absolutely-continuous and fills the positive half-axis $\lambda \geq 0$ with multiplicity dim $(\mathcal{E})$. The role of eigenfunctions of the spectral point $p^2 = \lambda > 0$ is played by the *scattered waves* $\overrightarrow{\Psi}_\nu$, $\overleftarrow{\Psi}_\nu$ labeled with vectors $\nu \in \mathcal{E}$. The components of the scattered waves $\Psi_\nu(p)$ in the *outer space* $L_2(\mathbf{R})$ are presented as linear combinations of exponentials:

$$\overrightarrow{\Psi}_\nu(p,x) = \begin{cases} e^{-ipx}\nu + e^{+ipx}\overleftarrow{R}(p)\nu, & x < 0, \\ e^{-ipx}\overrightarrow{T}(p)\nu, & x > 0, \end{cases}$$

$$\overleftarrow{\Psi}_\nu(x) = \begin{cases} e^{ipx}\nu + e^{-ipx}\overrightarrow{R}(p)\nu, & x > 0, \\ e^{-ipx}\overleftarrow{T}(p)\nu, & x > 0. \end{cases} \tag{3}$$

The matrix

$$\mathbf{S}_\beta(p) = \begin{pmatrix} \overrightarrow{T}(p) & \overrightarrow{R}(p) \\ \overleftarrow{R}(p) & \overleftarrow{T}(p) \end{pmatrix} \tag{4}$$

is called[3] the *scattering matrix* of the operator $\mathbf{A}_\beta$.

The evolution of the *wave function* of the quantum mechanical system with Hamiltonian $\mathbf{A}_\beta$ given by the equation

$$\frac{1}{i}\frac{\partial \Psi}{\partial t} = \mathbf{A}_\beta \Psi, \tag{5}$$

and proper initial condition

$$\Psi\Big|_{t=0} = \Psi_0,$$

can be described by the evolution operator constructed of the above scattered waves and square-integrable bound states Ψ_s which satisfy the homogeneous equation

$$\mathbf{A}_\beta \Psi_s = \lambda_s \Psi_s,$$

with negative eigenvalues λ_s. Bound states do not play an essential role in scattering processes, so we may assume that the initial state Ψ_0 is orthogonal to all bound states, and may be expanded in an analog of Fourier integral over the scattered waves

$$\Psi_0 = \frac{1}{2\pi}\int_R \sum_\nu \Psi_\nu(p)\langle \Psi_\nu(p), \Psi_0\rangle dp.$$

Then the evolution described by the solution of the equation (5) and the above initial data can be presented as a (continuous) linear combination

$$\Psi(t) = \frac{1}{2\pi}\int_{\mathbf{R}} \sum_\nu e^{ip^2 t}\Psi_\nu(p)\langle \Psi_\nu(p), \Psi_0\rangle dp$$

[3]The transmission coefficients appear on the main diagonal of the matrix to fit the physical meaning of the scattering matrix for small values of $|\beta|$, when it is reduced to the undisturbed transmission $\mathbf{S}(p) = \mathbf{I}$.

of *modes* incoming from infinity on the left ($-\infty$) and on the right ($+\infty$), and outgoing *scattered* to both directions $\pm\infty$ according to the solution of the time-dependent Schrödinger equation $\frac{1}{i}\frac{\partial \mathbf{U}}{\partial t} = \mathbf{AU}$:

$$\mathbf{S}_\beta(p) : e^{ip^2t}\begin{pmatrix} e^{-ipx}\nu_{left} \\ e^{ipx}\nu_{right}\end{pmatrix} \longrightarrow e^{ip^2t}\begin{pmatrix} e^{-ipx}(\overrightarrow{T}(p)\nu_{left} & + & \overrightarrow{R}(p)\nu_{right}) \\ e^{ipx}(\overleftarrow{R}(p)\nu_{left} & + & \overleftarrow{T}(p)\nu_{right})\end{pmatrix}.$$

In our case the scattered waves are calculated in an explicit form because the Hamiltonian is "solvable", that is its eigenfunctions (scattered waves)

and the evolution operator can be calculated explicitly. In particular, the corresponding scattering matrix is parametrized by an analytic function. If we choose an one-dimensional input space $\mathcal{E}$ and the *deficiency subspace* (see the Appendix) $\mathcal{N}_i$ spanned by the normalized vector $e = e_i$ and introduce the scalar function

$$\mathcal{M} = \langle \frac{I + \lambda A}{A - \lambda I} e, e\rangle,$$

then using (31) the scattering matrix becomes (4), with equal transmission and reflection coefficients $\overrightarrow{T}, \overleftarrow{T}, \overrightarrow{R}, \overleftarrow{R}$:

$$\overrightarrow{T}(p) = \overleftarrow{T}(p) = \frac{2ip}{2ip - |\beta|^2\mathcal{M}^{-1}},\ \overrightarrow{R}(p) = \overleftarrow{R}(p) = -\frac{|\beta|^2\mathcal{M}^{-1}}{2ip - |\beta|^2\mathcal{M}^{-1}}.$$

We observe first the behaviour of the scattering matrix at the resonance energy $\alpha_1^2 > 0$ in case the resonance level α_1^2 in the quantum dot is vacant, but α_2^2 is occupied, hence eliminated from the quantum picture (due to Pauli principle, as discussed below). In this case we have

$$\mathcal{M}_1 = \frac{1 + \alpha_1^2\lambda}{\alpha_1^2 - \lambda}|e_1|^2 + \sum_{l=3}^{N}\frac{1 + \alpha_l^2\lambda}{\alpha_l^2 - \lambda}|e_l|^2 = \frac{1 + \alpha_1^2\lambda}{\alpha_1^2 - \lambda}|e_1|^2 + \mathcal{M}_3,$$

where $|e_l|^2$ are the squares of the Fourier coefficients of the deficiency vector e with respect to the eigenvectors of the operator A_1.

Consider now the case when the resonance level α_1^2 is occupied, but the level α_2^2 is vacant. In this case

$$\mathcal{M}_2 = \frac{1 + \alpha_2^2\lambda}{\alpha_2^2 - \lambda}|e_2|^2 + \sum_{l=3}^{N}\frac{1 + \alpha_l^2\lambda}{\alpha_l^2 - \lambda}|e_l|^2 = \frac{1 + \alpha_2^2\lambda}{\alpha_2^2 - \lambda}|e_2|^2 + \mathcal{M}_3,$$

where $|e_l|^2$ are the squares of the Fourier coefficients of the deficiency vector with respect to the eigenvectors of the operator A_2.

The above constructed model corresponds to "spin-less" electrons (electrons with the constant spin, in absence of magnetic field) on a quantum circuit with the attached quantum dots. We may assume that the circuit lies on the surface of a semiconductor with the Fermi-level α_1^2 (see [35]) and the levels $\alpha_1^2,\ \alpha_2^2$ are (due to Pauli principle) alternatively occupied by electrons traveling on the circuit or by electrons transferred from one level to another inside the quantum dot under the resonance laser shining.[4]

[4]The manipulation of the resonance quantum dot is a sophisticated few-body problem of quantum scattering. A "solvable" model for it was discussed in [37]. The model involving the resonance laser shining as a tool of manipulation of the current through the quantum dot was introduced in [4].

We assume that the position of the electron on the level α_2^2 with the level α_1^2 vacant corresponds to $I_{in} = I_{out} = 0$ and the position of the electron on the level α_1^2 with the level α_2^2 vacant corresponds to $I_{in} = I_{out} = 1$. We identify these states of the system as the state S_1 and state S_2, respectively.

Using the statement (7.3) in case dim $\mathcal{E} = 1$, for every value β we obtain the following expressions for the transmission coefficients on the resonance electron's energy $\lambda = \alpha_1^2$:

$$\overrightarrow{T}(p) = \overleftarrow{T}(p) = \frac{2ip}{2ip + |\beta|^2 \mathcal{M}_1^{-1}} = 1.\ \overrightarrow{R}(p) = \overleftarrow{R}(p) = 0,$$

(at the resonance energy we have $\mathcal{M}_1^{-1} = 0$). In the second case, when the resonance level $p^2 = \lambda = \alpha_1^2$ is occupied, we obtain (due to Pauli principle) the following expression for the transmission coefficients of passing electrons with resonance energy:

$$\overrightarrow{T}(p) = \overleftarrow{T}(p) = \frac{2ip}{2ip - |\beta|^2 \mathcal{M}_2^{-1}(\lambda)} = \frac{2ip\left(\frac{1+\alpha_2^2\alpha_1^2}{\alpha_2^2-\alpha_1^2}|e_2|^2 - \sum_{l=3}^{N}\frac{1+\alpha_l^2\alpha_1^2}{\alpha_l^2-\alpha_1^2}|e_l|^2\right)}{2ip\left(\frac{1+\alpha_2^2\alpha_1^2}{\alpha_2^2-\alpha_1^2}|e_2|^2 + \sum_{l=3}^{N}\frac{1+\alpha_l^2\alpha_1^2}{\alpha_l^2-\alpha_1^2 a}|e_l|^2\right) + |\beta|^2},$$

and the corresponding expressions for the reflection coefficients:

$$\overrightarrow{R}(p) = \overleftarrow{R}(p) = \frac{|\beta|^2}{2ip\mathcal{M}_2 - |\beta|^2},$$

which may be approximated, for large enough β, as

$$\overrightarrow{T}(\alpha_1) = \overleftarrow{T}(\alpha_1) \approx 0,\ \overrightarrow{R}(\alpha_1) = \overleftarrow{R}(\alpha_1) \approx -1.$$

Hence the scattering matrix is equal to

$$\mathbf{S}_\beta(\alpha_1^2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

for any relatively large enough β, if the resonance level α_1^2 *is not occupied*, and is equal to

$$\mathbf{S}_\beta(\alpha_1^2) \approx \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{7}$$

if the resonance level α_1^2 *is occupied*, and the above conditions on the interaction of the quantum dot with environment are fulfilled.

Another way to obtain as scattering matrix the truth-table of C_{NOT} is to shift "locally" the Fermi level to fit some certain eigenvalue of the inner Hamiltonian[5]

$$A \to A + \Delta,\ \alpha_l^2 \to \alpha_l^2 + \Delta,$$

$$\mathcal{M}_2 \to \mathcal{M}_2^{\Delta} = \langle \frac{1 + \lambda(A + \Delta)}{A + \Delta - \lambda} e, e \rangle,$$

[5]We do not discuss here technical advantages and disadvantages of both methods, but we note that they may affect the time-characteristics of real devices.

such that $\mathcal{M}_2^{\Delta}(\alpha_1^2) = 0$. Then

$$\overrightarrow{T}(\alpha_1^2) = \overleftarrow{T}(\alpha_1^2) = \frac{2i\alpha_1\mathcal{M}_2^{\Delta}}{2ip\mathcal{M}_2^{\Delta} - |\beta|^2} = 0,\ \overrightarrow{R}(\alpha_1^2) = \overleftarrow{R}(\alpha_1^2) = -\frac{|\beta|^2}{2i\alpha_1\mathcal{M}_2^{\Delta} - |\beta|^2} = -1.$$

Hence in this case we have exactly

$$\mathbf{S}_{\beta}(\alpha_1^2) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

One can see, that, up to minor changes of notation, the direct sum of the above scattering matrices gives exactly the quantum truth-table of C_{NOT}, hence this quantum gate can be implemented by a resonance-manipulated quantum dot. Then, due to the universality of the C_{NOT} gate, any quantum circuit can be constructed using quantum dots and intermediate wires (phase-rotating elements). A similar two-body model for the simplest quantum gates was discussed in [37].

The life-time of an electron on the levels embedded into continuous spectrum is finite, hence the device described above may not be stable on large intervals of time (exceeding the life-time of the electron on the levels). We do not discuss here methods of stabilization of the electron on the levels; furthermore, in what follows we will assume (see Section 4), that the quantum system can be changed from one state into another state just by "switching on" or "switching off" an interaction defined by an operator-extension procedure. We will interpret each Merchant's Problem as a problem of distinguishing between the states of the system resulting from the observation of results of a scattering process with random input.

3 Probabilistic Solutions to Merchant Problems

In this section we briefly recall the probabilistic solution proposed in [12]. In both finite and infinite case we will adopt the following strategy. We are given a computable probability $\theta \in (0,1)$ and we assume that we work with a "device"[6] with sensitivity given by a computable real $\varepsilon \in (0,1)$. Then, we compute classically a time $T = T_{\theta,\varepsilon}$ and run the "device" on a random input for the time T. If we get a click, then the system has false coins (in the finite case the stack containing false coins can be located); if we don't get a click, then with probability greater than $1-\theta$ all coins are true. An essential part of the method is the requirement that the time limit T is *classically computable.*

For the finite case we consider $\mathbf{R}^N$ as quantum space. Denote by q_i the weight of a coin in the i-th stack; if the i-th stack contains true coins, then $q_i = 1$, otherwise, $q_i = 1+\gamma$.

Next we consider the operator $\mathbf{Q} = \sum_{i=1}^{N} q_i\mathbf{P}_i$, where $\mathbf{P}_i(\mathbf{x}) = (0,0,\ldots,x_i,0,\ldots,0)$ $\mathbf{Q}(\mathbf{x}) = (q_1\mathbf{P}_1,\ldots,q_N\mathbf{P}_N)(\mathbf{x}) = (q_1x_1,\ldots,q_Nx_N)$, and we construct the quadratic form induced by the t-th iteration of the operator $\mathbf{Q}$, $\langle\mathbf{Q}^t(\mathbf{x}),\mathbf{x}\rangle$, and consider its dynamics:

- if all coins are true $\langle\mathbf{Q}^t(\mathbf{x}),\mathbf{x}\rangle = \|\ \mathbf{x}\ \|^2$, for all $\mathbf{x}\in\mathbf{R}^N$;

[6] As in [12] we use quotation marks when referring to our mathematical "device".

- if there are false coins in some stack, for some $\mathbf{x} \in \mathbf{R}^N$, $\langle \mathbf{Q}^t(\mathbf{x}), \mathbf{x} \rangle > \| \mathbf{x} \|^2$, and the value increases with every new iteration.

We work with a "weighted Lebesgue measure" with proper non-negative continuous density ρ, for example, the Gaussian distribution

$$\rho(\mathbf{x}) = \frac{1}{\pi^{N/2}} \, e^{-\sum_{s=1}^{N} |x_s|^2}. \tag{8}$$

Hence the probability of the event $\{\mathbf{x} \mid x_1 \in \Omega\}$ is the integral $\mathrm{Prob}(\Omega) = \int_{\Omega \times \mathbf{R}^{N-1}} \rho dm$.

To solve our problem we assume that time is discrete, $t = 1, 2, \ldots$. The procedure is *probabilistic:* it indicates a method to decide, with a probability as close to one as we want, whether there exist any false coins; in the affirmative we will find the stack of false coins. Next we fix a computable real $\eta \in (0, 1)$ as probability threshold. Assume that both η and γ are computable reals. Choose randomly a "test" vector $\mathbf{x} \in \mathbf{R}^N$. Assume that we have a quantum "device" which measures the quadratic form and clicks at time T on $\mathbf{x}$ when

$$\langle \mathbf{Q}^T(\mathbf{x}), \mathbf{x} \rangle > (1 + \varepsilon) \| \mathbf{x} \|^2 .$$

In this case we say that the quantum "device" has sensitivity ε. In what follows we will assume that $\varepsilon > 0$ is a positive computable real. Two cases may appear:

1. If $\langle \mathbf{Q}^T(\mathbf{x}), \mathbf{x} \rangle > (1 + \varepsilon) \| \mathbf{x} \|^2$, then the "device" has clicked at time T and we know for *sure* that there exist false coins in the system.

2. If at time $T > 0$ the "device" hasn't (yet?) clicked, then either all coins are true, i.e., $\langle \mathbf{Q}^t(\mathbf{x}), \mathbf{x} \rangle = \| \mathbf{x} \|^2$, for all $t > 0$, or at time T the growth of $\langle \mathbf{Q}^T(\mathbf{x}), \mathbf{x} \rangle$ hasn't yet reached the threshold $(1 + \varepsilon) \| \mathbf{x} \|^2$.

In the first case the "device" will *never* click, so at each stage t the test-vector $\mathbf{x}$ produces "true" information; we can call $\mathbf{x}$ a "true" vector. In the second case, the test-vector $\mathbf{x}$ is "lying" at time T as we *do* have false coins in the system, but they were not detected at time T; we say that $\mathbf{x}$ produces "false" information at time T. Of course, the second case may be *dangerous*, and indeed, classically we cannot say anything in this case. If the system has false coins and they are located in the j-th stack, then each test-vector $\mathbf{x}$ whose j-th coordinate is 0 produces "false" information at any time.

If the system has false coins and they are located in the j-th stack, $x_j \neq 0$, but

$$\| \mathbf{x} \|^2 + ((1 + \gamma)^T - 1)|x_j|^2 \leq (1 + \varepsilon) \| \mathbf{x} \|^2,$$

then $\mathbf{x}$ produces "false" information at time T. If $|x_j| \neq 0$, then $\mathbf{x}$ produces "false" information only a finite period of time, that is, only for

$$T \leq \log_{1+\gamma} \left(1 + \frac{\varepsilon \| \mathbf{x} \|^2}{|x_j|^2} \right);$$

after this time the quantum "device" starts clicking. *The major problem is to distinguish between the absence/presence of false coins in the system, and in the last case to decide which stack contains false coins.*

We will show how to compute the time T such that when presented a randomly chosen test-vector $\mathbf{x} \in \mathbf{R}^N \setminus \{\mathbf{0}\}$ to a quantum "device" with sensitivity ε that fails to click in time T, then the system doesn't contain false coins with probability larger than $1-\eta$. Consider now the *indistinguishable set at time t*

$$\mathcal{F}_{\varepsilon,t} = \{\mathbf{x} \in \mathbf{R}^N \mid \langle \mathbf{Q}^t(\mathbf{x}), \mathbf{x}\rangle \leq (1+\varepsilon) \parallel \mathbf{x} \parallel^2\}.$$

If the system contains only true coins, then $\mathcal{F}_{\varepsilon,t} = \mathbf{R}^N$, for all $\varepsilon > 0, t \geq 1$. If there is one stack (say, the j-th one) containing false coins, then $\mathcal{F}_{\varepsilon,t}$ is a cone $\mathcal{F}_{\varepsilon,t,j}$ centered at the "false" plane $x_j = 0$: $((1+\gamma)^t - 1)\,|x_j|^2 \leq \varepsilon \parallel x \parallel^2$. A direct calculation shows that

$$\mathrm{Prob}(\mathcal{F}_{\varepsilon,t}) \leq \frac{3N^{3/4}\varepsilon^{1/4}}{\sqrt{\pi}((1+\gamma)^t - 1)^{1/4}} \tag{9}$$

hence,

$$\lim_{t\to\infty} \mathrm{Prob}(\mathcal{F}_{\varepsilon,t}) = 0. \tag{10}$$

In fact, the above limit is *constructive*, that is, from (9), for every computable $\eta \in (0,1)$ we can construct the computable bound

$$T_{\eta,\varepsilon} = \log_{1+\gamma}\left(\frac{3^4 N^3 \varepsilon}{\eta^4 \pi^2} + 1\right) \tag{11}$$

such that *assuming that the system contains false coins, if* $t \geq T_{\eta,\varepsilon}$, *then we get* $\mathrm{Prob}(\mathcal{F}_{\varepsilon,t}) \leq \eta$.

Let us now denote by $\mathcal{N}$ the event "the system contains no false coins" and by $\mathcal{Y}$ the event "the system contains false coins". By $P(\mathcal{N})$ ($P(\mathcal{Y})$) we denote the *a priori* probability that the system contains no false coins (the system contains false coins).

In the simplest case $P(\mathcal{Y}) = \frac{N}{N+1}, P(\mathcal{N}) = 1 - P(\mathcal{Y}) = \frac{1}{N+1}$. We can use Bayes' formula to obtain the *a posteriori probability that the system contains only true coins when at time t the quantum "device" didn't click*:

$$\begin{aligned} P_{\text{non-click}}(\mathcal{N}) &= \frac{P(\mathcal{N})}{P(\mathcal{N}) + (1 - P(\mathcal{N}))\mathrm{Prob}(\mathcal{F}_{\varepsilon,t})} \\ &\geq 1 - N \cdot \mathrm{Prob}(\mathcal{F}_{\varepsilon,t}). \end{aligned}$$

When $t \to \infty$, $\mathrm{Prob}(\Omega_{\varepsilon,t})$ goes to 0, so $P_{\text{non-click}}(\mathcal{N})$ goes to 1. More precisely, if $t \geq T_{\eta,\varepsilon}$, as in (11), then $P_{\text{non-click}}(\mathcal{N}) \geq 1 - \eta N$.

In conclusion,

> *for every computable* $\eta \in (0,1)$ *we can construct a computable time* $T = T_{\eta,\varepsilon}$ *such that picking up at random a test-vector* $\mathbf{x} \in \mathbf{R}^N \setminus \{\mathbf{0}\}$ *and using a quantum "device" with sensitivity* ε *up to time* T *either*

- ◇ *we get a click at some time $t \leq T$, so the system contains false coins; the jth stack, where j is the unique coordinate such that $(Q^T(\mathbf{x})/((1+\gamma)^T-1))_j > x_j$, contains false coins;*

- ◇ *we do not get a click in time T, so with probability greater than $1-\eta N$ all coins are true.*

For the infinite case we follow the same strategy but use more sophisticated mathematical tools: the "device" (with sensitivity ε) will distinguish the values of the iterated quadratic form $\langle \mathbf{Q}^t(\mathbf{x}), \mathbf{x}\rangle = \sum_{i=1}^{\infty} q_i^t\, |x_i|^2$, by observing the difference between the non-perturbed and perturbed sequences t_l, $\tilde{t}_l$ corresponding to two discrete stochastic processes. We work with the intersections of l_2 with the discrete Sobolev class l_2^1 of summable sequences with the square norm

$$| \mathbf{x} |_1^2 \; = \sum_{m=1}^{\infty} |x_{_m} - x_{_{m-1}}|^2, \tag{12}$$

and the discrete Sobolev class $\tilde{l}_2^1$ of weighted-summable sequences with the square norm

$$\| \mathbf{x} \|_1^2 \; = \sum_{m=1}^{\infty} \frac{1-\tilde{\delta}_m}{\tilde{\delta}_m} \; |x_{_m} - x_{_{m-1}}|^2. \tag{13}$$

We consider two discrete stochastic processes corresponding to the *equidistant sequence* of moments of time $t_l = l,\ l = 0, 1, \ldots,\ \delta_s = 1$ and to the *perturbed sequence* of moments of time $\tilde{t}_l = \sum_{m=0}^{l} \tilde{\delta}_m,\ \tilde{\delta}_m < 1$.

By natural extension from cylindrical sets we can define the Wiener measures $\tilde{W}$ and W on these spaces. In what follows we are going to use the following relation between $\tilde{W}$ and W (see [42]): due to the absolute continuity of the Wiener measure $\tilde{W}$ with respect to W, for every W–measurable set Ω,

$$\tilde{W}(\Omega) = \frac{1}{\prod_{l=1}^{\infty} \sqrt{\delta_l}} \int_{\Omega} e^{-\sum_{m=1}^{\infty} \frac{1-\tilde{\delta}_m}{\tilde{\delta}_m} |x_m - x_{m-1}|^2} dW. \tag{14}$$

If we assume that there exist false coins in the system, say at stack j, but the "device" does not click in time T, then the test-vector belongs to the *indistinguishable set*

$$\mathcal{F}_{\varepsilon,T} = \{\mathbf{x} \in l_2^1 \mid ((1+\gamma)^T - 1) \mid x_j \mid^2 < \varepsilon \parallel \mathbf{x} \parallel_1^2, \text{ for some } j\}.$$

The Wiener measure of the indistinguishable set tends to zero as

$$\tilde{W}(\mathcal{F}_{\varepsilon,T}) \leq \left(\frac{\varepsilon}{((1+\gamma)^T - 1 - \varepsilon) \cdot \prod_{m=1}^{\infty} \tilde{\delta}_m} \right)^{1/2}.$$

More precisely, $\tilde{W}(\mathcal{F}_{\varepsilon,T})$ *converges constructively to zero,*

$$\tilde{W}(\mathcal{F}_{\varepsilon,T}) \leq \eta, \text{ provided } t > \log_{1+\gamma} \left(\frac{\varepsilon}{\eta^2 \prod_{m=1}^{\infty} \tilde{\delta}_m} + 1 + \varepsilon \right).$$

Hence, the *a posteriori probability that the system contains only true coins when at time T the quantum "device" didn't click is*

$$P_{\text{non-click}}(\mathcal{N}) > 1 - \frac{1 - P(\mathcal{N})}{P(\mathcal{N})} \cdot \frac{\sqrt{\varepsilon}}{\sqrt{(1+\gamma)^T - 1 - \varepsilon}\sqrt{\prod_{m=1}^{\infty} \tilde{\delta}_m}}.$$

4 Scattered Waves and Scattering Data

In this section we start building a simple scattering system with input space $\mathcal{E}$, endowed with an appropriate probabilistic structure, which can be used to solve probabilistically the Finite and Infinite Merchant Problems. More precisely, we shall discuss the algebraic structure of scattered waves and some properties of scattering data in the input space of the system.

Note that introducing a proper probabilistic structure on the input subspace requires special preparation of the input beam of quantum particles used in the scattering experiment. In the finite-dimensional case we may assume that, for a given energy, the distribution of testing quantum particles in the input beam is isotropic with respect to the other quantum numbers. In the infinitely-dimensional case the probabilistic structure in the input space requires an infinite set of quantum numbers and a complete description of all trajectories of the Markov process used to implement the countably-additive measure. The problem of introducing a countably-additive measure on the input space (which is similar to a basic problem in Quantum Field Theory) will be discussed in Section 6.

We begin with the "zero-range" model which is described in Appendix, Theorem 7.4 where the scattering matrix is calculated (see equation (35)). We assume now that the deficiency subspace is one-dimensional, hence β is just a vector from the input space. Then

$$\mathbf{S}_\beta(p) = \mathbf{P}_\beta^\perp + \frac{ip\mathcal{M} - |\beta|^2}{ip\mathcal{M} + |\beta|^2}\mathbf{P}_\beta. \tag{15}$$

Here $\mathbf{P}_\beta$ is the one-dimensional projection from the input space onto the subspace spanned by the unit vector β_0, and $\beta = |\beta|\beta_0$ is the vector defining the boundary conditions (33). The orthogonal projection onto the complement $I - \mathbf{P}_\beta$ is denoted by $\mathbf{P}_\beta^\perp$.

We consider two states of the scattering system: the state S_1, when the resonance level α_1^2 is vacant, that is $\mathcal{M} = \mathcal{M}_1$ has a pole at α_1^2, and the state S_2 when the resonance level is occupied and $\mathcal{M}_2(\alpha_1^2) = 0$. In the first case the scattering matrix is trivial, that is $\mathbf{S} = I$; in the second case the scattering matrix is non-trivial:

$$\mathbf{S}(\alpha_1^2) = \mathbf{P}_\beta^\perp - \mathbf{P}_\beta = 1 - 2\mathbf{P}_\beta. \tag{16}$$

The eigenfunctions of the corresponding Schrödinger operator at the spectral point α_1^2 are

$$\psi(x) = e^{i\alpha_1 x}\nu + e^{-i\alpha_1 x}(P_\beta^\perp - P_\beta)\nu,$$

for every vector ν from the input space.

In the second case the eigenfunctions are trivial in the subspace orthogonal to the vector $\vec{\beta}$

$$\psi_\perp(x) = e^{i\alpha_1 x}\nu + e^{-i\alpha_1 x}\nu,\ \nu\perp\beta,$$

and non-trivial in the one-dimensional subspace spanned by the vector $\vec{\beta}$:

$$\psi_\beta(x) = e^{i\alpha_1 x}\vec{\beta} - e^{-i\alpha_1 x}\vec{\beta}.$$

Consider the scattering process starting with an input vector $\mathbf{x} \in \mathcal{E}$. The correlation between the input and output for the resonance energy is trivial in the first case $\langle \mathcal{S}_1\mathbf{x}, \mathbf{x}\rangle = |\mathbf{x}|^2$, but is non-trivial in the second case:

$$\langle \mathcal{S}_2\mathbf{x}, \mathbf{x}\rangle = |\mathbf{P}_\beta^\perp \mathbf{x}|^2 - |\mathbf{P}_\beta \mathbf{x}|^2 = |\mathbf{x}|^2 - 2|\mathbf{P}_\beta \mathbf{x}|^2. \tag{17}$$

A simple comparison reveals the difference between the results of scattering $\langle \mathcal{S}_1\mathbf{x}, \mathbf{x}\rangle - \langle \mathcal{S}_2\mathbf{x}, \mathbf{x}\rangle = 2|\mathbf{P}_{_\beta}\mathbf{x}|^2$. In the simplest case the "device" clicks if the above difference exceeds $\varepsilon|\mathbf{x}|^2$, but does not click otherwise. The set $\mathcal{F}_\varepsilon$ of vectors $\mathbf{x} \in \mathcal{E}$ such that

$$2|\mathbf{P}_\beta \mathbf{x}|^2 = 2|\langle \mathbf{x}, \beta\rangle|^2 < \varepsilon|\mathbf{x}|^2,$$

is invariant with respect to multiplication by constants (it is a cone). Following the terminology described in Section 3, $\mathcal{F}_\varepsilon$ is called an *indistinguishable set* because the non-click result of a single act of scattering may be caused either by the fact that the scattering system is in state S_1, or in state S_2, but the vector $\mathbf{x}$ is "lying", that is, its beta-component is relatively too small to produce the click. If we consider a probability measure μ on the input space and the cone $\mathcal{F}_\varepsilon$ is measurable, then we assume that

$$\mathrm{Prob}(\mathcal{F}_\varepsilon) = \mu(\mathcal{F}_\varepsilon). \tag{18}$$

If in two independent experiments we obtain non-click results, then either the system is in state S_1 or the system is in state S_1, but the test-vector $\mathbf{x}$ was chosen both times from the indistinguishable set $\mathcal{F}_\varepsilon$. The probability of choosing independently the test-vector $\mathbf{x}$ twice from the indistinguishable set is $(\mu(\mathcal{F}_\varepsilon))^2$. In general, the probability of choosing test-vectors from the indistinguishable set $\mathcal{F}_\varepsilon$ in l independent experiments is $(\mu(\mathcal{F}_\varepsilon))^l$ which converges to zero when $l \to \infty$.

If the *a priory* probability to find the scattering system in state S_1 (S_2) is P_1 (P_2, respectively), and $P_1 + P_2 = 1$, then the *a posteriori* probability that the scattering system is state S_1 after a series of l independent experiments each of which produces a non-click result can be calculated via Bayes' formula as:

$$\mathrm{Prob}_{l;\ \text{non-click}}(S_1) = \frac{P_1}{P_1 + P_2(\mu(\mathcal{F}_\varepsilon))^l}, \tag{19}$$

which guarantees the same type of exponential convergence to zero as in (10).

The proposed mathematical scheme essentially depends on the probability measure (see (18)). Note that following the strategy explained in the beginning of Section 3, it is not sufficient to prove that the process converges to zero, but that it *constructively converges to zero*.

The simplest choice for the *finite case* is the Gaussian measure with density (8) in the N-dimensional complex space $\mathcal{E} = \mathbf{C}^N$ of test-vectors. The Gaussian measure of the cone

$$\mathcal{E}_\varepsilon = \left\{\mathbf{x} : |\langle \mathbf{x}, \beta\rangle|^2 \le \frac{\varepsilon}{2}|\mathbf{x}|^2\right\},$$

in the N-dimensional complex space $\mathcal{E}$ of test-vectors can be calculated as follows. Without loss of generality we can assume that $N = 2n$, β is a basis vector co-linear with the axis N and

$$\Delta_\varepsilon = \sqrt{\frac{\varepsilon}{1-\varepsilon}}.$$

Put

$$R = \sqrt{\sum_{k=1}^{2n-2} x_k^2},\ \rho = \sqrt{x_{2n-1}^2 + x_{2n}^2}.$$

Then, if we denote by S_n the area of the sphere in $\mathbf{C}^n$, then

$$\begin{aligned}
\mathrm{Prob}(\mathcal{F}_\varepsilon) &= \frac{1}{\pi^n} \int ... \int_{\rho \le \Delta_\varepsilon R} e^{-\rho^2} dx_{n-1} dx_n \prod_{k=1}^{2n-2} e^{-x_k^2} dx_k \\
&= \frac{\mathrm{S}_{2n-2}}{\pi^n} \int_0^\infty e^{-R^2} R^{2n-3} \left(2\pi \int_0^{\Delta_\varepsilon R} e^{-\rho^2} \rho d\rho \right) dR \\
&= \frac{\mathrm{S}_{2n-2}}{\pi^{n-1}} \int_0^\infty \left(e^{-R^2} - e^{-\frac{1}{1-\varepsilon} R^2} \right) R^{2n-3} dR \\
&= 1 - (1-\varepsilon)^{n-1}.
\end{aligned}$$

For any "device" with sensitivity ε we have $\mathrm{Prob}(\mathcal{F}_\varepsilon) < 1$, hence the above calculation shows that if in the series of l independent experiments we always get a non-click result, then the *a posteriori* probability that the scattering system in state S_1 is

$$\mathrm{Prob}_{l;\ \text{non-click}}(S_1) = \frac{P_1}{P_1 + P_2 (1 - (1-\varepsilon)^{n-1})^l}. \tag{20}$$

It is easy to see that for every P_1 and P_2, $\mathrm{Prob}_{l;\ \text{non-click}}(S_1)$ *constructively converges to 1.* This result (which does not require any computability assumption on P_1 and P_2) is similar to the main result obtained in [12] for the the Finite Merchant's Problem (see Section 3).

Using the spherical symmetry of the Gaussian density we may sweep the measure of the whole input space $\mathcal{E}$ onto the unit sphere $\Sigma_1 \subset \mathcal{E}$ obtaining, in the finitely-dimensional case, the properly normalized Lebesgue measure on Σ_1. Then, the probability of set of normalized indistinguishable elements forming the "*belt*" $\left\{ \mathbf{x} : |\langle \mathbf{x}, \beta \rangle|^2 \le \frac{\varepsilon}{2} \right\}$ on the unit sphere $|\mathbf{x}| = 1$ coincides with the probability of the corresponding cone and depends on the "device" sensitivity ε and the dimension of the input space $\mathcal{E}$. But, unfortunately, from the formula (20) one can see that the Gaussian probability cannot be used in the infinitely-dimensional case (as an approximation when $n \to \infty$) since $(1-\varepsilon)^{n-1} \to 0$, for $0 < \varepsilon < 1$. This means that for a series of l experiments producing only non-click results the *a posteriori* probability is equal to the *a priori* probability P_1, independently of l. This fact is dual to the important fact from measure theory [23] cited in [12]: the measure defined by the Gaussian density $\rho(\mathbf{x}) = e^{-\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle}$, with a positive operator $\mathbf{A}$, on all cylindrical sets in a Hilbert space H with finitely-dimensional sections is Lebesgue-extendable as a countably-additive measure in H if and only if $\mathbf{A}$ is a positive Hilbert-Schmidt operator in H.

5 Resonance Amplification

In the solution (from [12]) described in Section 3 we can observe not only a single act of computation – the measurement of the quadratic form of the operator $\mathbf{Q}$, but also a measurement of quadratic forms of iterations $\langle \mathbf{Q}^t(\mathbf{x}), \mathbf{x}\rangle$ of the operator $\mathbf{Q}$. In contrast with the classical case, iterated experiments *with the same test-vector* do not increase the probability, but clarify the special role of resonances and permit to reduce the general case of the scattering system (with general rational R-function $\mathcal{M}$) to the special system with the scattering matrix (7), discussed in the previous section.

The estimation of the probability of absence of "false coins" in Section 3 was based on information regarding the weights of true and false coins. When trying to distinguish between the states $S(1)$ and $S(2)$ of our the quantum system we will assume that the state of the quantum system is is characterized by the frequency p_0 of the resonance, by the corresponding inverse life-time τ_0 and by the resonance vector β_0. In case the perturbed (non-trivial) scattering matrix $\mathbf{S}(p)$ is analytic in the lower half-plane this assumption means that

$$\mathbf{S}(p_0 - i\tau_0)\beta_0 = 0.$$

We assume now that the resonance data are not exact (so that the above equations are not necessarily fulfilled), but they are classically *computable* and given with proper precision. Further, assume that the incoming beam of particles is not mono-chromatic, but may be formed such that momenta $p = \sqrt{\lambda}$ of particles have a Breit-Wigner distribution, with density equal to the Poisson kernel:

$$\rho_{p_0,\tau_0}(p) = \frac{1}{\pi}\frac{\delta}{(p-p_0)^2 + \tau_0^2}.$$

Here $\zeta_0 = p_0 + i\tau_0$ is a complex point in the upper half-plane which may be chosen in a special way to maximize the "device' efficiency. The positive smooth density ρ is normalized $\int_{\mathbf{R}} \rho(p)dp = 1$ and converges if its width τ_0 goes to zero (limit of the mono-chromatic input beam). Averaging the correlation $\langle (\mathbf{S}^+)^T(p)\mathbf{x}, \mathbf{x}\rangle$ of the iterated scattering matrix (15)

$$\mathbf{S}_\beta(p) = \mathbf{P}_\beta^\perp + \frac{ip\mathcal{M}(\lambda) - |\vec{\beta}|^2}{ip\mathcal{M}(\lambda) + |\vec{\beta}|^2}\mathbf{P}_\beta,$$

over the Breit-Wigner distribution gives the following result:

$$\begin{aligned}
\int_{\mathbf{R}} \rho(p)\langle (\mathbf{S}^+)^T(p)\mathbf{x},\ \mathbf{x}\rangle dp &= \langle (\mathbf{S}^+)^T\mathbf{x},\ \mathbf{x}\rangle(p_0 + i\tau_0) \\
&= |\mathbf{P}_\beta^\perp\mathbf{x}|^2 + \left(\frac{i\zeta_0\mathcal{M}(\zeta_0^2) - |\beta|^2}{i\zeta_0\mathcal{M}(\zeta_0^2) + |\beta|^2}\right)^T |\mathbf{P}_\beta\mathbf{x}|^2 \\
&= |\mathbf{x}|^2 - \left(1 - \left(\frac{i\zeta_0\mathcal{M}(\zeta_0^2) - |\beta|^2}{i\zeta_0\mathcal{M}(\zeta_0^2) + |\beta|^2}\right)^T\right)|\mathbf{P}_\beta\mathbf{x}|^2.
\end{aligned}$$

The coefficient in front of $|\mathbf{P}_{\bar\beta}\mathbf{x}|^2$ is close to one if the input vector $\mathbf{x}$ is close to the resonance vector $\bar\beta$, and/or the point ζ_0 is chosen close to the expected resonance – the zero of the adjoint

scattering matrix $\mathbf{S}^+$ in the upper half-plane. Thus, for large T, the averaged correlation of the iterated scattering matrix is similar to the correlation (17) of the scattering matrix of the special system analyzed in Section 3. In particular, if the Breit-Wigner parameters p_0, τ_0 and the test-vector are chosen to be *equal* to the resonance parameters, we have

$$\int_{\mathbf{R}} \rho(p)\langle(\mathbf{S}^+)(p)\mathbf{x},\mathbf{x}\rangle dp = \langle(\mathbf{S}^+)(p_0+i\tau_0)\mathbf{x},\mathbf{x}\rangle = 0,$$

if the tested system is in state $S(2)$ and

$$\int_{\mathbf{R}} \rho(p)\langle(\mathbf{S}^+)(p)\mathbf{x},\mathbf{x}\rangle dp = 1,$$

if the tested system is in state $S(1)$, that is the scattering matrix is trivial, $\mathbf{S}^+ = \mathbf{S} = I$.

The resonance amplification based on the above idea of averaging will be used in the infinite-dimensional case in the next section.

6 A Brownian Solution Based on Resonance Amplification

In real quasi-one-dimensional nano-electronic circuits constructed with *cylindrical* quantum wires on the surface of a semiconductor the dimension of the input space (with a certain value of the momentum p) is finite. Indeed, for a nano-wire with finite diameter of the cross-section d, due to quantization of electrons on the Fermi level E_F (see [35]), the momentum can take only discrete values, which correspond to the open channels in the wire defined by the non-decreasing sequence of eigenvalues λ_k of the Laplace operator on the cross-section of the wire:

$$p_{_{F,1}} = \sqrt{\frac{2mE_F}{\hbar^2} - \lambda_1},\ p_{_{F,2}} = \sqrt{\frac{2mE_F}{\hbar^2} - \lambda_2}, ...,$$

$$p_{_{F,k_{\max}}} = \sqrt{\frac{2mE_F}{\hbar^2} - \lambda_{k_{max}}},\ \frac{2mE_F}{\hbar^2} - \lambda_k < 0 \text{ for } k > k_{\max}.$$

When studying properties of a microscopical quantum switch it is convenient to use *conical wires* in order to simplify the set up of macroscopical contacts of the wires with the power supply, see for instance [41, 14]. We consider the scattering problem with a few infinite conical quantum wires to obtain an infinitely-dimensional input space. Indeed, the continuous spectrum $\sigma_C = \{\lambda\}$, $\lambda = |\vec{p}|^2$ of the Laplace operator

$$-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} = l$$

in the truncated $2-d$ cone of angle Θ, $G_2 = \{r > 1, 0 < \theta < \Theta\}$ with zero boundary conditions has infinite multiplicity in $L_2(G_2)$ on the positive half-axis $0 < p^2 < \infty$. Unfortunately, this construction cannot be used to implement the resonance search described above, since the corresponding scattering matrix has a logarithmic singularity at the origin and the corresponding cuts in the complex plane. But for some $3-d$ cones the scattering matrix is an

analytic function of energy (see [34]): the role of cone is played by the whole $3-d$ space. To avoid technical complications we consider now an imaginable "device" represented by a quantum scattering system with an infinite-dimensional input space for which a solvable model is described in Theorem 7.4 (see the Appendix), as an extension of the orthogonal sum $A_0 \oplus l_0$ with boundary condition (31). We associate this extension with two states $S(1), S(2)$ of the total quantum system as a result of the interaction with inner and outer components, respectively: with the boundary condition (31), *switched on* if $\beta \neq 0$ and *switched off* if $\beta = 0$.

In what follows we will interpret the probabilistic solution of the Infinite Merchant's Problem as a problem of distinguishing between the states of the above described quantum system via a scattering experiment with a random input.

In scattering experiments we usually observe the asymptotic behaviour of scattered waves – the eigenfunctions of the absolutely-continuous spectrum of the operator – at infinity. This behaviour is defined in our case by the corresponding reflection coefficients ("stationary" scattering matrices) which depend on the "state" of the system. The reflection coefficient corresponding to the first state can be calculated via separation of variables; the calculation of the reflection coefficient in the second case is reduced to matching the boundary data (see the definitions in Appendix) $\vec{\xi}_{\pm}$ of the solution of the adjoint equation $A_0^+ u = \lambda u$ in the inner space E with the boundary data of the Anzatz

$$\Psi_\nu = e^{ipx}\nu + e^{-ipx}\mathbf{S}_\beta \nu, \tag{21}$$

combined of incoming and outgoing modes, see the Theorem 7.4.

We will choose the parameters of the Breit-Wigner distribution such that $\zeta_0 = p_0 + i\tau_0$ coincides with some zero of the scattering matrix . Then, averaging the correlation $\langle \mathbf{S}e,\, e\rangle(p)$ with a random test-vector over the Breit-Wigner distribution gives the following result:

$$\int_R \rho(p)\langle \mathbf{S}e,\, e\rangle(p)dp = \langle \mathbf{S}e,\, e\rangle(\zeta_0) = |e_\perp|^2. \tag{22}$$

Following the probabilistic strategy in ([12]) we compare the averaged correlation $\int_{\mathbf{R}} \rho(p)\langle \mathbf{s}e,\, e\rangle(p)dp$ of the relative scattering matrix in states $S(1)$ and $S(2)$. In the first state the relative scattering matrix coincides with identity, hence averaging gives $|e|^2$; in the second state the averaging on a random test-vector gives $|e_\perp|^2 = |e|^2 - |P_\beta e|^2$.

One may expect that the probability of the event $P_\beta e = 0$ is zero, hence choosing a random test-vector, with probability 1 the above correlation is strictly less than 1. To obtain the quantitative result we will say that the testing "device" distinguishing between the two states of the system "clicks" if

$$|P_\beta e|^2 > \varepsilon |e|^2.$$

Unfortunately, the above "device" is not sensitive enough, and we need a stronger norm in the right side of the last inequality. In our case the input space $\mathcal{E} = L_2(\Omega)$ is l_2 with the orthogonal basis of eigenfunctions of the Laplacian on the truncation domain Ω,

$$\mathbf{x} \longrightarrow \{x_m\}_{m=0}^{\infty},$$

where x_m are proper Fourier coefficients of the vector $\mathbf{x} \in \mathcal{E}$. Following ([12]) we consider the discrete Sobolev classes and norms introduced in Section 3, (12) and (13). in order to define the case when the "device" clicks. Following the pattern of the Section 4.1 in [12] we assume that the (complex) increments $x_m - x_{m-1}$ are independent. We are going to use, together with l_2 two more spaces of test-vectors. Both of them are stochastic spaces of all trajectories $\mathbf{x}(t)$ of a Brownian particle on the complex plane along different discrete sequences of moments of time ("stops"): the equidistant sequence $t_l = \sum_{m=1}^{l} \delta_m$, for the first space, and the sequence $\tilde{t}_l = \sum_{m=1}^{l} \tilde{\delta}_m$, on a pre-assigned sequence of "stops", for the second space. Both spaces are equipped with proper Wiener measures W, $\tilde{W}$ (see [42]). The measure W is defined on the algebra of all finite-dimensional cylindrical sets $C^{t_1,t_2,\ldots,t_N}_{\Delta_1,\Delta_2,\ldots,\Delta_N}$ of trajectories with fixed initial point $x_0 = 0$ and "gates" Δ_l, $l = 1, \ldots, N$ (which are open discs in the complex plane):

$$C^{t_1,t_2,\ldots,t_N}_{\Delta_1,\Delta_2,\ldots,\Delta_N} = \{\mathbf{x} \mid x_{t_l} \in \Delta_l, l = 1, 2, \ldots, N\},$$

via multiple convolutions of the Green functions $G(x_{l+1}, t_{l+1}|x_l, t_l)$ corresponding to the sequence $\delta_{l+1} = t_{l+1} - t_l$:

$$W^N(C^{t_1,t_2,\ldots,t_N}_{\Delta_1,\Delta_2,\ldots,\Delta_N}) = \frac{\int\ldots\int_{\Delta_N,\Delta_{N-1},\ldots,\Delta_1} \frac{dx_1 dx_2 \ldots dx_N}{\pi^{\frac{N}{2}}\delta_N\delta_{N-1}\ldots\delta_1} e^{-\frac{|x_N - x_{N-1}|^2}{\delta_N}} \ldots e^{-\frac{|x_1 - x_0|^2}{\delta_1}}}{\int\ldots\int_{\mathbf{R}_N,\mathbf{R}_{N-1},\ldots,\mathbf{R}_1} \frac{dx_1 dx_2 \ldots dx_N}{\pi^{\frac{N}{2}}\sqrt{\delta_N\delta_{N-1}\ldots\delta_1}} e^{-\frac{|x_N - x_{N-1}|^2}{\delta_N}} \ldots e^{-\frac{|x_1 - x_0|^2}{\delta_1}}}, \tag{23}$$

where $\mathbf{R}_N = \mathbf{R}_{N-1} = \ldots = \mathbf{R}_1 = \mathbf{R}$. Using the convolution formula, the denominator of (23) can be reduced to the Green function $G(x_N, t_N \mid 0, 0)$, for any $\tau \in (s, t)$:

$$G(x, t \mid y, s) = \int_{-\infty}^{\infty} G(x, t \mid \xi, \tau) G(\xi, \tau \mid y, s) d\xi.$$

Similarly, we can define the Wiener measure $\tilde{W}$ on trajectories with "stops" at the prescribed sequence moments of time $\tilde{t}_l$, $\tilde{t}_{l+1} - \tilde{t}_l = \Delta_l$. This perturbed Wiener measure $\tilde{W}$ is absolutely continuous with respect to the Wiener measure W associated with the equidistant sequence t_l, see (14).

Further we consider the class of *quasi-loops*, that is the class of all trajectories of the "perturbed process" which begin from $(x_0, \tilde{t}_0) = (0, 0)$ and for any t, $\max_{0<s<t} |x_s|^2 < Ct$. We note that

- every $\mathbf{x} \in l_2^1$ is a quasi-loop (with $C = |\mathbf{x}|_1^2$),
- due to the reflection principle (see [42], p. 221), the class of all quasi-loops has Wiener measure one, both in respect of W, $\tilde{W}$.

We assume that our "device" cannot identify the state of the system from the observation of the Breit-Wigner averaged correlation between the input and output of a single act of scattering when presented a randomly chosen input $\mathbf{x} \in \mathcal{E}$ if $|P_\beta \mathbf{x}|^2 < \varepsilon |\mathbf{x}|^2$. This means that the test-vector $\mathbf{x}$ belongs to the *indistinguishable set*

$$\begin{aligned} \mathcal{F}_{\varepsilon,T} &= \left\{ \mathbf{x} \in l_2 \cap l_2^1 \mid |P_\beta \mathbf{x}|^2 < \ \varepsilon \ \left(\sum_{m=1}^{\infty} \frac{1-\Delta_m}{\Delta_m} |x_{_m} - x_{_{m-1}}|^2 \right) \right\} \\ &= \{ \mathbf{x} \in l_2 \cap l_2^1 \mid |P_\beta \mathbf{x}|^2 < \ \varepsilon \parallel \mathbf{x} \parallel_1^2 \}. \end{aligned} \tag{24}$$

Note that the indistinguishable set depends only on the positive number ε, the vector $\beta \in \mathcal{E}$ defining the interaction in the quantum system, and the sequence Δ. We assume that the vector β has all non-zero components $\beta_l \neq 0$, and the vector $\mathbf{b} = \{b_l\}_{l=1}^{\infty}$, $b_l = \sum_{m=l}^{\infty} \beta_m$ belongs to l_2

$$|b|_{l_2}^2 = \sum_{m=1}^{\infty} m^2 |\beta_m|^2 < \infty. \tag{25}$$

Our main result is:

Theorem 6.1 *If the condition (25) is satisfied, then the Wiener probability $\tilde{W}(\mathcal{F}_{\varepsilon,1})$ of the indistinguishable set $\mathcal{F}_{\varepsilon,1}$ corresponding to a single act of scattering is finite and*

$$\tilde{W}(\mathcal{F}_{\varepsilon,1}) < \frac{\sqrt{\varepsilon}|\beta|}{\prod_l \Delta_l \sqrt{\varepsilon|\beta|^2 + |b|^2}}. \tag{26}$$

Following the calculation presented in [12], we approximate the indistinguishable set with finite-dimensional cylinder sets and reduce the estimation of the $\tilde{W}(\mathcal{F}_{\varepsilon,1})$ to the calculation of a Wiener integral with respect to the W measure on trajectories associated with "equidistant" intermediate points. We now have:

$$\begin{aligned} & \tilde{W}(\mathcal{F}_{\varepsilon,1}) \\ \leq & \int_{\text{quasi-loops}, x_0=0 \, |\langle \mathbf{x},\beta\rangle|^2 < \varepsilon \|\mathbf{x}\|_1^2} d\tilde{W} \\ = & \lim_{C\to\infty} \int_{sup_{s\leq l}|x_s| < C\sqrt{l}, l\leq N,\, x_0=0 |\langle \mathbf{x},\beta\rangle|^2 < \varepsilon \|\mathbf{x}\|_1^2} d\tilde{W} \\ = & \frac{1}{\prod_{l=1}^{\infty} \Delta_l} \lim_{C\to\infty} \lim_{N\to\infty} \\ & \frac{\int_{sup_{s\leq l}|x_s| < C\sqrt{l}, l\leq N,\, x_0=0} \int\int \ldots \int_{-\infty}^{\infty} dx_1 dx_2 \ldots dx_N e^{\frac{|x_N - x_{N-1}|^2}{\Delta_N}} e^{\frac{|x_{N-1}-x_{N-2}|^2}{\Delta_{N-1}}} \ldots e^{\frac{|x_1 - x_0|^2}{\Delta_1}}}{\pi^N \int_{|x_N| < C\sqrt{N}} dx_N e^{\frac{x_N - x_0}{t_N}}} \end{aligned}$$

$$\leq \frac{1}{\prod_{l=1}^{\infty}\Delta_l}\lim_{C\to\infty}\lim_{N\to\infty}$$
$$\frac{\int_{sup_{s\leq l}|x_s|<C\sqrt{l}, l\leq N,\, x_0=0}\int\int\ldots\int_{-\infty}^{\infty}dx_1dx_2\ldots dx_N e^{-\sum_{s=1}^{N}\frac{|x_s-x_{s-1}|^2}{\delta_s}-(1-\Delta_s)\frac{|x_s-x_{s-1}|^2}{\Delta_s}}}{\pi^N\int_{|x_N|<C\sqrt{N}}dx_N e^{\frac{x_N-x_0}{\tilde{t}_N}}}.$$

The integrand of the inner integral in the numerator contains the factor

$$e^{(1-\Delta_N)\frac{|x_N-x_{N-1}|^2}{\Delta_N}-(1-\Delta_{N-1})\frac{|x_{N-1}-x_{N-2}|^2}{\Delta_{N-1}}-(1-\Delta_1)\frac{|x_1-x_0|^2}{\Delta_1}},$$

which can be estimated by (24)

$$e^{-\frac{1}{\varepsilon}|P_\beta \mathbf{x}|^2}=e^{-\frac{1}{|\beta|^2\varepsilon}|\langle \mathbf{x},\,\beta\rangle|^2}.$$

Using this equality, the exponential in the numerator can be estimated from below by the quadratic form

$$\sum_{m=1}^{N}|x_m-x_{m-1}|^2+\frac{1}{|\beta|^2\varepsilon}|\langle \mathbf{x},\,\beta\rangle|^2. \tag{27}$$

This quadratic form can be simplified using new vector variables $\xi_m=x_m-x_{m-1}$:

$$\langle \beta,\,\mathbf{x}\rangle=\sum_{m=1}^{\infty}x_m\bar{\beta}_m=\sum_{m=1}^{\infty}\xi_m\overline{\sum_{l=m}^{\infty}\beta_l}.$$

Note that the vector $\mathbf{b}, b_m=\sum_{l=m}^{\infty}\beta_l$, belongs to l_2. Then, the quadratic form in the exponent of the numerator may be presented as a quadratic form of an operator

$$|\xi|^{\mathbf{2}}+\frac{\mathbf{1}}{|\beta|^{\mathbf{2}}\varepsilon}|\langle \mathbf{b},\xi\rangle|=\langle \xi,\left(\mathbf{I}+\frac{|\mathbf{b}|^{\mathbf{2}}}{|\beta|^{\mathbf{2}}\varepsilon}\mathbf{P_b}\right)\xi\rangle=\langle \xi,\mathbf{A}_\varepsilon\xi\rangle,$$

where $\mathbf{P_b}$ is the orthogonal projection onto the one-dimensional subspace in l_2 spanned by the vector $\mathbf{b}$. The obtained ratio of the N-dimensional Gaussian integral in the numerator, normalized by the factor π^{-N} and the Gaussian integral in the denominator can be expressed as

$$\frac{1}{\pi^N}\int\int\ldots\int e^{\langle,A_\varepsilon\xi\rangle}d\xi_1d\xi_2\ldots d\xi_N=\frac{1}{\sqrt{\det A_\varepsilon}}=\frac{\sqrt{\varepsilon}|\beta|}{\sqrt{\varepsilon|\beta|^2+|b|^2}}.$$

Finally, we obtain the announced result by taking into account the omitted factor $\prod_l\Delta_l$.

7 Appendix

In this section we give a brief presentation of the symplectic version of the operator-extension technique which may be useful for the design of nano-electronic devices and quantum circuits. In particular, we construct a solvable model of a quantum dot which can control resonance properties of the scattering matrix; the model can be used for both finite-dimensional and

infinite-dimensional input-output channels. Note that our model is obtained as an extension of a non-densely defined operator. Despite the danger of obtaining an operator-relation instead of an operator, [36], one may show (see [28, 30]) that this procedure gives actually a properly defined Hamiltonian with rich spectral properties.

Zero-range quantum models have appeared in mathematical physics with the paper [7], where the mathematical explanation of the singular potential introduced by Fermi [19] was suggested. More mathematical and physical details on zero-range potentials may be found in [3] and [15].

We construct a solvable model of the quantum controlled-NOT gate and a similar model for a "device" solving the Infinite Merchant's Problem. Both models are constructed as self-adjoint extensions of an orthogonal sum of a properly restricted finite Hermitian matrix $A : E \to E$ and the Schrödinger operator $l = -\frac{d^2}{dx^2}$ in the space $L_2(\mathbf{R}, \mathcal{E})$ of quadratically integrable vector functions taking values in a finite-dimensional or infinite-dimensional space $\mathcal{E}$.

The first model is realistic and may be easily implemented in the finite-dimensional case; there still remain problems related to the second model.

In the first case we restrict the Schrödinger operator $l \to l_0$ onto the class of all vector-functions vanishing at the origin, $v(0) = 0$, with continuous derivative $[v']|_0 = 0$. The adjoint operator l_0^+ is defined on continuous elements from the orthogonal sum of Sobolev classes

$$W_2^2(\mathbf{R}_-, \mathcal{E}) \oplus W_2^2(\mathbf{R}_+, \mathcal{E}) \subset L_2(\mathbf{R}, \mathcal{E})$$

with a possible jump of the derivative $[u']|_0$ at the origin. The *boundary form* is calculated via integration by parts as

$$J(U,\, V) = \langle l_0^+ U,\, V\rangle - \langle U,\, l_0^+ V\rangle = \langle [U'](0),\ V(0)\rangle_{\mathcal{E}} - \langle U(0),\ [V'](0)\rangle_{\mathcal{E}}. \tag{28}$$

We assume that the Hermitian operator $A : E \to E$ is defined by its spectral decomposition (on discrete spectrum)

$$A = \sum_\alpha \alpha^2 P_\alpha,$$

where α^2 are positive eigenvalues and P_α are the corresponding spectral projections, $P_\alpha = e_\alpha\rangle\langle e_\alpha$. We choose a finite-dimensional subspace $\mathcal{N}_i$, $\dim \mathcal{N}_i = d$, and restrict A onto the subspace $D_0 = \frac{I}{A-iI}[E \ominus \mathcal{N}_i]$. Any ortho-normal basis in $\mathcal{N}_i$ $\{e_l\} = \{e_{l,i}\}$, $l = 1, 2, \ldots d$, $d = \dim \mathcal{N}_i$ in $\mathcal{N}_i$ forms a set of deficiency vectors of the restricted operator A_0. The vectors $\frac{A+iI}{A-iI} e_l$ form an ortho-normal basis in the dual deficiency subspace $\mathcal{N}_{-i}$. We assume that the subspaces $\mathcal{N}_i$ and $\mathcal{N}_{-i}$ do not overlap, hence their (direct) sum $\mathcal{N} = \mathcal{N}_i + \mathcal{N}_{-i}$ has the dimension $2d = 2\ \dim \mathcal{N}_i$. Under this assumption the deficiency index of the restricted operator A_0 is equal to (d, d).

If $\mathcal{N}_i$ contains elements from the domain D_0 of A, then the restricted operator $A_0 = A|_{D_0}$ is not densely defined, hence the adjoint operator does not exist. However, under the above condition one may use the formal adjoint A_0^+ defined on linear combinations of elements from the domain D_0 and deficiency elements

$$u = u_0 + \sum_{l=1}^{d} [x_{l,i}\, e_{l,i} + x_{l,-i}\, e_{l,-i}],\ u_0 \in D_0$$

given by von-Neumann formula

$$A_0^+ u = Au_0 + \sum_{l=1}^{d} [-i\ x_{l,i}\, e_{l,i} + i\ x_{l,-i}\, e_{l,-i}].$$

We use the formal adjoint during the construction of extensions (see [26, 28, 30]) since the extension is actually developed in the sum $\mathcal{N} = \mathcal{N}_i + \mathcal{N}_{-i}$ of the deficiency subspaces. In order to use the symplectic version of the operator-extension techniques we introduce in the sum of deficiency subspace $\mathcal{N}$ a new basis $w_{\pm,l}$, on which the formal adjoint A_0^+ is correctly defined due to the above assumption:

$$w_{l,+} = \frac{e_{l,i} + e_{l,-i}}{2} = \frac{A}{A - iI} e_l,\ w_{l,-} = \frac{e_{l,i} - e_{l,-i}}{2i} = -\frac{I}{A - iI} e_l,$$

$$A_0^+ w_{l,+} = w_{l,-},\ A_0^+ w_l, - = -w_{l,+}.$$

The next statement (see [40]) permits to calculate the boundary form of a general operator via abstract integration by parts. It is used below to connect the inner Hamiltonian with the differential operator l by the boundary conditions (31).

Lemma 7.1 *Consider the elements from the domain of the (formal) adjoint operator A_0^+:*

$$u = u_0 + \sum_l [\xi^u_{l,+} w_{l,+} + \xi^u_{l,-} w_{l,-}] = u_0 + \frac{A}{A - iI} \vec{\xi}^u_+ - \frac{1}{A - iI} \vec{\xi}^u_-,$$

$$v = v_0 + \frac{A}{A - iI} \vec{\xi}^v_+ - \frac{1}{A - iI} \vec{\xi}^v_-,$$

with symplectic coordinates $\vec{\xi}^u_\pm$, $\vec{\xi}^v_\pm$:

$$\vec{\xi}^u_\pm = \sum_l \xi^u_{l,\pm} e_{l,i} \in \mathcal{N}_i.$$

Then the boundary form $\mathcal{J}(u, v)$ is

$$\mathcal{J}(u, v) = \langle A_0^+ u, v\rangle - \langle u, A_0^+ v\rangle = \langle \vec{\xi}^u_+ \vec{\xi}^v_- \rangle_E - \langle \vec{\xi}^u_- \vec{\xi}^v_+ \rangle_E. \tag{29}$$

The following statement helps to solve the non-homogeneous equations with the adjoint operator $(A_0^+ - \lambda I)u = f$. It is basic for the fundamental Krein Formula which describes all resolvents of the self-adjoint extensions ([27, 39]) of the operator A_0. In Theorem 7.3 and Theorem 7.4 below we use the version of the Krein formula for the scattering matrix derived in [2].

Lemma 7.2 *The symplectic coordinates $\vec{\xi}^u_\pm \in \mathcal{N}_i$ of components of the solution*

$$u = u_0 + \frac{A}{A - iI} \vec{\xi}^u_+ - \frac{1}{A - iI} \vec{\xi}^u_-$$

of the adjoint equation $[A_0^+ - \lambda I]u = 0$ are given by the formula

$$\vec{\xi}^u_- = -P_{\mathcal{N}i} \frac{I + \lambda A}{A - \lambda} \vec{\xi}^u_+. \tag{30}$$

The boundary form $\mathbf{J}(\mathbf{U},\mathbf{V})$ of the orthogonal sum $\mathbf{A}_0^+ = A_0^+ \oplus l_0^+$ of the reduced matrix and the differential operator on elements $(U,u) = \mathbf{U}$ on the orthogonal sum of the corresponding spaces is calculated as a sum of the forms (28) and (29). The self-adjoint extensions of the restricted operator $\mathbf{A}_0 = A_0 \oplus l_0$ are obtained as reductions of the adjoint operator $\mathbf{A}_0^+ = A_0^+ \oplus l_0^+$ onto Lagrangian planes of the form $\mathbf{J}(\mathbf{U},\mathbf{V})$. These planes are defined by the boundary conditions connecting the symplectic coordinates $[U'](0)$, $U(0)$, $\vec{\xi}_+^u$, $\vec{\xi}_-^u$ of the corresponding elements in the deficiency subspaces. For instance, one may select a d-dimensional operator $\beta : E \to \mathcal{E}$ and define the Lagrangian plane $\mathbf{L}_\beta$ by the boundary condition

$$\begin{pmatrix} [U'](0) \\ -\vec{\xi}_- \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ \beta^+ & 0 \end{pmatrix} \begin{pmatrix} U(0) \\ \vec{\xi}_+ \end{pmatrix}. \tag{31}$$

This condition defines a self-adjoint extension $\mathbf{A}_\beta$ of the above orthogonal sum of the restricted operators $A_0 \oplus l_0$ in $L_2(\mathbf{R},\mathcal{E}) \oplus E$. The spectrum σ_β of the operator $\mathbf{A}_\beta$ is absolutely-continuous and fills the positive half-axis $\lambda \geq 0$ with multiplicity $\dim \mathcal{E}$. The scattered waves $\overrightarrow{\Psi}_\nu$, $\overleftarrow{\Psi}_\nu$ labeled with vectors $\nu \in \mathcal{E}$ play the role of eigenfunctions for the spectral point $p^2 = \lambda > 0$. The components of the scattered waves in the *outer space* $L_2(\mathbf{R})$ are presented as linear combinations of exponentials in (3). The scattering matrix (4) can be calculated by inserting the above Ansatz (3) into the boundary conditions (31). Let us denote by $\mathbf{P}_\beta$ the orthogonal projection in $\mathcal{E}$ onto the d-dimensional subspace $\mathcal{E}_\beta = \beta E$ and by $\mathbf{P}_\beta^\perp$ the orthogonal projection onto the complement $\mathcal{E} \ominus \mathcal{E}_\beta$. Then the following statement is true:

Theorem 7.3 *We have*

$$\overrightarrow{T}(p) = \overleftarrow{T}(p) = \mathbf{P}_\beta^\perp + \frac{2ip}{2ip - \beta\mathcal{M}^{-1}\beta^+}\mathbf{P}_\beta,$$

$$\overrightarrow{R}(p) = \overleftarrow{R}(p) = \frac{\beta\mathcal{M}^{-1}\beta^+}{2ip - \beta\mathcal{M}^{-1}\beta^+}\mathbf{P}_\beta.$$

The expression for the scattering matrix may be presented via projections $\mathbf{P}^{sym,\,asym}$ *onto the subspaces of symmetric*

$$\mathbf{N}_{sym} = \begin{pmatrix} \nu \\ \nu \end{pmatrix}$$

and skew-symmetric

$$\mathbf{N}_{asym} = \begin{pmatrix} \nu \\ -\nu \end{pmatrix}$$

incoming waves,

$$\mathbf{P}^{sym,\,asym} = \frac{1}{2}\begin{pmatrix} \mathbf{P}_\beta & \pm\mathbf{P}_\beta \\ \pm\mathbf{P}_\beta & \mathbf{P}_\beta, \end{pmatrix},$$

as

$$\mathbf{S}_\beta(p) = \begin{pmatrix} \mathbf{P}_\beta & 0 \\ 0 & \mathbf{P}_\beta \end{pmatrix} + \mathbf{P}^{asym} + \frac{2ip - \beta\mathcal{M}^{-1}\beta^+}{2ip - \beta\mathcal{M}^{-1}\beta^+}\mathbf{P}^{sym}.$$

The inner component of the scattered wave $\overrightarrow{\Psi}_\nu$ *(inside the dot) is equal to*

$$\overrightarrow{\psi}_\nu = -\frac{A + iI}{A - \lambda I}e\frac{2ip\langle\beta,\,\nu\rangle}{2ip\mathcal{M} - |\beta|^2}.$$

The formulae in Theorem (7.3) permit the calculation of the quantum conductance from the transmission coefficient based on the Landauer formula. For instance, if $d = 1$, then the transmission coefficient differs from the unity only on the one-dimensional subspace $\mathbf{P}_\beta\mathcal{E}$ where it is equal to

$$T_\beta = \frac{2ip}{2ip - \langle\beta, \beta\rangle_\mathcal{E}\mathcal{M}^{-1}}.$$

Then the quantum conductance in this subspace $\nu\mathbf{P}_\beta\mathcal{E}$ is calculated as

$$\sigma_\beta = \frac{e^2}{2\pi\hbar}\frac{|T_\beta|^2}{1 - |T_\beta|^2},$$

see, for instance [31, 5].

For the mathematical "device" solving the Infinite Merchant's Problem using a scattering process we assume that the input space $\mathcal{E}$ is infinitely-dimensional but the deficiency subspace $\mathcal{N}_i$ is one-dimensional. The role of the operator β is played by the one-dimensional operator $\beta\rangle\langle e$, where $\beta \in \mathcal{E}$ $e \in \mathcal{N}_i$, $|e| = 1$.

We construct a solvable quantum-mechanical model of the "device" as an extension of the operator presented in the form of an orthogonal sum of the differential operator l_0 defined in $L_2(\mathbf{R}_+, \mathcal{E})$ on elements vanishing near the origin and the operator A_0 described above. The "inner" boundary form coincides with $J(u, v) = \xi_+^u\bar{\xi}_-^v - \xi_-^u\bar{\xi}_+^v$, and the "outer" boundary form of the adjoint operator l_0^+ is

$$J(U,\, V) = \langle l_0^+U,\, V\rangle - \langle U,\, l_0^+V\rangle = \langle U'(0),\ V(0)\rangle_\mathcal{E} - \langle U(0),\ V'(0)\rangle_\mathcal{E}. \tag{32}$$

The sum of the boundary forms $J(U,V) + J(u,v)$ is a symplectic Hermitian form which vanishes on the plane given by the boundary condition defined by a vector $\beta \in \mathcal{E}$:

$$\begin{pmatrix} U'(0) \\ -\xi_- \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ (\beta)^+ & 0 \end{pmatrix}\begin{pmatrix} U(0) \\ \xi_+ \end{pmatrix}. \tag{33}$$

Theorem 7.4 *The boundary conditions (33) define a self-adjoint extension $\mathbf{A}_\beta$ of the orthogonal sum of the Hermitian operators $l_0 \oplus A_0$. The spectrum σ_β of the operator $\mathbf{A}_\beta$ is absolutely continuous and fills the positive half-axis $\lambda = p^2 \geq 0$ with constant multiplicity* $\dim\mathcal{E}$. *The eigenfunctions of the absolutely-continuous spectrum are scattered waves Ψ_ν whose component in $L_2(\mathbf{R}_+, \mathcal{E})$ can be presented in the form*

$$\Psi_\nu(p) = e^{ipx}\nu + e^{-ipx}\mathbf{S}_\beta(p)\nu, \tag{34}$$

with the reflection coefficient

$$\mathbf{S}_\beta(p) = \mathbf{P}_\beta^\perp + \frac{ipM + |\vec{\beta}|^2}{ipM - |\vec{\beta}|^2}\mathbf{P}_\beta, \tag{35}$$

and the inner component of the scattered wave Ψ_ν in the space E is equal to

$$\psi_\nu = \frac{2ip\langle\beta,\, \nu\rangle}{|\beta|^2 - ip\mathcal{M}}\frac{A + iI}{A - \lambda I}e. \tag{36}$$

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