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The Broadcasting Problem For Bounded-Degree Directed Networks

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CDMTCS-193
September 2002

Centre for Discrete Mathematics and
Theoretical Computer Science

The Broadcasting Problem For Bounded-Degree Directed Networks

by

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE
in
Computer Science

University of Auckland

ABSTRACT

Increasingly, the design of computer networks and multi-processor configurations are now considered critical applications of computer science. Communication supported by network design between different nodes are important for many applications. There are some constraints in network design which usually created by economic and physical limitations. One constraint is the bounded degree, which is the limited number of connections between one node to others. Another possible constraint is a bound on the time that a message can afford to take during a "broadcast". The topic of this thesis will apply group theory, already used in network design, to design bounded-degree communication-efficient directed networks. We present, for the first time, the largest-known directed networks satisfied special bounds on node degree and broadcast time. The thesis also presents a family of optimal (Δ, Δ+1) broadcast digraphs. That is, digraphs with a proven maximum number of nodes, having degree Δ and broadcast time at most Δ+1.

ACKNOWLEDGEMENTS

I wish to extend my great thanks and appreciations to my supervisor, Dr. Michael. J. Dinneen, for guidance, and for his innumerable helpful comments, his creative suggestions, and his patient and unwavering support.

Special thanks go to my friends, Linda Jiang and Michael Zhou, for providing a constant source of encouragement and support, and for being there for me at all times.

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Chapter 1

Introduction

In this thesis, the topic that we study is the construction of directed networks with order as large as possible for a given maximum node degree and broadcast time by exploring uses of group theory.

The development of the recent technology for applications demand an efficient topological design of the network. Some fundamental design problems related with the topology of networks have been widely studied. The techniques developed from those studies provides the best-known constructions for those network design problems. Some examples will be given in the next section.

One of those design problems is the study of network construction under the constraint of a bound on the maximum node degree imposed by economic and physical limitations. The constraint gives a limited number of physical connections with other nodes to each node in the interconnected network. A lot of research has been done recently on the design of networks with the largest order satisfying that constraint (see [CF95, CM96, D91, D98, DPW98]).

An important feature characterizing the "quality" of an interconnection network for parallel computing is the ability to effectively disseminate the information among its processors [D92]. One of main problems of information dissemination is broadcasting, which is the process of sending a message originating at one node of a network to all other nodes. There exists two kinds of network connection models: point-to-point model and multi-cast model. The minimum time of broadcasting in the interconnection network for those two different models may not be the same. With these two connection models, two different basic design sub-problems occur from the original problem satisfying the constraint of bounded node degree.

Those two basic design problems have been given the several best-known constructions, which we describe as follows. We view any interconnection network as a connected undirected graph, where the vertices in the graph correspond to the processors and the edges correspond to the physical links of the network. A good survey paper [D98] has discussed these two problems.

1. *The Degree/Diameter Problem.*

The Degree/Diameter problem is the problem of the design of the network with largest possible order satisfying the bounds on node degree and the diameter. In this problem, the network connection model we consider is the multi-cast connection model. That is, each node can communicate with all of its neighbor nodes in one time step. For example, a node can forward the message it received to all nodes simultaneously. The diameter is the maximum time delay for broadcasting a message throughout the whole network under this model.

Fig 1.1(a) shows a broadcasting scheme in a simple graph for the multi-cast connection model. If **A** is the origin node to broadcast, node **G** will receive the message originated from **A** at lest 3 time steps. Then we can find that for any node which is the origin node to send the message, the time delay for flowing the message through the whole network is the same. So the diameter is 3.

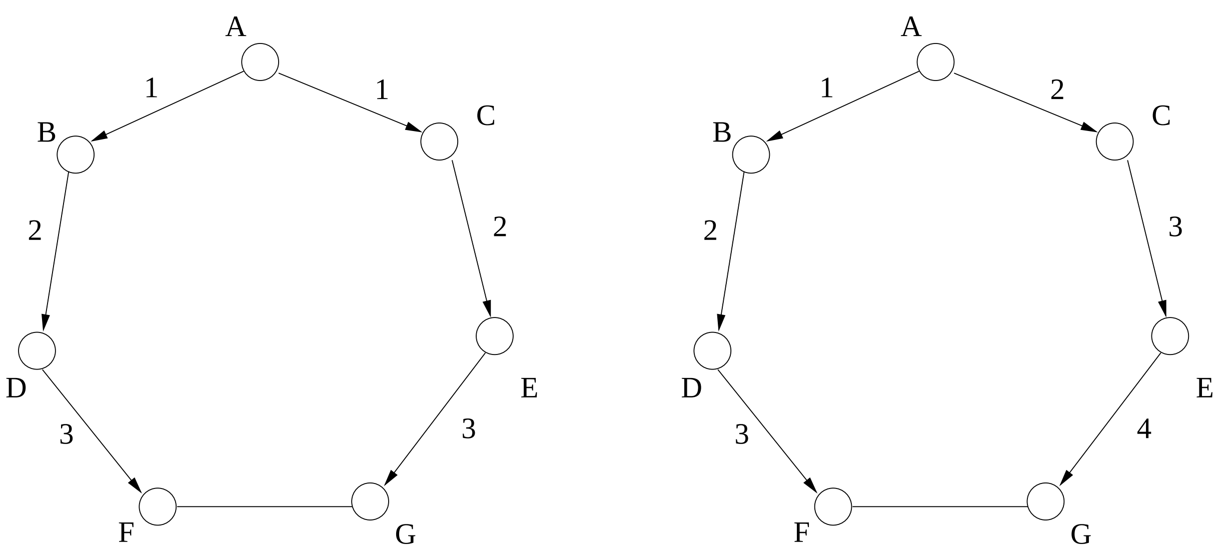

Figure 1.1: A comparison between (a) diameter and (b) broadcast time for a simple graph (order is 7).

2. *The Degree/Broadcast-Time Problem.*

The Degree/Broadcast-Time problem provides the construction of network with largest possible order satisfying the bounds on node degree and maximum broadcast time. In this problem, the network connection model we consider is point-to-point connection model. That is each mode communicate with just one of its neighbor nodes at one time step. Here, the broadcast time is the maximum time for disseminate a message throughout the whole network under that model. A good survey paper about broadcasting problem is [HKMP96].

Fig 1.1(b) shows a broadcasting scheme in a simple graph (order is 7) for the point-to-point model. If the origin node to broadcast is **A**, **A** forward the message to **B** in first step. In the second time step, **A** and **B** send the message to node **D** and **C**. After **D** and **C** received the message, node **E** and **F** will be the next two nodes who get the message on next time step. So it costs 4 time steps for node **G** to get the message. We can find that for any node which is the origin node to send the message, the time delay for flowing the message through the whole network is the same. So the broadcast time is 4.

Generally a network's broadcast time is larger than its diameter. It is obviously since communication for diameter is multi-cast while that for broadcast time is point-to-point. On the other side, in the point-to-point model, the node has more choices than it in the multi-cast model because the node is able to choose any one of its neighbor nodes to be the next node to receive the message. So computing the broadcast time of the network is harder than computing the diameter when the topology of the network becomes more complicated.

There are many ways to explore those two problems. Nowadays the use of the group-theoretic methods for designing networks becomes more and more popular [D91]. There are some advantages of using group theory in the design of connected networks. First of all, the most important advantage is node symmetry, also called vertex transitive, which makes the routing scheme independent of node. It means that for a given routing scheme, the time for disseminating the information through the whole network will be the same in spite of different original node sending the information. Most connection networks with node symmetry are based on Cayley graphs. In the next section, we will give the definition. Other advantages of using group theory for the design of network include: line symmetry, hierarchical structure, and/or high fault tolerance.

Some good results have been presented by using group-theoretic methods for designing networks. For the undirected case, a table of largest-known (Δ, D) graphs is shown on [H95]. Michael. J. Dinneen and his colleagues give the table of the largest known (Δ, T) broadcast graphs on the paper [DPW98]. For the directed case, Faber and Moore in [FM88] study families of digraphs on permutations and give a table of largest-known vertex symmetric (Δ, D) digraphs for the Degree/Diameter problem. Another result of largest known (Δ, D) digraphs is presented on [CF95]. However, there is no such result for the Degree/Broadcast-Time problem (directed case).

1.1 Our Research Aims

As discussed in the previous section, the research for large graphs/digraphs which have the additional property of being node symmetric has been considered more recently. However, results for the Degree/Broadcast time problem (directed case) has not yet been presented. Our research aims to establish a table of the largest-known (Δ, T) digraphs through the research of the Degree/Broadcast-Time problem.

The original broadcast design problem was introduced by Farley in [F77]. It is slightly different from what we discuss in this thesis. This is the problem of finding graphs for a given order with least number of edges that one can broadcast from each vertex in minimum time. We can find that the minimum time for broadcast in a network of order n is $log_2\ n$, because the number of vertices received the message double at most at each time step on the broadcasting schedule. The current study for the minimum broadcast problem is presented in [DVWZ97] and [F00].

For the Degree/Broadcast-Time problem we discuss in here, we constrain both the degree Δ and broadcast time T while maximizing the order of the network instead of fixing the order and minimizing the number of edges. Specially, we will focus on the broadcast directed network (a detailed definition will be given in next chapter).

Our research will begin with the study of some traditional good digraphs such as *the de Brujin digraph* and *the Kautz digraphs* which are two of the earliest known large (Δ, D) digraphs. The (Δ, D) de Brujin digraph has order Δ^D while the (Δ, D) Kautz digraphs has order $(\Delta^D + \Delta^{D-1})$. There are also some advantages for those classic digraphs. Firstly, it is interesting to see different constructions for these large digraphs. Secondly, these various constructions give us a strong impression that these digraphs achieve the maximum possible order. For example, the Kautz digraphs of diameter 2 are optimal. It's helpful for us to study the construction of good broadcast directed networks. Normally, people start to study the Degree/Broadcast-Time problem by studying the (Δ, D) networks because computing the diameter of network is much easier than computing the broadcast time and there already have some good results concerning the (Δ, D) problem. Additionally, if a digraph's diameter is known, we will get the lower bound for the broadcast time of that digraph because the diameter is smaller than or equal to broadcast time of a digraph. Studying those classic (Δ, D) digraphs will help us to establish the basic idea of construction the optimal broadcast directed networks.

After observing the classic digraphs, we will study the cycle prefix digraph, which is a Cayley coset digraph. Group theory will help us know how to apply abstract algebra into the the construction of network. The Cayley coset digraphs are an interesting family of vertex symmetric digraphs, defined by Faber and Moore in [FM88], and may be viewed as well as family of digraphs on alphabets [FMC93]. We will try to compute the broadcast time of those Cayley coset digraphs and find the largest known (Δ, T) broadcast digraphs from it.

At last, we will use the group theory to explore the Cayley digraphs based on the semi-product group. This construction way was initiated in [DFF91]. We hope to establish the table of largest-known (Δ, T) broadcast digraphs. We will present the generators for each Cayley digraph. The most important results of our work is finding a family of optimal (Δ, Δ+1) broadcast digraphs. We will give a simple proof that these broadcast digraphs are optimal.

1.2 Thesis Outline

An outline of the thesis is given as follows:

- **Chapter 2:** This chapter presents graph-theoretical and group-theoretical definitions. We also discuss some important theorems related to these definitions.

- **Chapter 3:** This chapter provides the two classic families of digraphs. In the first part, we will discuss the *de Brujin digraphs* and present their structure and some other specialties, such as each vertex labeled with the words from a alphabet of size Δ. In the second part, we will present the *Kautz digraphs* and their structure and specialties. Here we try to compare the *Kautz digraphs* with the *de Brujin digraphs*. For example, the *Kautz digraphs* have vertices labeled with words belongs to a alphabet of size Δ+1. And for a given degree and diameter, the order of the *Kautz digraphs* is larger than that of the *de Brujin digraphs*. In the last part, we will discuss broadcasting in those digraphs and presents an upper bound of the broadcast time of those digraphs.

- **Chapter 4:** In this chapter, we analyze *Cycle prefix digraphs*, a type of *Cayley coset digraphs*. Those digraphs are defined on an alphabet of Δ+1 symbols. *Cycle prefix digraphs* have some relevant properties. We will discuss the recursiver structure of *Cycle prefix digraphs* and present the broadcasting algorithm. At last, we also give some testing results for the broadcasting time for those digraphs. These results will help us to establish the table of largest known (Δ, T) digraphs.

- **Chapter 5:** In this main chapter of thesis, we analyze *Cayley digraphs* and use a random algorithm to compute the upper bound of the minimum broadcast time for those digraphs. At first, we discuss the properties of *Cayley digraphs* and the algebra construction methods. Secondly, we will present an algorithm to compute the broadcast time of those digraphs and give the table of largest known (Δ, T) digraphs. At last, we will give a family of optimal (Δ, Δ+1) broadcast digraphs and prove it.

- **Chapter 6:** This chapter gives some conclusions of our thesis. It also discusses some future research extensions.

Chapter 2

Definitions and Background

We are ready to give some standard definitions and notations for our thesis.

2.1 Graph Theoretic Preliminary Definitions

We present some basic concepts of graph theory that will be used in this thesis. Most of our definition and further information could be found in [C79], [GY86], [M01] and [G88].

Definition 1 *A **graph** G=(V, E) is a mathematical structure consisting of two sets V and E. The elements of V are called **vertices** (or **nodes**), and the elements of E are called **edges**. Each edge has set of one or two vertices associated to it, which are called its **endpoints**.*

Definition 2 *A **digraph** G=(V, E) is a mathematical structure consisting of two sets V of vertices and E of ordered pairs of distinct vertices of G called **arcs**.*

Definition 3 *Given a graph G=(V, E), the number of vertices in V is called the **order** of G, which is also denoted by $|G|$. And the number of edges in E is called the **size** of G, which is written by $|E|$.*

Definition 4 *In a graph G=(V, E), a **walk** from vertex v_0 to vertex v_n is an alternating sequence W=$\langle v_0, e_1, v_1, e_2, \ldots, v_n - 1, e_n, v_n \rangle$ of vertices and edges, such that endpoints(e_i)=$\{v_i - 1, v_i\}$, for i=1,...,n.*

Definition 5 *In a digraph G=(V, E), a* ***directed walk*** *from vertex v_0 to vertex v_n is an alternating sequence W=$\langle v_0, e_1, v_1, e_2, \ldots, v_n - 1, e_n, v_n\rangle$ of vertices and arcs, such that tails(e_i)=$v_i - 1$ and heads(e_i)=v_i, for i=1,...,n.*

Definition 6 *The* ***length*** *of a walk or directed walk is the number of edge-steps in the walk sequence. A* ***path*** *is a walk in which no vertex is repeated.*

Definition 7 *Vertex v is* ***reachable from*** *vertex u if there is a walk from u to v. Two vertices u and v in a digraph D are said to be* ***mutually reachable*** *if D contains both a directed u-v walk and a directed v-u walk.*

Definition 8 *A graph is* ***connected*** *if for every pair of vertices u and v, there is a walk from u to v. A digraph is* ***strongly connected*** *if every two of its vertices are mutually reachable.*

Definition 9 *In a graph, the* ***distance*** *from vertex u to vertex v is the length of a shortest walk from u to v, or ∞ if there is no walk from u to v. For digraphs, the* ***directed distance*** *is the length of a shortest directed walk.*

Definition 10 *A* ***graph isomorphism*** *f: G $\longrightarrow$ H is a pair of bijections $f_V : V_G \longrightarrow V_H$ and $f_E : E_G \longrightarrow E_H$. An* ***automorphism*** *of a graph G is an isomorphism f that maps G to itself. A graph (digraph) G=(V, E) is* ***vertex symmetric*** *(or vertex transitive) if for all vertex pairs u, v $\in$ V, there is an automorphism of G that maps u to v.*

Definition 11 *In a graph, the* ***degree*** *of a vertex v, denoted by deg(v), is the number of edges incident to v. For digraphs, the* ***out-degree*** *of a vertex v is the number of vertices adjacent from v and its* ***in-degree*** *is the number of vertices adjacent to v.*

Definition 12 *The* ***diameter*** *of a connected graph (or a strong connected digraph) G=(V, E) is the least integer D such that for all vertices u and v in G, the distance from u to v $\leq$ D.*

Definition 13 *For a connected graph (or a strongly connected digraph) G=(V, E) and vertex v in V, let* ***broadcast***(v) *be the minimum time needed to broadcast a message originating from vertex v. The* ***broadcast time*** *T of G is the max$\{$ $broadcast(v) \mid v \in V\}$.*

Definition 14 *A* (Δ, D) ***graph*** *is a graph G=(V, E) satisfying: (1)* $deg(v) \leq \Delta$ *for all vertices v in V and (2) diameter of G less than or equal to D. A* (Δ, D) ***digraph*** *is defined similarly; in this case, all in-degree and out-degree must be bounded by* Δ*. A* (Δ, D) *graph (digraph) is* ***optimal*** *if it have the maximum order possible for* (Δ, D) *graph (digraph).*

Definition 15 *A* (Δ, T) ***broadcast graph*** *is a graph G=(V, E) satisfying: (1)* $deg(v) \leq \Delta$ *for all vertices v in V and (2) broadcast time of G less than or equal to T. A* (Δ, T) ***broadcast digraph*** *is defined similarly; in this case, all in-degree and out-degree must be bounded by* Δ*. A* (Δ, T) *graph (digraph) is* ***optimal*** *if it have the maximum order possible for* (Δ, T) *graph (digraph).*

2.2 Group Theoretic Preliminary Definitions

In this section, we give the necessary background in abstract algebra. The book [C37], [M94], [F82] and [S00] are good references for the subject.

Definition 16 *A set G with a binary operation on it is called a* ***group*** *if it has the following properties:*

1. *The operation is associative, i.e., for any* $\alpha, \beta, \gamma \in G$*, We have* $(\alpha\beta)\gamma = \alpha(\beta\gamma)$*;*
2. *There is an identity element, written* $1 \in G$*, i.e., for any* $\alpha \in G, \alpha \cdot 1 = 1 \cdot \alpha = \alpha$*;*
3. *Every element in G has an inverse, i.e., for every* $\alpha \in G$ *there exists an element written* $\alpha^{-1} \in G$ *such that* $\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = 1$*.*

A group G is ***abelian*** *if it is also commutative, that is* $x \cdot y = y \cdot x$ *for any x and y belongs to G.*

Definition 17 *A subset H of a group G is a* ***subgroup*** *of G if and only if*

1. *H is closed under the binary operation of G,*
2. *The identity e of G is in H,*
3. *For all* $a \in H$ *it is true that* $a^{-1} \in H$ *also.*

*The **index** of a subgroup H of a finite group G is $|G|/|H|$.*

Definition 18 *Let G be a group and let $a \in G$. then $H = \{a^n \mid n \in Z\}$ is a subgroup of G and is the smallest subgroup of G that contains a. The group H is the **cyclic subgroup** of G **generated by** a, and will be denoted by $\langle a \rangle$. An element a of a group G **generates** G and is a **generator for** G if $\langle a \rangle = G$. A group is **cyclic** if there is some element a in G that generates G. Let a be an element of group G. If the cyclic subgroup $\langle a \rangle$ of G is finite, the the **order of** a is the order $|\langle a \rangle|$ of this cyclic subgroup. Otherwise, we say that a is of **infinite order**.*

Definition 19 *A (commutative) **ring** $\langle\ R,\ +,\ \cdot\rangle$ is a set R together with two binary operations $+$ and $\cdot$, which we call addition and multiplication , defined on R such that the following axioms are satisfied:*

1. *$\langle\ R,\ +,\ \cdot\rangle$ is an abelian group.*
2. *Multiplication is associative (and commutative).*
3. *For all $a,\ b,\ c \in R$, the **left distributive law**, a(b+c)=(ab)+(ac), and the **right distributive law**, (a+b)c=(ac)+(bc),hold.*

*A multiplicative identity in a ring is an **unity** element.*

Definition 20 *Let R and R' be rings. A map $\phi : R \longrightarrow R'$ is a **homomorphism** if the following two properties are satisfied for all $a,\ b \in R$:*

1. *$\phi(a + b) = \phi(a) + \phi(b)$.*
2. *$\phi(ab) = \phi(a)\phi(b)$.*

*An isomorphism $\phi : R \longrightarrow R'$ from a ring R to a ring R' is a **homomorphism** that is one to one and onto R'. The rings R and R' are then **isomorphic.***

Definition 21 *Let R be a ring with unity. An element u in R is a **unit of** R if it has a multiplicative inverse in R. If every nonzero element of T is a unit, then R is a **division ring**. A **field** is a commutative division ring.*

Definition 22 *The unit of a ring R form the* ***group of unites*** *of R under multiplication and we denote this group by U(R). If a generator exists for the group U(R) then it is called a* ***primitive root****.*

Definition 23 *Let H be a subgroup of a group G. The subset* $aH=\{ah \mid h \in H\}$ *of G is the* ***left coset*** *of H containing a, while* $Ha=\{ha \mid h \in H\}$ *of G is the* ***righ coset*** *of H containing a.*

Definition 24 *The* ***symmetric group*** S_n *is the collection of permutations of the set* $\{0, 1, \ldots, n-1\}$ *where group multiplication is defined by composition of permutations.*

Definition 25 *Given two groups G and H, we define their* ***direct product*** *to be the set* $G{\times}H$ *with the operation:*

$$\big((\alpha_1, \beta_1), (\alpha_2, \beta_2)\big) \longmapsto (\alpha_1\alpha_2, \beta_1\beta_2)$$

where $\alpha_1, \alpha_2 \in G$ *and* $\beta_1, \beta_2 \in H$*.*

Definition 26 *Give two groups (G,·) and (H,∘), and supposed there exists a homomorphism* $\sigma : G \to AUT(H)$ *of G into the group of automorphisms of H. The set of all ordered pairs* $\{(\alpha, \beta) \mid \alpha \in G, \beta \in H\}$ *can be made into a group if we define products by:*

$$(\alpha_1, \beta_1) * (\alpha_2, \beta_2) = (\alpha_1 \cdot \alpha_2, \beta_1 \circ (\sigma(\alpha_1))(\beta_2))$$

This group is called the ***semi-directed product*** *of G and H, relative to the homomorphism* σ*, and denoted* $A \times_\sigma B$*.*

Definition 27 *Given a group A and a set S of generators for A the* ***Cayley digraph*** *G=(V, E), denoted by* $\langle A, S\rangle$*, is constructed as follows:*

1. *The elements of the group A are the vertices V of digraph G.*
2. *An edge (a,b) is in E if and only if ag=b for some generator g in S.*

If we also requires $S{=}S \cup S^{-1}$ *then G is a* ***Cayley graph****.*

The semi-directed product methods for the Cayley graphs (digraphs) is a very useful designing for efficient directed networks. The definition and notation of a Cayley coset graph (digraph) will be given in Chapter 4.

Chapter 3

Classic Directed Network Constructions

Before we discuss how to use group-theoretic methods for designing directed broadcast networks, we would like to present some other well-known digraph constructions. Vertex symmetry is a main design requirement for large (Δ, T) broadcast digraphs. In this section, however, the families of digraphs we discuss are not vertex symmetry.

In this chapter, the de Bruijn digraphs and the Kautz digraphs will be presented. Both of them were the earliest known large (Δ, D) digraphs. It would help us to design large (Δ, T) broadcast digraphs by observing those network constructions. We will give a brief description of the de Bruijn and the Kautz digraphs and show their properties. Afterwards, the broadcasting problem in those digraphs will be discussed. Some new efficient broadcasting protocols will help us find the minimum broadcasting time of those digraphs for a given degree and diameter. A table comparing the upper bounds of the broadcast time between the de Bruijn digraphs and the Kautz digraphs will be given in the last part of this chapter.

3.1 The de Bruijn digraphs

An import family of digraphs with a larger number of vertices and small diameter are the de Bruijn digraphs [Q95]. There are three ways to define de Bruijn digraphs. One of them is based on alphabets. For given integer $\Delta \geq 2$ and $D \geq 1$, the de Brujin digraphs B(Δ, D) have vertices labeled with words $x_1 x_2 \ldots x_D$ where x_i belongs to an alphabet of size Δ. Vertices are connected if and only if the label of one is left-shifted label of the other. That means there is an arc from any vertex $x_1x_2 \ldots x_D$ to the Δ vertices $x_2 \ldots x_D x_{D+1}$, where x_{D+1} is any letter of the alphabet.

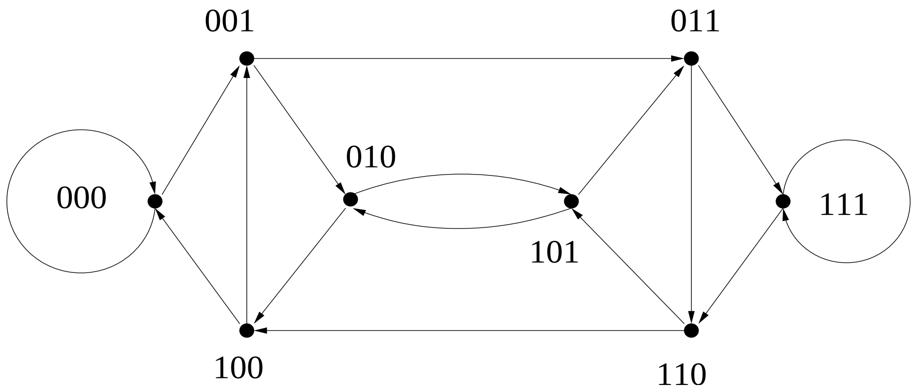

Figure 3.1: The de Bruijn digraph B(2, 3).

Consider the node labelled 001 in Fig 3.1. It is connected to itself right-shifted, 010 and 011. On the other side, there are two arcs from its left-shifted, 000 and 100 to itself. For vertex 001, its out-degree and in-degree is 2. Notice that there is a self-loop for vertex 000 and 111 according to the defined connection model.

Another way to define the de Brujin digraphs is by using the line digraph iterations [FYMC84]. If G is a digraph, its digraph L(G) will be given as follows: Every arc E in G is represented by a vertex e in L(G); vertex v is adjacent to vertex x in L(G) if and only if the arc E is incident to the arc F in G. The line digraph of B(Δ, D) is B(Δ,D+1).

B(Δ, D) are strong connected digraph and also are Δ-regular, which have Δ^D vertices and diameter D. B(Δ, D) are connectivity Δ-1 (due to the self-loop). De Bruijn digraphs are neither vertex- or edge- transitive. And they are not recursively scalable. That is, B(Δ, D+1) are not composed of Δ copies of B(Δ,D).

3.2 The Kautz digraphs

The Kautz digraphs are the earliest known large (Δ, D) digraphs, which were discovered by Kautz [K69]. One of the definitions of the Kautz digraphs is also based on alphabets. Let Δ and D is the positive integers, the vertices of the Kautz digraphs K(Δ, D) labeled with words $x_1x_2 \ldots x_D$ where x_i belongs to an alphabet of Δ+1 letters and $x_i \neq x_{i+1}$ for $1 \leq i \leq D-1$. A vertex labeled with $x_1x_2 \ldots x_D$ connected to the Δ vertices $x_2x_3 \ldots x_Dx_{D+1}$, where $x_{D+1} \neq x_D$ and x_{D+1} belongs to the alphabets. It is easy to see that $k(\Delta, D)$ is a Δ-regular digraph with in- and out-degree Δ. The order of $k(\Delta, D)$ is $\Delta^D + \Delta^{D-1}$.

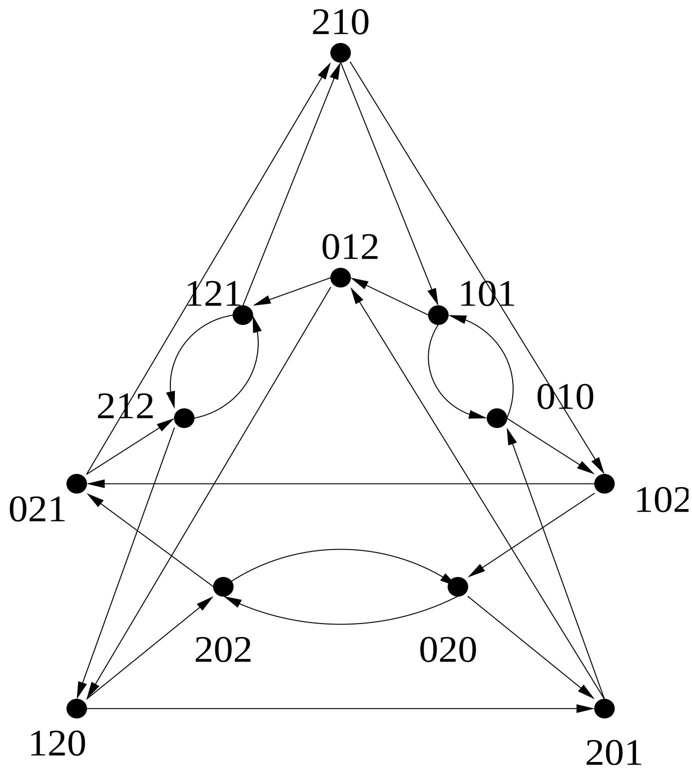

Figure 3.2: The Kautz digraph K(2, 3).

Fig 3.2 demonstrates the construction for the Kautz K(2,3) digraph. For the vertex 202, there are two arcs from vertex 120 and 020 to itself. On the other side, it connects to its left-shifted, 021 and 020. The out- and in- degree is 2. And there is no self-loop for every vertex.

A class of digraphs that generalize the Kautz digraphs is defined by Imase and Itoh [II83] using arithmetic congruence: The vertices are numbered with integers modulo n. If the out-degree is d, then vertex v is joined to vertices u ≡ -dv-α (mod n), for $1 \leq \alpha \leq d$. The diameter of the resulting digraph is at most $log_d n$. Furthermore, if n=$\Delta^D + \Delta^{D-1}$ these digraphs are isomorphic to K(Δ, D). We call those digraphs the generalized Kautz digraphs, denoted by GK(Δ, D).

K(Δ, D) are strong-connected and enable very simple routing based on left shifting. K(Δ, D) have connectivity Δ. The Kautz digraphs are neither vertex- nor edge-transitive except that For $D = 2$ the Kautz digraphs are vertex symmetric. Similarly, the Kautz digraphs are also not recursively scalable.

3.3 Broadcasting in Kautz/de Bruijn digraphs

For a given degree Δ and diameter D, the de Bruijn digraphs and the Kautz digraphs have the a number of vertices. Studying the broacasting in the de Bruijn digraphs and the Kautz digraphs means finding the minimum broadcast time of those digraphs for the given degree and diameter.

In this problem, people always first try to give the smallest upper bound of the broadcast time of those digraphs. Then, people can find more smaller results for the minimum broadcasting time by establishing a efficient broadcasting scheme. A lot of efficient broadcasting protocols have been published. For example, Bermond and his colleagues give a good protocol and improve their results in [BP92]. In [KMPS92], there is another paper discussing a broadcasting scheme.

We will give the best-known upper bounds on the broadcast time of the de Bruijn digraphs and the Kautz digraphs.

Theorem 28 *[BP88] The broadcast time for de Bruijn digraphs B(Δ, D),* $2 \leq D \leq 14$*, is bounded as follows.*

$$b(B(\Delta, D)) \leq \frac{(\Delta+1)(D+1)}{2}$$

Theorem 29 *[HOS92] The broadcast time for Kautz digraphs K(Δ, D) is bounded as follows.*

$$\begin{aligned} b(K(2,D)) &\leq 2D, \\ b(K(3,D)) &\leq 3D, \\ b(K(\Delta,D)) &\leq \begin{cases} \frac{(\Delta+3)(D+1)}{2}, & \text{if } 4 \leq \Delta \leq 12, \Delta \neq 9 \\ min\{2Dlog_2\Delta, 3Dlog_3\Delta\}, & \text{if } \Delta = 9 \text{ or } \Delta \geq 13 \end{cases} \end{aligned}$$

Table 3.1: Comparative values of upper bound of the broadcast time of de Bruijn digraphs and Kautz digraphs for small value of the degree.

Δ	B(Δ, 3)		K(Δ, 3)	
	|v|	$\beta(B(\Delta,\ 3))$	|v|	$\beta'(K(\Delta,\ 3))$
3	27	8	36	9
4	64	10	80	14
5	125	12	150	16
6	216	14	252	18
7	343	16	392	20
8	512	18	576	22
9	720	20	810	24
10	1000	22	1100	26

From Theorem 28 and 29, we get the smallest upper bound of the broadcast time of the de Bruijn digraphs and the Kautz digraphs. In Table 3.1, we compare the broadcast time and the order of the de Bruijn digraphs with those of the Kantz digraphs for diamter 3 and small values of the degree (3-10). In here, $|v|$ is the order of digraph. $\beta(B(\Delta,\ D))$ is the best-known upper bounds on the broadcast time of the de Bruijn digraphs while $\beta'(k(\Delta,\ D))$ is that of the Kautz digraphs.

Chapter 4

Cayley Coset Directed Network Constructions

In the last chapter, we have discussed two classic directed network constructions, the de Bruijn digraphs and the Kautz digraphs. For the given degree and diameter, the order of those digraphs are not the largest. In most cases, Cayley coset digraphs have the largest known orders for fixed degree and diameter.

In this chapter, we will give the definition of Caycley coset digraphs and discuss the methods to construct a special family of (Δ, D) digraphs using the Cayley coset digraphs. Very often, vertex symmetric digraphs may be described as digraphs on an alphabet. Cycle prefix digraphs were defined by Faber and Moore as Cayley coset digraphs and have some nice properties. We will discuss the recursive structure of the Cycle prefix digraphs and the upper bounds on the broadcast time of them. At last, some testing results for the broadcast time of those digraphs will be given .

4.1 Algebra Foundation of Cayley Coset Digraphs

The digraphs mentioned in this section are the result of Faber and Moore in [FM88]. Cayley coset digraphs are defined as follows:

Definition 30 *Let G be a group, H a subgroup, and S a subset. Suppose*

1. $G = \langle S \cup H \rangle$,
2. $HSH \subseteq SH$, *(For well-defined arcs)*

3. S is a set of distinct non-identity coset representatives of H in G.

Then we can form the Cayley coset Digraph Γ=(G,S,H) with vertices given by the set of left cosets $\{gH \mid g \in G\}$ and a arc $(g_1H,\, g_2H)$ whenever $g_1sH{=}g_2H$ for some s ∈ S.

From the definition, we know that a set of generator S ⊆ $G\backslash$H and the arc should be well-defined. Next, we will proof that Cayley coset digraph is vertex symmetric.

Theorem 31 *Every Cayley coset digraph* $\langle G, S, H\rangle$ *is vertex symmetric.*

Proof. Let g_1H, g_2H are the two elements from $\{gH \mid g \in G\}$, where $g_1, g_2 \in G$. We must show an adjacency-preserving automorphism ϕ of $\{gH \mid g \in G\}$ mapping g_1H to g_2H. Defined $\phi(x) = (g_2g_1^{-1})x$ for all x ∈ $\{gH \mid g \in G\}$. Clearly ϕ maps g_1H to g_2H since

$$\phi(g_1H) = (g_2g_1^{-1})g_1H = g_2(g_1^{-1}g_1)H = g_2eH = g_2H \ (associativity).$$

The map ϕ is injective since if $\phi(x) = \phi(y)$ then $(g_2g_1^{-1})x = (g_2g_1^{-1})y$ and so

$$x = (g_2g_1^{-1})^{-1}(g_2g_1^{-1})x = (g_2g_1^{-1})^{-1}(g_2g_1^{-1})y = y \ (inverses).$$

Similarly ϕ is surjective since for any x in $\{gH \mid g \in G\}$, $\phi(g_1g_2^{-1}x){=}(g_2g_1^{-1})(g_1g_2^{-1}$ x)=x. So ϕ is bijection. Finally, ϕ maps vertices adjacent to g_1H to vertices adjacent to g_2H.

$$\phi(g_1s_iH) = (g_2g_1^{-1})(g_1s_iH) = g_2s_iH \ for\ all\ s_i \in S.$$

□

Moreover, every vertex symmetric digraph is a Cayley coset digraph, as shown in [S69]. But can every vertex symmetric digraph can be represented as a Cayley digraph? The answer is no. i.e., the petersen graph is vertex symmetric but is not a Cayley graph. In particular, a Cayley coset digraph Γ is a Cayley digraph if and only if H=$\{e \mid e\ is$ the identity of $G\}$.

Cayley coset digraph have some other properties. It is connected, and $|S|$-regular. $|s|$-regular means the out-degree and in-degree of Cayley coset digraph is $|S|$. The most important properties is as below.

Theorem 32 *For each $\Delta \geq D$, there exists a vertex symmetric digraph with degree Δ, diameter D, and $\frac{(\Delta+1)!}{(\Delta+1-D)!}$*

The proof of this theorem is presented in [FM88]. The idea of Faber and Moore's construction as follow: Let G be any k-transivtive group on $\Delta+1$ letters, denoted by symmetric group $S_{\Delta+1}$. And $T_k = \{0, 1, 2, \ldots, D-1\}$. The subgroup $H \subset S_{\Delta+1}$ be the set of elements $\{h \mid h \in S_{\Delta+1},\ h(i) = i \text{ for all } 0 \leq i \leq D-1\}$. The set of generators $S = \{g_i \mid 1 \leq i \leq D\}$. For each $i \neq 0$, choose $g_i \in S_{\Delta+1}$ so that

$$g_i(0) = i,$$

$$if\ 1 \leq j \leq D-1,\ g_i(j) = \begin{cases} j-1, & \text{for } l \leq j \leq i \\ j, & \text{for j < i} \end{cases}$$

We can denoted the coset of H in G by k-tuples$(a_1, a_2, \ldots, a_{D-1})$ with all a_i distinct since each coset gH is completely determined by its action on T_k. To see this, note that is $a_1 = a_2\ on T_k$, then $a_1^{-1}a_2$ is the identity on T_k. that is $a_1 a_2 \in H$. Thus, the total number of cosets is $\frac{(\Delta+1)!}{(\Delta+1-D)!}$. Given any $(a_1, a_2, \ldots, a_{D-1})$ in at most D steps using the generators as needed, the new representative becomes our coset identity $(0, 1, \ldots, D-1)$. So the diameter is D.

4.2 Composition of Cayley Coset Digraphs

From the last section, we know the Cayley coset digraphs have degree larger than or equal to diameter. In [H95], P.R. Hafner give the largest-known (Δ, D) digraphs. The bold numbers in Table 4.1 are the orders of those largest known (Δ, D) digraphs which are Cayley coset digraphs. We find that the Cayley coset digraphs give the majority of the entries listed in Table 4.1. (When D=2, the largest known (Δ, D) digraphs is Kautz digraphs that are Cayley coset digraphs also). We need to find largest symmetric digraphs with diameter larger than the degree. A lot of methods have been used to find those digraphs. For example, F.Comellas and M.A. Fiol give a methods to construct a special family of (Δ, D) digraphs by using the Cayley coset digraphs in their paper [CF95]. Before discussing the construction method, we will give some definitions and theorems.

Definition 33 *A digraph G=(V, E) is **K-reachable** if there exists a uv-path of length k for all u, $v \in V$.*

Theorem 34 *The digraphs $\Gamma_\Delta(D)$ are D-reachable for $D \geq 3$.*

This theorem was proved by Comellas and Fiol. It is possible to apply the new version of Conway and Guy's theorem [CG82] into the construction. Comellas and Fiol present two ways to construct larget (Δ, D) digraphs. The first method is to increase of the diameter of digraph.

Theorem 35 *IF there is a vertex symmetric Δ-regular k-reachable digraph with N vertices then, for all n and m a multiple of n, there exists a vertex symmetric Δ-regular digraph with mN_n vertices and diameter at most kn+m+1.*

In their paper, Comellas and Fiol proved Theorem 35 by presenting the construction of a new digraph. Let G=(V, E) be a digraph satisfying the hypotheses of the theorem (vertex symmetric, Δ-regular k-reachable). A new digrtaph G'=$(V',\ E')$ may be constructed as follows: The vertex set V' has elements $(\alpha \mid p_0p_1 \ldots p_{n-1})$ with $\alpha \in Z/mZ$ and $p_i \in V$. And the arc is determined by adjacencies of G' $(\alpha \mid p_0p_1 \ldots p_{n-1})$ to $(\alpha+1 \mid p_0p_1 \ldots p_{n-1})$ where all the indices of the vertices of G are taken modulo n and p_α is adjacent to q_α in the digraph G.

Comellas and Fiol showed that we can constructed the large (Δ, D) digraphs from Cayley coset digraphs according to this theorem. For (Δ, 3) digraphs, they obtain (Δ, 7) digraphs with order $2(\frac{(\Delta+1)!}{(\Delta-2)!})^2$ and (Δ, 11) digraphs with order $3(\frac{(\Delta+1)!}{(\Delta-2)!})^3$. Moreover, they construct (Δ, 9) digraphs with order $2(\frac{(\Delta+1)!}{(\Delta-3)!})^2$ from the (Δ,4) digraphs.

Comellas and Foil give anther way to construct the largest digraph by increasing the diameter and degree [CF95].

Theorem 36 *IF there is a vertex symmetric Δ-regular k-reachable digraph with N vertices then, for all positive integers b, n and m a multiple of n, there exists a vertex symmetric $\Delta+1$-regular digraph with mN_n vertices and diameter kn+d with d being the diameter of the fixed 2-step digraph $Cay(Z/mZ,\ \{1, b\})$.*

The proof of this theorem follows similar step in Theorem 35. The new digraph $G' = (V',\ E')$ is construted as follows: The vertex set V' has elements $(\alpha \mid p_0p_1 \ldots p_{n-1})$ with $\alpha \in Z/mZ$ and $p_i \in V$. The arcs of E' are given by $(\alpha \mid p_0p_1 \ldots p_{n-1})$ to $(\alpha+1 \mid p_0p_1 \ldots p_{n-1})$ and $(\alpha+b \mid p_0p_1 \ldots p_{n-1})$ where p_α is adjacent to q_α in the digraph G.

With this constructions, a vertex-symmetric $(\Delta, 10)$ digraphs with order $3(\frac{\Delta!}{(\Delta-3)!})^3$ can be created from Cayley coset digraphs $\Gamma_{\Delta-1}(3)$.

In Table 4.1, the italic number is in the order of the largest-known (Δ, D) digraph created by Theorem 35 an 36.

Table 4.1: Largest-known vertex symmetric (Δ, D) digraphs.

Δ\ D	2	3	4	5	6	7	8
2	**6**	10	20	27	72	144	171
3	**12**	27	60	165	333	*1152*	1860
4	**20**	**60**	168	444	1260	*7200*	12090
5	**30**	**120**	**36**0	*1152*	3582	*28800*	54505
6	**42**	**210**	**840**	**252**0	7776	*88200*	170898
7	**56**	**336**	**1680**	**6720**	**20160**	*225792*	521906
8	**72**	**504**	**3024**	**15120**	**60480**	*508032*	1371582
9	**90**	**720**	**504**0	**30240**	**151200**	**1036800**	2965270
10	**110**	**99**0	**7920**	**55400**	**332640**	*1960220*	**6652800**

4.3 Broadcasting in Cayley Coset Digraphs

The cycle prefix digraphs $\Gamma_\Delta(D)$ were introduced as Cayley coset digraphs by Faber and Moore in 1988. In [CM96], Comellas and Mitjana presented new details concerning cycle prefix digraphs' structure that are used to design a communication scheme leading to give a upper bounds on the broadcast time of cycle prefix digraphs.

From the definition of the Cayley coset digraphs given in the beginning of this chapter, [FMC93] and [CF95] define a cycle prefix digraph on a alphabet of Δ +1 symbols as follows: For a given digraph G=(V, E), each vertex $v_1, v_2, \ldots, v_D$ is a sequence of distinct symbols from the alphbet. The arc is determined by the adjacencies which are given by

$$v_1 v_2 \ldots v_D \rightarrow \begin{cases} v_2 v_3 v_4 \ldots v_D v_{D+1}, & \text{if } v_{D+1} \neq v_1, v_2, \ldots, v_D \\ v_1 v_2 \ldots v_{k-1} v_{k+1} \ldots v_D v_k, & 1 \leq k \leq D-1 \end{cases}$$

The first kind of adjacency is called a **shift** because of introducing a new symbol. The second adjacency will be called **rotation**.

Consider the node labeled 21 in Fig 4.1. It is connected to itself shifted node, 10. On the other side, there is arc from 21 to 12 which is a rotation. By using the

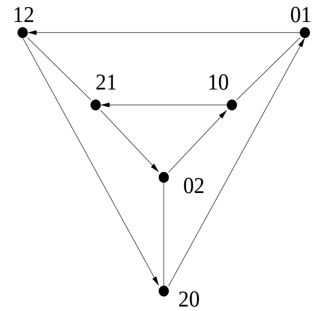

Figure 4.1: The Cayley Coset Digraph $\Gamma_2(2)$.

definition as digraphs on an alphabet, it is possible to give a recursive decomposition for the cycle prefix digraphs. Comellas and Mitjana have presented the next Lemma and proved it in [CM96].

Lemma 37 *The cycle prefix digraph* $\Gamma_\Delta(D)$ *decomposes into* $\binom{\Delta+1}{D}$ *subdigraphs each isomorphic for* $\Gamma_{D-1}(D-1)$.

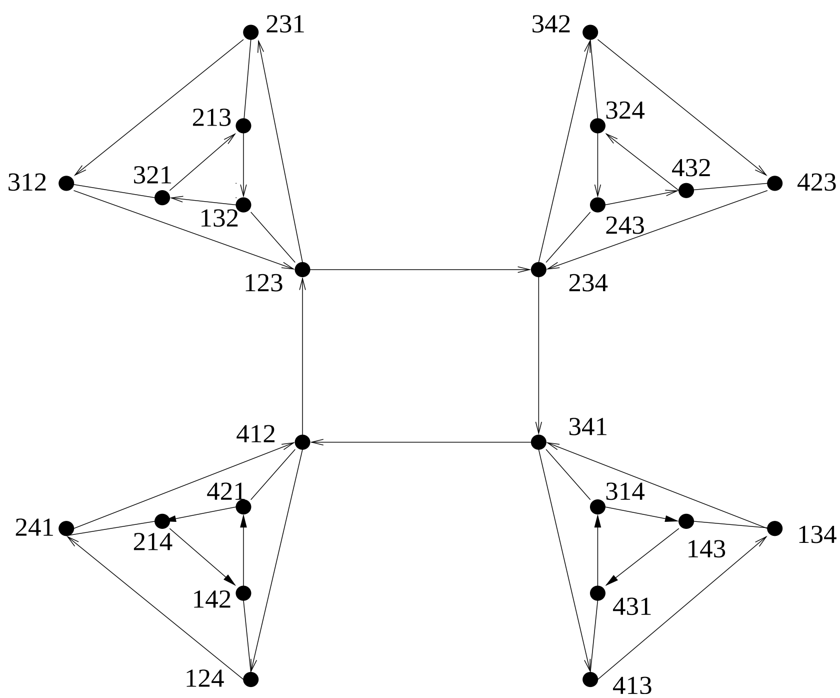

Figure 4.2: The Cayley coset digraph $\Gamma_3(3)$ composed by 4 subdigraphs which isomorphic to $\Gamma_2(2)$.

Fig 4.2 shows $\Gamma_3(3)$ can be decomposed into 4 subdigraphs each isomorphic to $\Gamma_2(2)$. To show the decomposition clearly, we have omit some arcs that join the nodes of the 4 terminal subdigraphs.

Comellas and Mitjana try to create a broadcast scheme and give the upper bounds of broadcast time of the cycle prefix digraphs by using the decomposition into subdigraphs. The first step of their broadcast scheme is the origin node distributes the message to $\binom{\Delta+1}{D}$ -1 other vertices. Those vertices belong to different subdigraphs of the decomposition. In the second step, the message is transmitted inside of each subdigraph. The two steps are executed in each subdigraphs in parallel.

It is very difficult to construct the set of vertices which receive the message in the first step. Comellas and Mitjana solved this problem by building a structure in $\Gamma_\Delta(D)$ containing the set of vertices such that, any two vertices in the set differ in at least one symbol. And the set have $\binom{\Delta+1}{D}$ elements. Then, we can construct a broadcast tree in the first step of the broadcast scheme.

Lemma 38 *For any cycle prefix digraph $\Gamma_\Delta(D)$ with $\Delta \geq D$, and any vertex x, there exists a tree T rooted at x with $\binom{\Delta+1}{D}$ vertices, depth D, and maximum degree $\Delta + 1 - D$, such that any two vertices in T differ in at least one symbol.*

We will give the basic broadcast tree for $\Gamma_4(3)$ in Fig 4.3 to shows how to construct the set of vertices in the first step of the broadcast scheme. Boldface number in Fig 4.3 indicate the broadcast order.

At first, the origin node is labeled 123. The vertices of level one are obtained by shifts that add one of the symbols $\{4, 5\}$ to the end of root 123. The two vertices are 234 and 235.

At level two, two vertices are ended with 1 adjacent from two nodes in level 1, which are 341 and 351. The other vertex in level two should be $34v_1$ and $v_1 \in 5, 6$. There is only 1 choice for v_1 because that vertex is adjacent from 234. So v_1 is 5.

At level three, we obtain three vertices $\{412, 452, 512\}$ from vertices of level two by shifts that add 2. The other vertex of level three must be $41v_1$. Because there is a vertex 421 adjacent from 341, v_1 should not be 2. The only choice from v_1 is 5 so that vertex is 415.

From that broadcast scheme, Comellas and Mitjana give the upper bound of the broadcast time of $\Gamma_\Delta(D)$ in the next theorem.

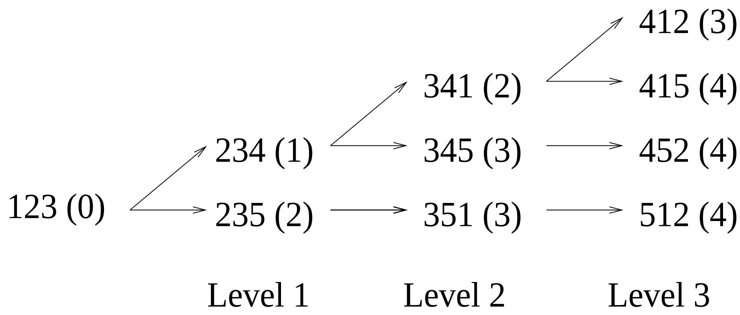

Figure 4.3: The basic broadcast tree for $\Gamma_4(3)$.

Theorem 39 *The broadcast time for* $\Gamma_\Delta(D)$*,* $\Delta \leq D$*, is bounded as follows:*

$$b(\Gamma_\Delta(D)) \leq \Delta + \frac{D(D+1)}{2}$$

By using a random algorithm described in next chapter, we got some new results of the broadcast time of the cycle prefix digraphs for small diameter and degree. Those results are smaller than or equal to the upper bound in Theorem 39. In Table 4.2, $B(\Gamma_\Delta(D))$ is our new results. The bold number indicate our new results is smaller than the upper bounds. we also give the corresponding upper bounds of the broadcast time of $\Gamma_\Delta(D)$, which is denoted by b$(\Gamma_\Delta(D))$. Furthermore, $|v|$ denotes the order of digraph.

Table 4.2: New results of the broadcast time of the cycle prefix digraphs for small values of the degree and diameter.

(Δ, D)	|V|	$b(\Gamma_\Delta(D))$	$B(\Gamma_\Delta(D))$
(2,2)	6	3	3
(3,2)	12	4	4
(3,3)	24	6	**5**
(4,2)	20	5	5
(4,3)	60	7	7
(4,4)	120	10	**8**
(5,2)	30	6	**5**
(5,3)	120	8	8
(5,4)	360	11	**10**
(5,5)	720	15	**11**
(6,2)	42	7	**6**
(6,3)	210	9	9
(6,4)	840	12	**11**
(7,2)	56	8	**6**
(7,3)	336	10	**9**
(8,2)	72	9	**7**
(8,3)	504	11	**10**
(9,2)	90	10	**7**
(9,3)	720	12	**11**
(10,2)	110	11	**8**

Chapter 5

Efficient Broadcast Directed Network Constructions

In this chapter, we present the largest-known broadcast directed networks satisfying the bounds on maximum vertex degree and the broadcast time. As we discussed in Chapter 2, broadcasting is a process that disseminating the message from the origin node to all other nodes in a connected network, with the restriction that each node can only forward the message to one neighbor at a time. It is a point-to-point communication model, which is similar to the telephone system.

We have already given some definitions in the previous chapters, such as the broadcast time and (Δ, T) broadcast digraphs. We now give another definition in preparation for our directed broadcast network construction.

Definition 40 *A **broadcast protocol (scheme)** for a vertex v (called the originator) for a graph G=(V, E) may be presented as a sequence* $V_0 = \{v\},\ E_1,\ V_1,\ E_2,\ V_2, \ldots,\ E_t,\ V_t = V$ *such that each* $V_i \subseteq V$*, each* $E_i \subseteq E$*, and for* $1 \leq i \leq t$*,*

1. *Each edge in* E_i *has exactly one endpoint in* V_{i-1}*.*

2. *No two edges in* E_i *share a common endpoint.*

3. $V_i = V_{i-1} \cup \{w \mid \{u, w\} \in E_i\}$*.*

The main purpose of the Degree/Broadcast-Time problem is to provide constructions of the largest possible (Δ, T) broadcast graphs (digraphs), which maximize the number of nodes for fixed constraints of maximum vertex degree and broadcast time.

The problem is slightly different from minimum broadcast problem studied by other researchers (e.g, [F77]).

In [D91] [DFF91] and [DPW98], Dinneen and his colleagues analyzed this problem. They introduced Cayley graphs as a model and presented algebra methods to construct the largest-known (Δ, T) broadcast graphs. Here we would like to use those same algebra methods to construct the largest-known (Δ, T) broadcast digraphs.

5.1 Cayley Digraph Construction

In this section, we will give the main group construction for finding large (Δ, T) broadcast digraphs. Most of digraphs listed in Appendix A are created by using semi-direct products of groups.

As explained in [D91], for given two cyclic group Z_m and Z_n, a semi-direct product group G= $Z_m \times_\sigma Z_n$ is formed by defining an appropriate homomorphism $\sigma : Z_m \to Aut(Z_n)$. In here, Z_n is a communitative ring with a group of units $U(Z_n)$. We define a mapping $\sigma'(k) = (r_c)_k = r_{ck}$ where r is belonged to $Aut(Z_n)$ and c is chosen so that $r_{cm} = 1$. The multiplication table of the semi-direct product group G is defined by

$$(a_0, a_1) *_\sigma (b_0, b_1) = (a_0 + b_0 \ mod\ m, (a_1 + \sigma'(a_0) \cdot b_1)\ mod\ n).$$

For $a \in Z_m$ and $b \in Z_n$, $(\sigma(a))b$= $\sigma'(a) \cdot b$ is a suitable homomorphism. Note that (0,0) is the group identity for a semi-direct product group constructed as above.

In Fig 5.1, we give a example of the above construction. The digraph is a largest known (2,4) Cayley digraph with 12 vertices based on the group $Z_4 \times_\sigma Z_3$. For $a \in Z_4$ and $b \in Z_3$, a vertex of the digraph whose corresponding group element is [a,b] is labeled 4a+3. The normal edge presents the use of the first generator while the dashed edges represent the second generator.

(Δ, T)	Order	Group	Generators	Order of Generators
(2,4)	12	$Z_4 \times_\sigma Z_3$ $\sigma'(1)$=2	[2 2] [3 1]	6 4

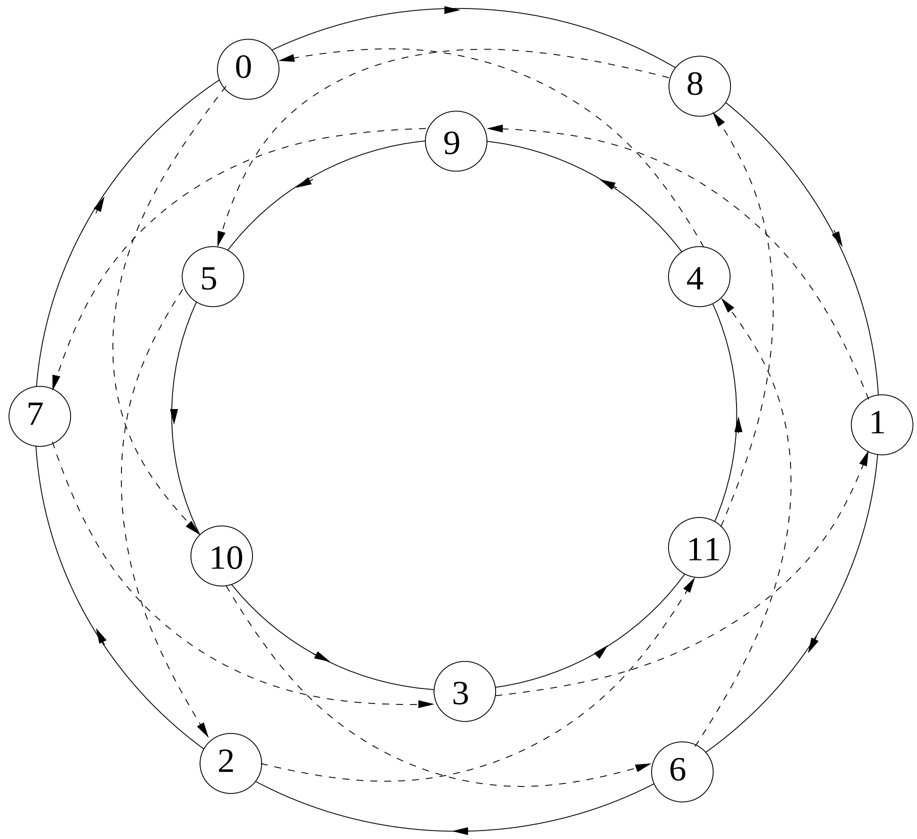

Figure 5.1: A largest known (2, 4) broadcast digraph.

5.2 Upper Bounds and Low Bounds

In [D91], Dinneen presents, as noted by others, a recurrence relation of the upper bound of the maximum order of broadcast directed networks. Let f(Δ, T) is the branch-out upper bound of the maximum order with out-degree Δ and Broadcast time T of a directed tree.

$$f(\Delta, 0) = 1$$

$$f(\Delta, T) = \sum_{i=1} min(\Delta, T) f(\Delta, T - i) + 1$$

From above recurrence relation, we get the table of some (Δ, T) broadcast directed networks upper bounds, Table 5.1.

Table 5.1: Some (Δ, T) directed broadcast network upper bounds.

Δ\T	2	3	4	5	6	7	8	9	10
2	4	7	12	20	33	54	88	143	232
3	4	8	15	28	52	96	177	326	600
4	4	8	16	31	60	116	224	432	833
5	4	8	16	32	63	124	244	480	944
6	4	8	16	32	64	127	252	508	992
7	4	8	16	32	64	128	255	511	1012
8	4	8	16	32	64	128	256	512	1020
9	4	8	16	32	64	128	256	512	1023
10	4	8	16	32	64	128	256	512	1024

For a given (Δ, T) broadcast directed network, a lower bound for the order of the larger broadcast directed networks can be obtain from the following theorem. The proof of this theorem is similar to that of the theorem for the undirected networks.

Theorem 41 *B(Δ, T) is the order of a (Δ, T) broacast directed network G, then the order of (Δ+1, T+1) broadcast directed network B(Δ+1, T+1)* $\geq 2 \cdot B(\Delta, T)$.

Proof. For a given (Δ, T) broadcast digraphs, we take the Cartesian product of two copies of it. That is , an arc for two directions is added between each vertex v and v' in two identical digraph G and G'. So we have a new digraph G''. The routing scheme of G'' uses one of the (v, v') edge during the first broadcast time. After that, the message from the originator broadcast in G and G' in parallel. Notice that the broadcast time for G'' is T+1 while the degree for G'' is Δ+1. So G'' is a (Δ+1, T+1) broadcast digraph. Obviously, the order of G'' is twice as order of G'. That is B(Δ+1, T+1) $\geq 2 \cdot B(\Delta, T)$. □

Similarly, we have a more general theorem as follows:

Theorem 42 *B(Δ+n, T+n)* $\geq 2^n \cdot B(\Delta, T)$, $n \geq 2$

Proof. The proof is simple. From Theorem 42, we have B(Δ+1, T+1) $\geq 2 \cdot B(\Delta, T)$. Then for a (Δ+n, T+n) broadcast digraph, B(Δ+n, T+n) should be larger than or equal to $2 \cdot B(\Delta + (n-1), T + (n-1))$. Next, we also get B(Δ+(n-1), T+(n-1)) $\geq 2 \cdot B(\Delta + (n-2), T + (n-2))$. So, B(Δ+n, T+n) $\geq 2^n \cdot B(\Delta, T)$, $n \geq 2$. □

5.3 Numerical Results and Broadcast Algorithm

Table 5.2 shows that the current largest known broadcast digraphs. For the reader's convenience the bold entries in Table 5.2 shows where the upper bounds have been achieved. The asterisks in the table denote where the random search algorithm has been used. Other entries show the results that have been achieved in [DPW98] for the undirected case, which are implicitly a given lower bounds for the directed case.

Table 5.2: The largest known (Δ, T) directed broadcast networks.

D\T	2	3	4	5	6	7	8	9	10
2	**4**	**7**	12*	20*	27*	42*	64*	84*	126*
3		**8**	**15**	28*	48*	80*	110*	220*	328*
4			**16**	**31**	56*	96*	165*	300*	506*
5				**32**	**63**	116	210	390	686
6					**64**	**127**	234	440	840
7						**128**	**255**	486	952
8							**256**	**512**	1000
9								**512**	**1023**
10									**1024**

Our computer search was processed in the Linux environment. The compiler is GCC. We now describe how our computer search for efficient directed networks was conducted. Only the basic algorithm will be discussed and we present only the principle algorithm used to calculate the broadcast time of the Cayley digraphs.

In [D92], Dinneen proved that Bounded-Degree Minimum Broadcast Time (BDM BT) is NP-complete. Since exact algorithms are impractical for large networks, a algorithm heuristics have been proposed. Also, because of the hardness of this problem, some research were restricted to some specific families of "nice" graphs (digraphs). So we choose an heuristics algorithm to do computer search on the broadcast time of the Cayley digraphs.

Our algorithm is simple. At first, it read the structure of input Cayley digraph G=(V, E) and record the label of each vertex and its neighbors. We set the original node to send the message is the vertex labeled 0. After the initial state is read, the algorithm divides all vertices into two sets. One set is the vertices that have received the message from the original node while The other set is the vertices who haven't received the message.

Then, the algorithm will check each vertex in the first set to find whether it has any neighbor didn't receive the message. If it has, a random vertex is picked from its neighbors and that neighbor node is included to the first set.

After that, all vertex in first set have been searched. Then the broadcast time is added 1. Repeating the step again until the number of elements in first set is equals to the order of the digraph. That means all vertices have received the message. The process of broadcasting is finished. We get an upper bound on the broadcast time for that digraph.

We may repeat the steps listed before for a fixed time. Then, we compare the results we achieved and choose the minimum one as an upper bound on the minimum broadcast time for that digraph.

Table 5.3: One routing scheme for an optimal (2, 5) directed broadcast network.

Cayley group digraph: $Z_4 \times_\sigma Z_5$	
Generators:	Order of the generator
[3 1] [1 0]	4 4
Routing scheme from identity element 0.	
$0 \to 5$	1st broadcast: $\|V_1\|$=2
0→16 5→10	2nd broadcast: $\|V_2\|$=4
5→1 10→6 16→2	3rd broadcast: $\|V_3\|$=7
1→19 2→9 6→4 10→15 16→14	4th broadcast: $\|V_4\|$=12
1→7 2→17 4→18 6→12 9→3 14→8 15→11 19→13	5th broadcast: $\|V_5\|$=20

We give an example in Table 5.3, Which shows a routing alogrithm for optimal (2,5) broadcast digraph. And Fig 5.2 is the digraph.

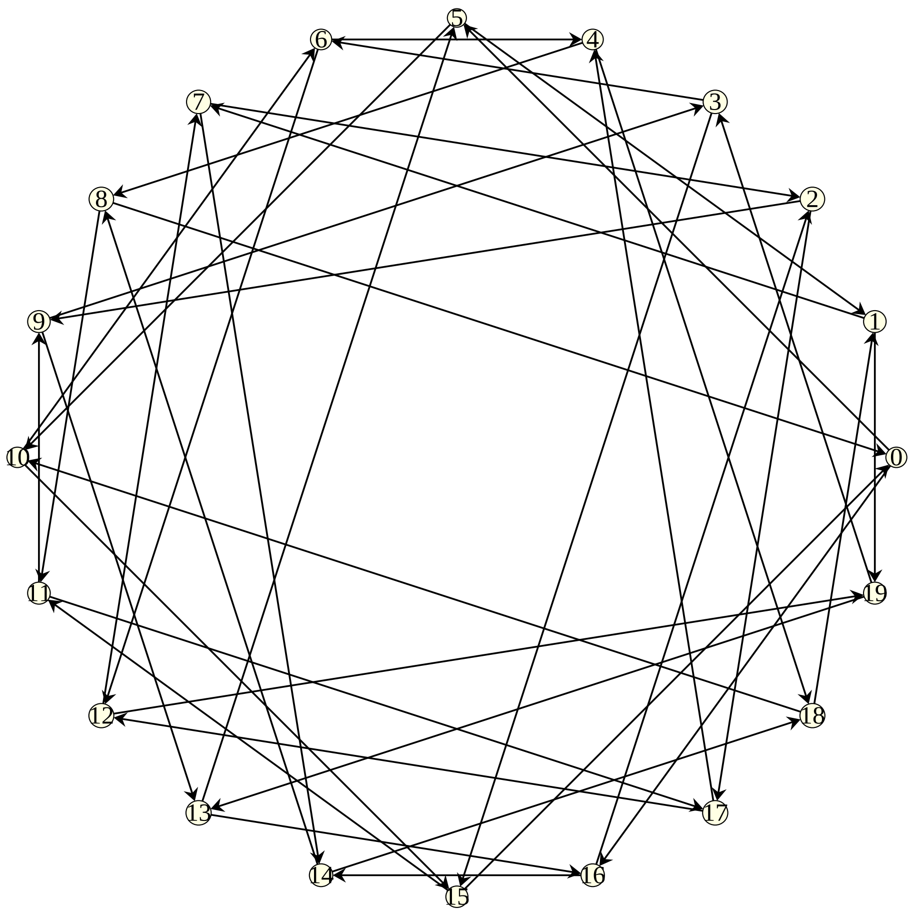

Figure 5.2: An Optimal (2, 5) directed broadcast network.

5.4 Algebra Construction Techniques

We will explain what broadcast digraphs correspond to the various listed in Table 5.1. As we see, we omit the entries below the diagonal since those entries are follow from the (Δ, Δ) broadcast digraphs. All entries in Table 5.1 are Cayley digraphs. In last section, we have mentioned that all results flagged with an asterisk have been computed by our random search algorithm. All digraphs are based on the semi-direct product of cyclic groups. We give the detail information of the group and its generators which created digraphs in Appendix A.

Next, some theorems will be given to cover some known optimal broadcast digraphs.

Theorem 43 *The directed hypercube Q_Δ is an optimal (Δ, Δ) broadcast digraph.*

Proof. Dinneen, and others, have proved that the hypercube Q_Δ is an optimal (Δ, Δ) broadcast graph (see [D91]). In this standard proof, the hypercube Q_Δ can be represented as a Cayley graph using the abelian group $(Z_2)^n$ with generators $\{e_i \mid 1 \leq i \leq n\}$ where $e_i = (\underbrace{0,\ldots,0}_{i-1},1,\underbrace{0,\ldots,0}_{n-i})$. The routing scheme is all vertices with the broadcast message route by using generator e_i at time i. The first vertex to receive the message is $(1,0,\ldots,0)$ $=(0,0,\ldots,0)+e_1$. So after n time every vertex will have received the message original from $(0,0,\ldots,0)$. So the broadcast time is equal to degree. The hypercube Q_Δ is an optimal broadcast graph since 2^n is the order of Q_Δ and also reach the upper bound.

In here, the same construction and routing scheme will be used. We just change the every edge in hypercube Q_Δ to two-directional arc. So the out-degree and in-degree of new digraph are equal to the degree of hypercube Q_Δ (undirected). Then, the hypercube Q_Δ (directed) is an optimal broadcast digraph. $\square$

The following construction is our main result of the thesis.

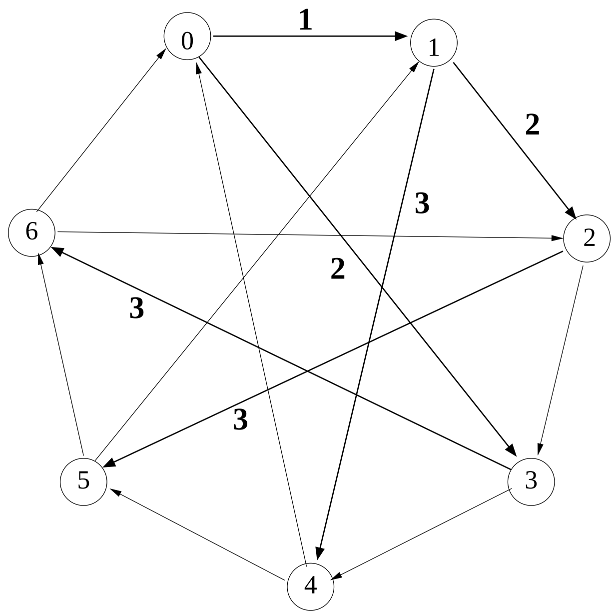

Figure 5.3: An optimal (2, 3) directed broadcast network.

Theorem 44 *The Cayley digraphs from the cyclic groups Z_{2^n-1} with generators $\{g_1, g_2, \ldots, g_{n-1}\}$ where $g_1 = 1, g_2 = 3, g_3 = 7, \ldots,$ and $g_{n-1} = 2^{n-1} - 1$ form optimal (Δ-1, Δ) broadcast digraphs $\{\mathcal{A}_{\Delta-1}\}$.*

Proof. Before proving the general version of this theorem, we illustrate it by building a simple (2,3) broadcast digraph $\mathcal{A}_2$ in Fig 5.3. The bold edges in Fig 5.3 denote the routing scheme from node 0. For this graph, the generators are 1 and 3. Let V_i={ v | vertex v has received the message at time i }. We start the broadcasting at the identity $V_0 = \{0\}$. Using 1 as the first generator yields $V_1 = V_0 \cup \{1\} = \{0, 1\}$. Then, using 1 as the generator for node 1 and using 3 as another generator for node 0 so that

$$V_2 = V_1 \cup \{1 + 1, 0 + 3\} = \{0, 1\} \cup \{2, 3\} = \{0, 1, 2, 3\}.$$

Finally, using 3 as the third generator and final broadcast time yields

$$V_3 = V_2 \cup \{0 + 3, 1 + 3, 2 + 3, 3 + 3\} = \{0, 1\} \cup \{3, 4, 5, 6\} = \{0, 1, 2, 3, 4, 5, 6\} = Z_7.$$

This shows that the broadcast time for $\mathcal{A}_2$ is 3.

We now give the complete proof of this theorem. At first, we defined

$$f_i = \{Spanning\ broadcast\ tree\ for\ time\ i+1\ in\ \mathcal{A}_i\}.$$

Proof by induction,

1. When i=1, f_i is given by the next figure:

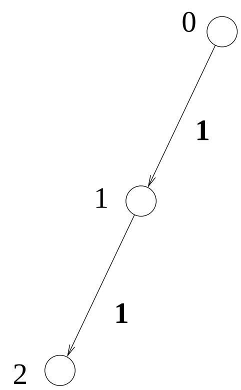

Figure 5.4: A spanning broadcast tree of time 2 in (1, 2) broadcast digraph $\mathcal{A}_1$.

   In Fig 5.4, The message is originated from vertex 0. When the time is 1, there exists an arc from vertex 0 to vertex 1 by using generators 1. When the time is 2, vertex 1 sends the message to vertex 2 by using generator 1 again. The bold number in Fig 5.4 is the generator being used. As we know, f_1 is a spanning broadcast tree of time 2 in $\mathcal{A}_1$. Obviously, $|f_1|$=3. That values equals to the upper bound of the order of (1,2) broadcast digraph.

2. When i=2, we defined as the followings,

   We show f_2 in Fig 5.5. When the time is 1, routing scheme is the same as that in f_1. There is a slight difference in time 2. We add one more arc from vertex 0 to vertex 3 by using a new generator 3. When the time is 3, 3 arcs from vertices 1, 2 and 3 to vertices 4, 5 and 6, respectively. Here, we use generator g_2=3 again. Then $|f_2|$=4+3=7. That is $|\mathcal{A}_2|$=7. So the order of $\mathcal{A}_2$ is equal to the upper bound for an optimal (2,3) broadcast digraph.

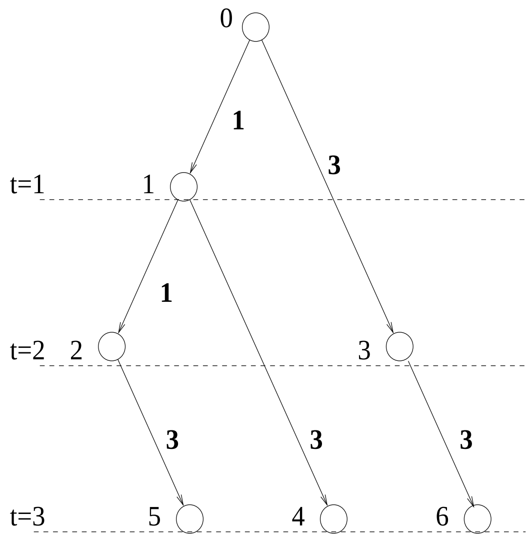

Figure 5.5: A spanning broadcast tree of time 3 in (2, 3) broadcast digraph $\mathcal{A}_2$.

3. When $i > 1$, we do the followings,

In Fig 5.6, f_i is the spanning broadcast tree of time $i+1$ in $\mathcal{A}_i$ while f_{i+1} is a new spanning broadcast tree of time $i+2$ in $\mathcal{A}_{i+1}$. We construct f_{i+1} from f_i. The routing scheme is designed by adding one more arc from vertex 0 to vertex $2^{i+1}-1$ by using the new generator $2^{i+1}-1$ when the time is i. Then, when time is $i+1$, there are arcs from each vertex from f_i (except 0) to the new $2^{i+1}-1$ vertices of f_i' by using generator $2^{i+1}-1$. So we have

$$\begin{aligned} f_i' &= \{1+2^{i+1}-1, 2+2^{i+1}-1, \ldots, 2^{i+1}+2^{i+1}\} \\ &= \{2^{i+1}, 2^{i+1}+1, \ldots, 2^{i+2}\} \end{aligned}$$

Then,

$$|f_{i+1}| = 2|f_i| + 1 = 2(2^{i+1}-1) + 1 = 2^{i+2} - 1$$

After time i+1, all vertices in $\mathcal{A}_{i+1}$ have received the message. Because the order of $\mathcal{A}_{i+1}$ equals to $2^{i+2}-1$ is the upper bound and $i+1$ is the degree of $\mathcal{A}_{i+1}$, each $\mathcal{A}_\Delta$ is optimal (Δ, Δ+1) directed broadcast network. $\square$

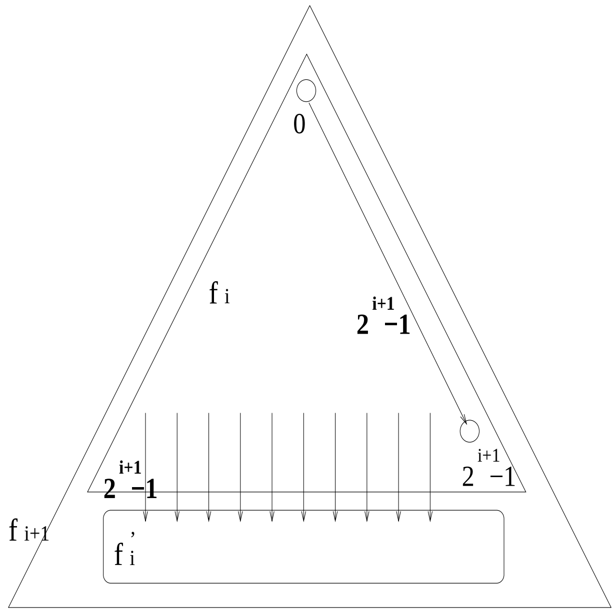

Figure 5.6: A spanning broadcast tree of time $i+2$ in $(i+1,\ i+2)$ broadcast digraph $\mathcal{A}_{i+1}$.

We give another example of the optimal broadcast directed networks $\mathcal{A}_{\Delta}$. Fig 5.7. shows an optimal (3,4) broadcast digraph $\mathcal{A}_3$ with 15 vetices. In that digraph, 3 generators $\{1, 3, 7\}$have been used. That is, there are 3 arcs from vertex 0 to vertex 1, 3 and 7 by using generators 1, 3 and 7 respectively. Its broadcast time is 4.

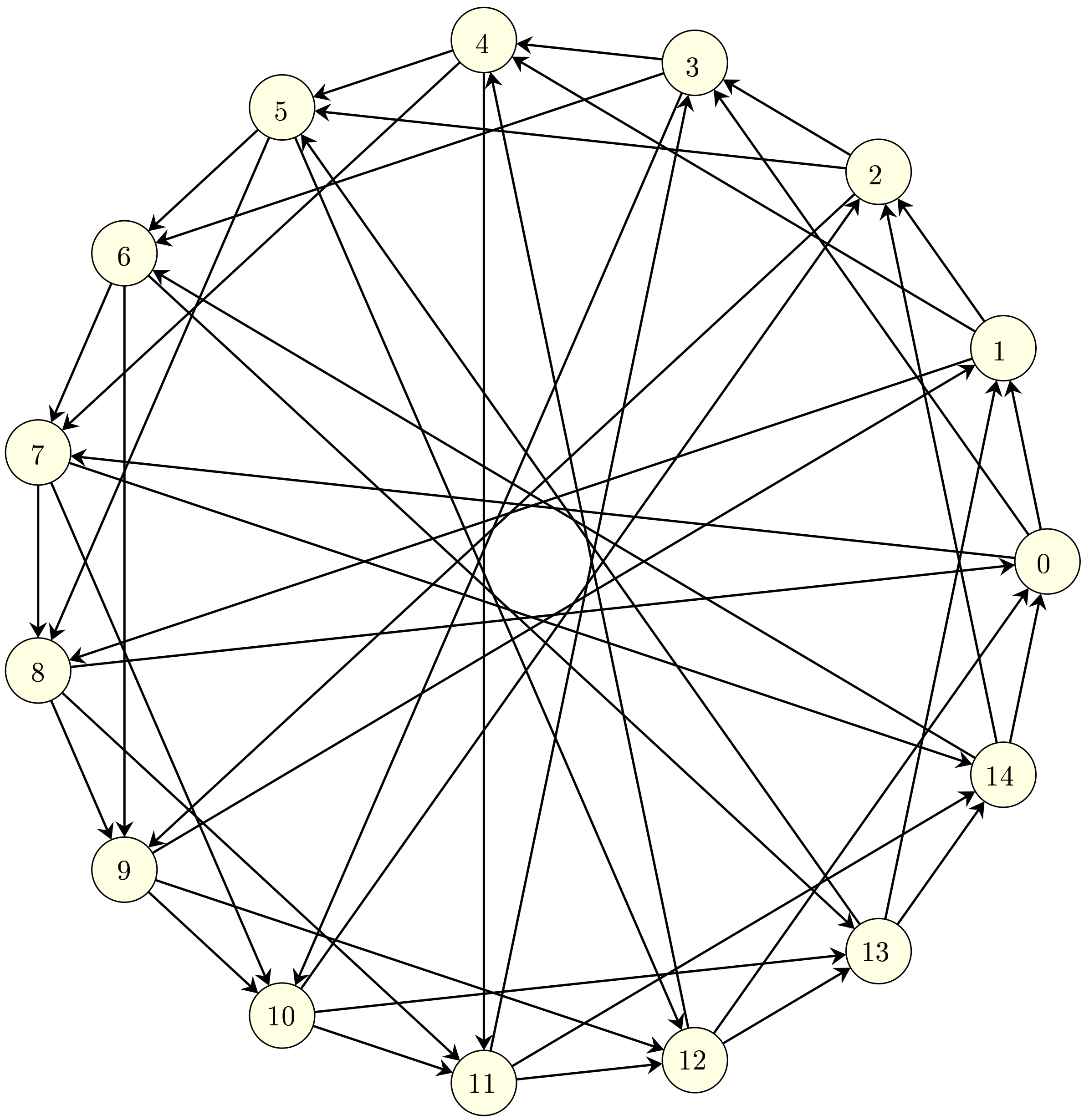

Figure 5.7: An optimal (3, 4) directed broadcast network.

Chapter 6

Conclusion

We now summarize our thesis as follows.

We have discussed some directed network constructions to study the broadcast time of those. At first, we study two classic directed networks. One is the Kautz digraphs and another is the de Bruijn digraphs. Both of them are the earliest-known large (Δ, D) digraphs. The (Δ, D) de Brujin digraph has (Δ^D) order while the (Δ, D) Kautz digraphs has ($\Delta^D + \Delta^{D-1}$) order. We presented upper bounds of the broadcast time of those digraphs to help us to study the construction for directed broadcast networks.

After observing those classic digraphs, we study the Cayley coset digraphs. From the algebra point, we discuss the definition of Cayley coset digraph and show that it is vertex symmetric. We first show an upper bound of the broadcast time of these digraphs, and then we have computed the broadcast times of cycle prefix digraphs, a type of Cayley coset digraphs. Some new results are smaller than upper bounds.

At last, the group theory has been used to discuss Cayley digraphs based on the semi-product groups. we establish the largest-known (Δ, T) broadcast digraphs table for small degree and broadcast time. It is very interesting to find some (Δ,Δ+1) optimal broadcast digraphs. As we have showed before, $\mathcal{A}_\delta$ is a new family of optimal (Δ,Δ+1) directed broadcast networks. Those digraphs are Cayley digraphs from the cyclic groups Z_{2^n-1} with generators $\{1, 3, 7, \ldots, 2^{n-1} - 1\}$.

In this paper, we show the advantage of using group theory in designing broadcast directed networks. The broadcast digraphs we have discussed are all vertex symmetric. As mentioned, routing algorithm is node independent in vertex symmetric digraphs. It can help us to establish an broadcast scheme and get the minimum broadcast time for those digraphs.

Some problems occurred in the course of this work. We mention some of them as follows:

1. We just use the random algorithm to compute the broadcast time of a given Cayley digraph. The efficiency of the algorithm is not so satisfactory. We need to apply other methods to get upper bounds of the minimun broadcast time. In Haobi Wang's paper [W99] and Fang Guo's thesis [F01], he presents a heuristic broadcast algorithm based on the theory of maximum matching. We can use their idea to design a heuristic broadcast algorithm for (Δ, T) digraphs.

2. As we know, the broadcast time of a (Δ, T) digraph is longer than the diameter of that. Does there exists another bound or a good polynomial-time algorithm to predict whether a Cayley digraph has a smaller broadcast time?

3. It seems that the largest known (Δ, T) dirceted broacast networks are all Cayley digraphs. Does there exist an non-vertex-symmetric broadcast digraph with largest order for given degree and broadcast time?

Appendix A

Broadcast (Δ, T) Digraph Results

(Δ, T)	Order	Group	Generators	Order of Generators
(2,4)	12	$Z_4 \times_\sigma Z_3$ $\sigma'(1)=2$	[2 2] [3 1]	6 4
(2,5)	20	$Z_4 \times_\sigma Z_5$ $\sigma'(1)=2$	[3 1] [1 0]	4 4
(2,6)	27	$Z_3 \times_\sigma Z_9$ $\sigma'(1)=2$	[1 8] [1 4]	9 9
(2,7)	42	$Z_3 \times_\sigma Z_{14}$ $\sigma'(1)=3$	[0 3] [1 3]	14 6
(2,8)	64	$Z_4 \times_\sigma Z_{16}$ $\sigma'(1)=3$	[2 7] [3 11]	16 8
(2,9)	84	$Z_6 \times_\sigma Z_{14}$ $\sigma'(1)=3$	[1 0] [0 5]	6 14
(2,10)	126	$Z_{18} \times_\sigma Z_7$ $\sigma'(1)=3$	[1 4] [12 5]	18 21

(Δ, T)	Order	Group	Generators	Order of Generators
(3,5)	28	$Z_4 \times_\sigma Z_7$ $\sigma'(1)$=3	[1 1] [2 1] [0 4]	4 14 7
(3,6)	48	$Z_{16} \times_\sigma Z_3$ $\sigma'(1)$=2	[2 1] [15 0] [9 2]	24 16 16
(3,7)	80	$Z_8 \times_\sigma Z_{10}$ $\sigma'(1)$=3	[4 6] [7 9] [1 4]	10 8 8
(3,8)	110	$Z_{10} \times_\sigma Z_{11}$ $\sigma'(1)$=2	[5 10] [4 4] [2 9]	2 5 5
(3,9)	220	$Z_{20} \times_\sigma Z_{11}$ $\sigma'(1)$=2	[10 2] [3 4] [14 0]	22 20 10
(3,10)	328	$Z_8 \times_\sigma Z_{41}$ $\sigma'(1)$=6	[2 7] [7 27] [1 28]	4 8 8

(Δ, T)	Order	Group	Generators	Order of Generators
(4,6)	56	$Z_4 \times_\sigma Z_{14}$ $\sigma'(1)$=3	[1 0] [3 13] [0 12] [1 8]	4 4 7 4
(4,7)	96	$Z_6 \times_\sigma Z_{16}$ $\sigma'(1)$= 3	[2 7] [3 11] [3 14] [5 6]	48 16 8 24
(4,8)	165	$Z_{15} \times_\sigma Z_{11}$ $\sigma'(1)$=2	[2 7] [0 5] [12 0] [8 8]	15 11 5 15
(4,9)	300	$Z_{12} \times_\sigma Z_{25}$ $\sigma'(1)$=2	[9 21] [11 10] [8 2] [1 15]	4 12 75 12
(4,10)	506	$Z_{22} \times_\sigma Z_{23}$ $\sigma'(1)$= 5	[7 15] [5 11] [0 3] [8 0]	22 22 23 11

(Δ, T)	Order	Group	Generators	Order of Generators
(5,7)	110	$Z_5 \times_\sigma Z_{22}$ $\sigma'(1)$=7	[2 7] [0 5] [2 0] [3 8] [4 18]	10 22 5 5 5
(5,8)	188	$Z_4 \times_\sigma Z_{47}$ $\sigma'(1)$=5	[3 17] [3 6] [0 33] [2 35] [2 39]	4 4 47 94 94
(5,9)	340	$Z_{20} \times_\sigma Z_{17}$ $\sigma'(1)$=3	[2 7] [15 10] [17 8] [13 5] [9 13]	10 4 20 20 20
(5,10)	520	$Z_{20} \times_\sigma Z_{26}$ $\sigma'(1)$=7	[2 7] [15 1] [17 6] [13 8] [9 10]	10 4 20 20 20

(Δ, T)	Order	Group	Generators	Order of Generators
(6,8)	224	$Z_{14} \times_{\sigma} Z_{16}$	[2 7]	112
		$\sigma'(1)= 3$	[1 11]	112
			[1 14]	56
			[9 6]	56
			[9 4]	28
			[4 7]	112
			[9 3]	112
(6,9)	396	$Z_{18} \times_{\sigma} Z_{22}$	[2 7]	198
		$\sigma'(1)= 7$	[15 5]	6
			[3 0]	6
			[17 8]	18
			[17 8]	18
			[8 17]	198
			[15 19]	6
			[1 4]	18
(6,10)	770	$Z_{22} \times_{\sigma} Z_{35}$	[2 7]	55
		$\sigma'(1)$=2	[15 27]	154
			[13 5]	154
			[13 14]	22
			[21 22]	154
			[10 32]	385
			[17 11]	154
			[17 33]	154
			[10 17]	385

(Δ, T)	Order	Group	Generators	Order of Generators
(7,9)	450	$Z_{18} \times_\sigma Z_{25}$	[2 7]	225
		$\sigma'(1)$=2	[15 2]	6
			[3 10]	6
			[17 19]	18
			[17 17]	18
			[8 17]	225
			[15 16]	6
			[1 23]	18
			[0 2]	25
(7,10)	820	$Z_{20} \times_\sigma Z_{41}$	[2 7]	10
		$\sigma'(1)$=6	[15 27]	4
			[17 28]	20
			[13 7]	20
			[9 16]	20
			[6 6]	10
			[11 35]	20
			[9 6]	20
			[8 38]	5
			[5 33]	4

(Δ, T)	Order	Group	Generators	Order of Generators
(8,10)	882	$Z_{18} \times_\sigma Z_{49}$	[2 7]	9
		$\sigma'(1)$=2	[15 27]	6
			[3 12]	6
			[17 0]	18
			[17 22]	18
			[8 18]	9
			[15 39]	6
			[1 33]	18
			[0 24]	49
			[9 8]	2

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