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Abstract

The hypermetric cone HYP_n is the set of vectors $(d_{ij})_{1\leq i<j\leq n}$ satisfying the inequalities

$$\sum_{1\leq i<j\leq n} b_i b_j d_{ij} \leq 0 \text{ with } b_i \in \mathbb{Z} \text{ and } \sum_{i=1}^{n} b_i = 1\,.$$

A Delaunay polytope of a lattice is called *extremal* if the only affine bijective transformations of it into a Delaunay polytope, are the homotheties; there is correspondance between such Delaunay polytopes and extreme rays of HYP_n. We show that unique Delaunay polytopes of root lattices A_1 and E_6 are the only extreme Delaunay polytopes of dimension at most 6. We describe also the skeletons and adjacency properties of HYP_7 and of its dual.

1 Introduction

A vector $(d_{ij})_{1\leq i<j\leq n} \in \mathbb{R}^N$ with $N = \binom{n}{2}$ is called an *n-hypermetric* if it satisfy the following *hypermetric inequalities*:

$$\sum_{1\leq i<j\leq n} b_i b_j d_{ij} \leq 0 \text{ with } b = (b_i) \in \mathbb{Z}^n \text{ and } \sum_{i=1}^{n} b_i = 1\ . \tag{1}$$

The set of vectors satisfying (1) is called the *hypermetric cone* and denoted by HYP_n.
We have the inclusions $CUT_n \subset HYP_n \subset MET_n$, where MET_n denotes the cone of all semimetrics on n points and CUT_n (see section 3 below and chapter 4 of [DeLa97]) is the cone of all semimetrics on n points, which are isometrically embeddable into some space l_1^m. In fact, the triangle inequality $d_{ij} \leq d_{ik} + d_{jk}$ is the hypermetric inequality with vector b such that $b_i = b_j = 1$, $b_k = -1$ and $b_l = 0$, otherwise.
For $n \leq 4$ all three cones coincide and $HYP_n = CUT_n$ for $n \leq 6$; so the cone HYP_7 is the first proper hypermetric cone[1]. See [DeLa97] for detailed study of those cones and their numerous applications in Combinatorial Optimization, Analysis and other areas of mathematics. In particular, the hypermetric cone had direct applications in Geometry of Quadratic Forms; see section 2.
In fact, HYP_n is a polyhedral cone (see [DGL93]). Lovasz (see [DeLa97] pp. 201-205) gave another proof

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[1] Apropos, MET_7 has 46 orbits of extreme rays and not 41 as, by a technical mistake, was given in [Gr92] and [DeLa97]

of it and the bound $max|b_i| \leq n!2^n \binom{2n}{n}^{-1}$ for any vector $b = (b_i)$, defining a facet of HYP_n.
The group of all permutations on n vertices induces a partition of the set of i-dimensional faces of HYP_n into orbits. Baranovskii using his method presented in [B70] found in [Ba99] the list of all facets of HYP_7: 3773 facets, divided into 14 orbits. On the other hand, in [DGL92] were found 29 orbits of extreme rays of HYP_7 by classifying the basic simplexes of the Schlafli polytope of the root lattice E_6.
In section 3 we show that the 37170 extreme rays containing in those 29 orbits are, in fact, the complete list. It also implies that the Schlafli polytope (unique Delaunay polytope of E_6) and the segment α_1 (the Delaunay polytope of A_1) are only extreme Delaunay polytopes of dimension at most six.
In section 4 we give adjacency properties of the skeletons of HYP_7 and of its dual.
The computations were done using the programs *cdd* (see [Fu]) and *nauty* (see [MK]).

2 Hypermetrics and Delaunay polytopes

For more details on the material of this section see chapters 13–16 of [DeLa97].
Let $L \subset \mathbb{R}^k$ be a k-dimensional lattice and let $S = S(c, r)$ be a sphere in $\mathbb{R}^k$ with center c and radius r. Then, S is said to be an *empty sphere* in L if the following two conditions hold:

$$\|v - c\| \geq r \text{ for all } v \in L \text{ and the set } S \cap L \text{ has affine rank } k + 1.$$

Then, the center c of S is called a *hole* in [CS]. The polytope P which is defined as the convex hull of the set $P = S \cap L$ is called a *Delaunay polytope*, or (in original terms of Voronoi who introduced them in [Vo]) *L-polytope*.
On every set $A = \{v_1, \ldots, v_m\}$ of vertices of a Delaunay polytope P we can define a distance function $d_{ij} = \|v_i - v_j\|^2$. The function d turns out to be a metric and, moreover, a hypermetric. It follows from the following formula (see [As82] and [DeLa97] p. 195) :

$$\sum_{i,j \in A} b_i b_j d(i,j) = 2(r^2 - \|\sum_{i \in A} b_i v_i - c\|^2) \leq 0\,.$$

On the other hand, Assouad has shown in [As82] that *every finite* hypermetric space is a square euclidean distance on a generating set of vertices of a Delaunay polytope of a lattice.
For example, in dimension two there are two kinds of combinatorial types of Delaunay polytopes: triangle and rectangle. Since $HYP_3 = MET_3$, we see that a triangle can be made a set of vertices of a Delaunay polytope if and only if it has obtuse angles.
A Delaunay polytope P is said to be *extreme* if the only (up to orthogonal transformations and translations) affine bijective transformations T of $\mathbb{R}^k$ for which $T(P)$ is again a Delaunay polytope, are the homotheties. [DGL92] show that the hypermetric on generating subsets of a extreme Delaunay polytope (see above) lie on extreme ray of HYP_n and that a hypermetric, lying on an extreme ray of HYP_n, is the square of euclidean distance on generating subset of extreme Delaunay polytope of dimension at most $n-1$.
In [DeLa97] p. 228 there is a more complete dictionary translating the properties of Delaunay polytopes into those of the corresponding hypermetrics.
Remind that E_6 and E_7 are *root* lattices defined by

$$E_6 = \{x \in \mathbb{Z}^8 : x_1 + x_2 = x_3 + \cdots + x_8 = 0\},\ E_7 = \{x \in \mathbb{Z}^8 : x_1 + x_2 + x_3 + \cdots + x_8 = 0\}$$

The skeleton of unique Delaunay polytope of E_6 is 27-vertex strongly regular graph, called the *Schlafli graph*. In fact, the 29 orbits of extreme rays of HYP_7, found in [DGL92], were three orbits of extreme rays of CUT_7 (*cuts*) and 26 ones, corresponding to all sets of seven vertices of Schlafli graph, which are affine bases (over $\mathbb{Z}$) of E_6. The root lattice E_7 has two Delaunay polytopes: 7-simplex and 56-vertex polytope, called *Gosset polytope*. In [Du] were found all 374 orbits of affine bases for the Gosset polytope.

3 Computing the extreme rays of HYP_7

All 14 orbits F_m, $1 \le m \le 14$, of facets of HYP_7, found by Baranovskii, are represented below by the corresponding vector b^m (see (1)):

$b^1 = (1,1,-1,0,0,0,0)$; $b^2 = (1,1,1,-1,-1,0,0)$; $b^3 = (1,1,1,1,-1,-2,0)$,
$b^4 = (2,1,1,-1,-1,-1,0)$; $b^5 = (1,1,1,1,-1,-1,-1)$; $b^6 = (2,2,1,-1,-1,-1,-1)$,
$b^7 = (1,1,1,1,1,-2,-2)$, $b^8 = (2,1,1,1,-1,-1,-2)$; $b^9 = (3,1,1,-1,-1,-1,-1)$,
$b^{10} = (1,1,1,1,1,-1,-3)$; $b^{11} = (2,2,1,1,-1,-1,-3)$, $b^{12} = (3,1,1,1,-1,-2,-2)$,
$b^{13} = (3,2,1,-1,-1,-1,-2)$, $b^{14} = (2,1,1,1,1,-2,-3)$.

It gives the total of 3773 inequalities. The first ten orbits are the orbits of hypermetric facets of the cut cone CUT_7; first four of them come as 0-*extension* of facets of the cone HYP_6, i.e. the vector has zero components x_{ij} for some $1 \le i \le 7$ and all $1 \le j < i$, $i < j \le 7$. The orbits F_{11}–F_{14} consist of some 19-dimensional simplex faces of CUT_7, becoming simplex facets in HYP_7.
The proof (see [B70] and [RB]) was in terms of volume of simplices; his proof implies that for facet of HYP_7 holds $|b_i| \le 3$ (compare with the bound in introduction). In [RB] the repartitioning polytopes (connected implicitly in [Vo] to facets of HYP_n) found for facets of HYP_7.
Because of the large number of facets of HYP_7, it is difficult to find extreme rays just by application of existing programs (see [Fu]). So, let us consider in more detail the cut cone CUT_7.
Call *cut cone* and denote by CUT_n the cone generated by all *cuts* δ_S defined by

$$(\delta_S)_{ij} = 1 \text{ if } |S \cap \{i,j\}| = 1 \text{ and } \delta_S(i,j) = 0, \text{ otherwise},$$

where S is any subset of $\{1,\dots,n\}$. The cone CUT_n has dimension $\binom{n}{2}$ and $2^{n-1}-1$ non-zero cuts as generators of extreme rays. There are $\lfloor \frac{n}{2} \rfloor$ orbits, corresponding to all non-zero values of $min(|S|, n-|S|)$. The skeleton of CUT_n is the complete graph $K_{2^{n-1}-1}$. See part V of [DeLa97] for a survey on facets of CUT_n.
The 38780 facets of the cut cone CUT_7 partitioned in 36 orbits. In [Gr90] was shown that the known list of 36 orbits was complete (See [DDL94] and chapter 30 of [DeLa97] for details). Of these 36 orbits ten are hypermetric. We computed the diameter of the skeleton of dual CUT_7: it is exactly 3 (apropos, the diameter of the skeleton of MET_n, $n \ge 4$, is 2, see [DeDe94]). So, we have $CUT_n \subset HYP_n$ and cones CUT_7, HYP_7 have ten common (hypermetric) facets: F_1–F_{10}.
Each of 26 orbits of non-hypermetric facets of CUT_7 consists of simplex cones, i.e. those facets are incident exactly to 20 cuts or, in other words, adjacent to 20 other facets. It turns out that the 26 orbits of non-hypermetric facets of CUT_7 correspond exactly to 26 orbits of non-cut extreme rays of HYP_7.
In fact, if d is a point of an extreme ray of HYP_7, which is not a cut, then it violates one of the non-hypermetric facet inequalities of CUT_7. More precisely, our computation consist of the following steps:

1. If d belongs to an non-cut extreme ray of HYP_7, then $d \notin CUT_7$.

2. So, there is at least one non-hypermetric facet F of CUT_7 with $F(d) < 0$.

3. Select a facet F_i for each non-hypermetric orbit of O_i with $1 \le i \le 26$ and define 26 subcones C_i, $1 \le i \le 26$, by $C_i = \{d \in HYP_7 : F_i(d) \le 0\}$.

4. The initial set of 3773 hypermetric inequalities is non-redundant, but adding the inequality $F_i(d) \le 0$ yields a highly redundant set of inequalities. We remove the redundant inequalities using invariant

group (of permutations preserving the cone C_i) and linear programming (see polyhedral FAQ[2] in [Fu]).

5. For each of 26 subcones we found, by computation, a set of 21 non-redundant facets, i.e. each of the subcones C_i is a simplex. We get 21 extreme rays for each of the 26 subcones.

6. We remove the 20 extreme rays, which are cuts, from each list and get, for each of these subcones, exactly one non-cut extreme ray.

So, we get an upper bound 26 for the number of non-cuts orbits of extreme rays. But [DGL92] gave, in fact, a lower bound 26 for this number. So, we get:

Proposition 3.1 *The hypermetric cone HYP_7 has 37170 extreme rays, divided into three orbits corresponding to cuts and 26 orbits corresponding to hypermetrics on 7-vertex affine bases of the Schlafli polytope.*

Note that the above computation proves again that the list of 14 orbits of hypermetric facet is complete. If not, there would exist an hypermetric facet that is violated by one extreme ray belonging to the 29 found orbits, but this would imply that the Schlafli polytope or the 1-simplex have interior lattice points, which is false.

Corollary 3.2 *The only extreme Delaunay polytopes of dimension at most six are the 1-simplex and the Schlafli polytope.*

This method computes precisely the difference between HYP_7 and CUT_7.
The observed correspondence between the 37107 non-hypermetric facets of CUT_7 and the 37107 non-cut extreme rays of HYP_7 is presented in Table 1. The first line of Table 1 indicates ij position of the vector, defining facets and generators of extreme rays. By double line we separate 26 pairs (facet and corresponding extreme ray) into five *switching* classes. Two facets F and F' of CUT_7 are called *switching equivalent* if there exist

$$S \subset \{1, \dots, 7\}, \text{ such that } F(\delta_S) = 0,$$
$$F_{ij} = -F'_{ij} \text{ if } |S \cap \{i, j\}| = 1 \text{ and } F_{ij} = F'_{ij}, \text{ otherwise.}$$

See section 9 of [DGL92] for details on the switchings in this case. In the first column of Table 1 is given, for each of five switching classes, the cut δ_S such that corresponding facet is obtained by the switching by δ_S from the first facet of the class. The non-hypermetric orbits of facets of CUT_7 are indicated by O_i and the corresponding non-cut orbits of extreme rays of HYP_7 are indicated by E_{i+3}. For any extreme ray we indicate also the corresponding graph $\overline{G}_j$ (in terms of [DGL92] and chapter 16 of [DeLa97]).
The five switching classes of Table 1 correspond, respectively, to the following five classes of non-hypermetric facets of CUT_7, in terms of [DDL94] and chapter 30 of [DeLa97]: *parachute* facets $P1-P3$; *cycle* facets $C1, C4-C6$; *Grishukhin* facets $G1-G7$; cycle facets $C2, C7-C12$; cycle facets $C3, C13-C16$.
[DG93] consider extreme rays of HYP_n which corresponds, moreover, to the path-metric of a graph; the Delaunay polytope, generated by such hypermetrics belongs to an integer lattice and, moreover, to a root lattice. They found, amongst 26 non-cut orbits of extreme rays of HYP_7, exactly twelve which are graphic: E_4, E_5, E_8, E_9, E_{10}, E_{15}, E_{16}, E_{17}, E_{22}, E_{23}, E_{24}, E_{29}. For example, E_{10}, E_{23} and E_{29} correspond to graphic hypermetrics on $K_7 - C_5$, $K_7 - P_4$ and $K_7 - P_3$, respectively. Three of above twelve extreme hypermetrics correspond to polytopal graphs: 3-polytopal graph corresponding to E_4 and 4-polytopal graphs $K_7 - C_5$, $K_7 - P_4$. Remark also that the footnote and figures on pp. 242-243 of [DeLa97] mistakingly attribute the graph $\overline{G}_{18}$ to the class $q = 11$ (fourth class in our terms); in fact, it belongs to the class $q = 12$ (our third class) as it was rightly given originally in [DGL92].

[2]http://www.ifor.math.ethz.ch/ fukuda/polyfaq/polyfaq.html

	12	13	14	15	16	17	23	24	25	26	27	34	35	36	37	45	46	47	56	57	67
O_1	-1	-1	0	0	1	1	-1	0	1	0	1	1	0	1	0	1	-1	1	1	-1	0
$E_4;\overline{G}_{24}$	2	2	2	2	1	1	2	1	1	2	1	1	1	1	2	1	2	1	1	2	2
$O_2;\delta_{\{\underline{3},5,6\}}$	-1	1	0	0	-1	1	1	0	-1	0	1	-1	0	1	0	-1	1	1	1	1	0
$E_5;\overline{G}_4$	2	1	2	1	2	1	1	1	2	1	1	2	1	1	1	2	1	1	1	1	1
$O_3;\delta_{\{\underline{3},5,4\}}$	-1	1	0	0	1	1	1	0	-1	0	1	1	0	-1	0	1	1	-1	-1	1	0
$E_6;\overline{G}_{23}$	2	1	1	1	1	1	1	2	2	2	1	1	1	2	1	1	1	2	2	1	2
O_4	-1	-1	-1	1	1	1	-1	0	0	1	1	0	1	0	1	1	1	1	0	-1	-1
$E_7;\overline{G}_{25}$	2	2	2	1	1	2	2	1	2	1	1	1	1	2	1	1	1	1	1	2	2
$O_5;\delta_{\{\underline{3},7\}}$	-1	1	-1	1	1	-1	1	0	0	1	-1	0	-1	0	1	1	1	-1	0	1	1
$E_8;\overline{G}_5$	2	1	2	1	1	1	1	1	2	1	2	2	2	1	1	1	1	2	1	1	1
$O_6;\delta_{\{\underline{2},3,7\}}$	1	1	-1	1	1	-1	-1	0	0	-1	1	0	-1	0	1	1	1	-1	0	1	1
$E_9;\overline{G}_{26}$	1	1	2	1	1	1	2	2	1	2	1	2	2	1	1	1	1	2	1	1	1
$O_7;\delta_{\{\underline{1},5,6\}}$	1	1	1	1	1	-1	-1	0	0	-1	1	0	-1	0	1	-1	-1	1	0	1	1
$E_{10};\overline{G}_1$	1	1	1	1	1	1	2	1	1	2	1	1	2	1	1	2	2	1	1	1	1
O_8	-1	-1	-1	0	1	2	-1	0	1	1	2	0	1	1	2	-1	1	1	0	-1	-2
$E_{11};\overline{G}_{22}$	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	2	1	1	1	2	2
$O_9;\delta_{\{\underline{1},4,6\}}$	1	1	-1	0	1	-2	-1	0	1	-1	2	0	1	-1	2	1	1	-1	0	-1	2
$E_{12};\overline{G}_{21}$	1	1	2	1	2	2	1	2	1	2	1	2	1	2	1	1	1	2	2	2	1
$O_{10};\delta_{\{\underline{5}\}}$	-1	-1	-1	0	1	2	-1	0	-1	1	2	0	-1	1	2	1	1	1	0	1	-2
$E_{13};\overline{G}_{20}$	2	2	2	1	2	1	1	1	2	1	1	1	2	1	1	1	1	1	2	1	2
$O_{11};\delta_{\{\underline{3},5\}}$	-1	1	-1	0	1	2	1	0	-1	1	2	0	1	-1	-2	1	1	1	0	1	-2
$E_{14};\overline{G}_{19}$	2	1	2	1	2	1	2	1	2	1	1	2	1	2	2	1	1	1	2	1	2
$O_{12};\delta_{\{\underline{1},7\}}$	1	1	1	0	-1	2	-1	0	1	1	-2	0	1	1	-2	-1	1	-1	0	1	2
$E_{15};\overline{G}_7$	1	1	1	1	1	1	1	1	1	1	2	1	1	1	2	2	1	2	1	1	1
$O_{13};\delta_{\{\underline{7},4,1\}}$	1	1	-1	0	-1	2	-1	0	1	1	-2	0	1	1	-2	1	-1	1	0	1	2
$E_{16};\overline{G}_8$	1	1	2	1	1	1	1	2	1	1	2	2	1	1	2	1	2	1	1	1	1
$O_{14};\delta_{\{\underline{6},4\}}$	-1	-1	1	0	-1	2	-1	0	1	-1	2	0	1	-1	2	1	1	-1	0	-1	2
$E_{17};\overline{G}_{18}$	2	2	1	2	1	1	1	2	1	2	1	2	1	2	1	1	1	2	2	2	1
O_{15}	-1	-1	-2	1	1	2	0	-1	1	1	2	-2	1	1	1	2	2	3	-1	-2	-2
$E_{18};\overline{G}_{14}$	2	2	2	2	2	1	1	1	1	1	1	2	1	1	2	1	1	1	1	2	2
$O_{16};\delta_{\{\underline{5},3\}}$	-1	1	-2	-1	1	2	0	-1	-1	1	2	2	1	-1	-1	-2	2	3	1	2	-2
$E_{19};\overline{G}_{15}$	2	1	2	1	2	1	2	1	2	1	1	1	1	2	1	2	1	1	2	1	2
$O_{17};\delta_{\{\underline{5},4\}}$	-1	-1	2	-1	1	2	0	1	-1	1	2	2	-1	1	1	2	-2	-3	1	2	-2
$E_{20};\overline{G}_{17}$	2	2	1	1	2	1	1	2	2	1	1	1	2	1	2	1	2	2	2	1	2
$O_{18};\delta_{\{\underline{7},2,6\}}$	-1	1	2	-1	-1	2	0	1	-1	-1	2	-2	1	1	-1	2	2	-3	-1	2	2
$E_{21};\overline{G}_{13}$	2	1	1	1	1	1	2	2	2	2	1	2	1	1	1	1	1	2	1	1	1
$O_{19};\delta_{\{\underline{7},4,1\}}$	1	1	-2	-1	-1	2	0	1	1	1	-2	2	1	1	-1	-2	-2	3	-1	2	2
$E_{22};\overline{G}_6$	1	1	2	1	1	1	1	2	1	1	2	1	1	1	1	2	2	1	1	1	1
$O_{20};\delta_{\{\underline{1},7\}}$	1	1	2	-1	-1	2	0	-1	1	1	-2	-2	1	1	-1	2	2	-3	-1	2	2
$E_{23};\overline{G}_2$	1	1	1	1	1	1	1	1	1	1	2	2	1	1	1	1	1	2	1	1	1
$O_{21};\delta_{\{\underline{4},5,6\}}$	-1	-1	2	-1	-1	2	0	1	-1	-1	2	2	-1	-1	1	2	2	-3	-1	2	2
$E_{24};\overline{G}_{16}$	2	2	1	1	1	1	1	2	2	2	1	1	2	2	2	1	1	2	1	1	1
O_{22}	-1	-1	-2	1	2	3	-1	-2	1	2	3	-2	1	2	3	2	3	5	-2	-3	-5
$E_{25};\overline{G}_{11}$	1	1	2	2	1	1	1	2	2	1	1	2	2	1	1	1	2	1	2	2	2
$O_{23};\delta_{\{\underline{3},6\}}$	-1	1	-2	1	-2	3	1	-2	1	-2	3	2	-1	2	-3	2	-3	5	2	-3	5
$E_{26};\overline{G}_{10}$	1	2	2	2	2	1	2	2	2	2	1	1	1	1	2	1	1	1	1	2	1
$O_{24};\delta_{\{\underline{7},4\}}$	-1	-1	2	1	2	-3	-1	2	1	2	-3	2	1	2	-3	-2	-3	5	-2	3	5
$E_{27};\overline{G}_9$	1	1	1	2	1	2	1	1	2	1	2	1	2	1	2	2	1	1	2	1	1
$O_{25};\delta_{\{\underline{5}\}}$	-1	-1	-2	-1	2	3	-1	-2	-1	2	3	-2	-1	2	3	-2	3	5	2	3	-5
$E_{28};\overline{G}_{12}$	1	1	2	1	1	1	1	2	1	1	1	2	1	1	1	2	2	1	1	1	2
$O_{26};\delta_{\{\underline{5},4\}}$	-1	-1	2	-1	2	3	-1	2	-1	2	3	2	-1	2	3	2	-3	-5	2	3	-5
$E_{29};\overline{G}_3$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	1	2

Table 1: non-hypermetric facets of CUT_7 and non-cut extreme rays of HYP_7

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	Inc.	Size
E_1	90	150	150	180	20	15	15	180	30	30	180	180	120	120	1460	7
E_2	80	130	80	220	20	60	0	180	40	10	240	100	320	10	1490	21
E_3	75	126	96	180	18	36	12	156	30	12	162	132	240	84	1359	35
E_4	13	7	0	0	0	0	0	0	0	0	0	0	0	0	20	2520
E_5	14	6	0	0	0	0	0	0	0	0	0	0	0	0	20	2520
E_6	13	7	0	0	0	0	0	0	0	0	0	0	0	0	20	2520
E_7	14	5	0	0	1	0	0	0	0	0	0	0	0	0	20	2520
E_8	15	4	0	0	1	0	0	0	0	0	0	0	0	0	20	1260
E_9	14	5	0	0	1	0	0	0	0	0	0	0	0	0	20	1260
E_{10}	15	5	0	0	0	0	0	0	0	0	0	0	0	0	20	252
E_{11}	11	7	1	1	0	0	0	0	0	0	0	0	0	0	20	2520
E_{12}	11	7	0	2	0	0	0	0	0	0	0	0	0	0	20	2520
E_{13}	12	6	2	0	0	0	0	0	0	0	0	0	0	0	20	2520
E_{14}	11	7	0	2	0	0	0	0	0	0	0	0	0	0	20	2520
E_{15}	12	7	0	1	0	0	0	0	0	0	0	0	0	0	20	1260
E_{16}	12	6	0	2	0	0	0	0	0	0	0	0	0	0	20	1260
E_{17}	12	6	2	0	0	0	0	0	0	0	0	0	0	0	20	630
E_{18}	10	6	0	2	1	0	0	1	0	0	0	0	0	0	20	2520
E_{19}	11	5	1	1	1	0	0	1	0	0	0	0	0	0	20	2520
E_{20}	10	6	0	2	1	1	0	0	0	0	0	0	0	0	20	1260
E_{21}	10	6	1	1	1	1	0	0	0	0	0	0	0	0	20	840
E_{22}	11	6	1	0	1	0	0	1	0	0	0	0	0	0	20	840
E_{23}	11	6	0	2	1	0	0	0	0	0	0	0	0	0	20	420
E_{24}	11	6	2	0	0	0	1	0	0	0	0	0	0	0	20	420
E_{25}	7	6	1	3	0	1	0	1	0	0	0	0	1	0	20	840
E_{26}	8	5	2	2	0	0	0	2	0	0	1	0	0	0	20	630
E_{27}	8	6	0	3	0	0	0	2	0	0	0	1	0	0	20	420
E_{28}	8	6	4	0	0	0	1	0	0	0	0	0	0	1	20	210
E_{29}	8	6	0	4	0	2	0	0	0	0	0	0	0	0	20	105
Size	105	210	210	420	35	105	21	420	105	42	630	420	840	210	3773	37170

Table 2: Incidence between extreme rays and orbits of facets of HYP_7

4 Adjacency properties of the skeleton of HYP_7 and of its dual

We start with Table 2 giving incidence between extreme rays and orbits of facets, i.e. on the place ij is the number of facets from the orbit F_j containing fixed extreme ray from the orbit E_i.

It turns out, curiously, that each of 19-dimensional hypermetric faces F_{11}-F_{14} of 21-dimensional cone CUT_7 (which became simplex facets in HYP_7) is the intersection of a triangle facet and one of cycle facets corresponding, respectively, to orbits O_{23}, O_{24}, O_{22}, O_{25} of table 1.

Proposition 4.1 *We have the following properties of adjacency of extreme rays:*

(i) The restriction of the skeleton of HYP_7 on the union of cut orbits $E_1 \cup E_2 \cup E_3$ is the complete graph.

(ii) Every non-cut extreme ray of HYP_7 has adjacency 20 (namely, it is adjacent to 20 cuts lying on corresponding non-hypermetric facet of CUT_7); see on Table 3 the distribution of those 20 cuts amongst the cut orbits.

(iii) Any two simplex extreme rays are non-adjacent; any simplex extreme ray (i.e. non-cut ray) has local graph (i.e. the restriction of the skeleton on the set of its neighbors) K_{20}.

(iv) The diameter of the skeleton of HYP_7 is 3.

Proposition 4.2 *We have the following properties of adjacency of facets of HYP_7:*

	E_1	E_2	E_3	Adj.	Size
E_1	6	21	35	15662	7
E_2	7	20	35	12532	21
E_3	7	21	34	10664	35
E_4	3	6	11	20	2520
E_5	4	7	9	20	2520
E_6	3	7	10	20	2520
E_7	3	7	10	20	2520
E_8	4	7	9	20	1260
E_9	3	8	9	20	1260
E_{10}	5	5	10	20	252
E_{11}	3	6	11	20	2520
E_{12}	2	8	10	20	2520
E_{13}	4	5	11	20	2520
E_{14}	2	8	10	20	2520
E_{15}	4	7	9	20	1260
E_{16}	3	9	8	20	1260
E_{17}	4	4	12	20	630
E_{18}	2	8	10	20	2520
E_{19}	3	7	10	20	2520
E_{20}	1	10	9	20	1260
E_{21}	2	9	9	20	840
E_{22}	4	6	10	20	840
E_{23}	4	7	9	20	420
E_{24}	5	1	14	20	420
E_{25}	1	9	10	20	840
E_{26}	2	8	10	20	630
E_{27}	3	6	11	20	420
E_{28}	5	1	14	20	210
E_{29}	2	12	6	20	105
Size	7	21	35		37170

Table 3: Adjacencies of extreme rays of HYP_7

(i) See Table 4, where on the place ij we have the number of facets from orbit F_j adjacent to fixed facet of orbit F_i.

(ii) Any two simplex facets are non-adjacent; any simplex facet (i.e. from F_9–F_{14}) have local graph K_{20}.

(iii) The diameter of the skeleton of dual HYP_7 is 3.

(iv) The symmetry of the skeleton of dual HYP_7 is the symmetric group $Sym(7)$.

5 Final remarks

In order to find extreme rays of HYP_8 the same methods will, probably, work with more computational difficulties but in dimension $n \geq 9$ polyhedral methods may fail.
The list of 374 orbits of non-cut extreme rays of HYP_8 (containing 7126560 extreme rays), found in [Du], will be confronted there with the list of at least 2169 orbits of facets of CUT_8, found in [ChRe98]. Exactly 55 of above 374 orbits corresponds to path-metrics of a graph. It was shown in [DG93] that any graph, such that its path-metric lies on an extreme ray of a HYP_n, is a subgraph of the skeleton of Gosset or Schafli polytopes.
It turns out (it can also be found in chapter 28 of [DeLa97]) that exactly 26 of those orbits consist of hypermetric inequalities; ten are 0-extensions of the hypermetric facets of CUT_7 and 16 come from the

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	Adj.	Size
F_1	86	168	110	216	35	56	13	196	14	6	54	36	64	18	1072	105
F_2	84	116	62	114	3	5	1	18	0	0	15	12	24	6	460	210
F_3	55	62	9	20	1	1	1	4	1	1	6	0	4	4	169	210
F_4	54	57	10	25	2	2	0	6	1	0	3	3	6	0	169	420
F_5	105	18	6	24	0	3	0	12	0	0	0	0	0	0	168	35
F_6	56	10	2	8	1	2	0	8	0	0	0	0	8	0	95	105
F_7	65	10	10	0	0	0	0	0	0	0	0	0	0	10	95	21
F_8	49	9	2	6	1	2	0	5	0	0	3	2	2	0	81	420
F_9	14	0	2	4	0	0	0	0	0	0	0	0	0	0	20	105
F_{10}	15	0	5	0	0	0	0	0	0	0	0	0	0	0	20	42
F_{11}	9	5	2	2	0	0	0	2	0	0	0	0	0	0	20	630
F_{12}	9	6	0	3	0	0	0	2	0	0	0	0	0	0	20	420
F_{13}	8	6	1	3	0	1	0	1	0	0	0	0	0	0	20	840
F_{14}	9	6	4	0	0	0	1	0	0	0	0	0	0	0	20	210
Size	105	210	210	420	35	105	21	420	105	42	630	420	840	210		3773

Table 4: Adjacency of facets of HYP_7

following vectors b (see (1)):

$$(2,1,1,1,-1,-1,-1,-1), \quad (3,1,1,1,-1,-1,-1,-2), \quad (2,2,1,1,-1,-1,-1,-2),$$
$$(4,1,1,-1,-1,-1,-1,-1), \quad (3,2,2,-1,-1,-1,-1,-2),$$

representing, respectively, switching classes of sizes 2, 4, 3, 2, 5.

There is one to one correspondance between non-hypermetric facets of CUT_7 and non-cut extreme rays of HYP_7; namely every such facet is violated by exactly one such ray. Apropos, there is one to one correspondance between the ten non-cut extreme rays(in fact, path metric $K_{2\times 3}$) of MET_5 and the ten non-triangle facets (in fact, pentagonal) of CUT_5; namely every non-cut extreme ray is violated by a non-triangle facet. There is no such thing between MET_n and CUT_n for $n > 5$ but we hope that the correspondance exist for CUT_8 and HYP_8.

Another direction for further study is to find *all* faces of HYP_7. While the extreme rays of HYP_n yields the extreme Delaunay polytopes of dimension $n-1$, the study of all faces of HYP_n will provide the list of all (combinatorial types of) Delaunay polytopes of dimension less or equal $n-1$. See [BK00] for description of the method and computations for $n \leq 4$; in fact, Kononenko (submitted) found all Delaunay polytopes of dimension five.

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