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**The Bridge Crossing
Problem**

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Calude: Draft Form**

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The Bridge Crossing Problem: Draft Form

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Abstract

In this note we solve the general case of the Bridge Crossing Problem.

1 The Problem

The following problem is known as the *Bridge Crossing Problem*.¹

Four people want to cross a bridge. They all begin on the same side. It is night, and they have only one flashlight with them. At most two persons can cross the bridge at a time, and any party who crosses, either one or two people, must have the flashlight with them. The flashlight must be walked back and forth: it cannot be thrown, etc. Each person walks at a different speed. A pair must walk together at the speed of the slower person. Person A needs 1 minute to cross the bridge, person B needs 2 minutes, person C needs 5 minutes, and person D needs 10 minutes.

Question: How can all four persons cross the bridge in 17 minutes?

2 The Classical Solution

First, person A and person B walk across the bridge. This takes 2 minutes. After this, person A walks back with the flashlight. This takes 1 minute. Then C and D walk across the bridge. This takes 10 minutes. After this, person B walks back with the flashlight. This takes 2 minutes. Finally, A and B walk across the bridge. This takes 2 minutes as before. In total: $2 + 1 + 10 + 2 + 2 = 17$ minutes.

3 Other “Solutions”

The origin of the problem is apparently unknown.² The problem has attracted quite a bit of interest and various “solutions” have been proposed. For example, Karen L. Lingel has posted to *Newsgroups: rec.puzzles* (on 17 Sep 1997) the following “rhyme solution” titled *Four Hungry Men Cross a Bridge*:

Four men start out to cross the sea
And yet they all walk different speeds!
The first, a sprinter, he goes fast

¹Apparently, it has been proposed as a test to candidates applying for a job with Microsoft. See, for example, <http://www.dse.nl/puzzle/math/index.html#thebridge>.

²It has been cited in S. X. Levmore, E. E. Cook. *Super Strategies For Puzzles and Games*, Doubleday & Company, Inc, Garden City, New York 1981, p. 3. Hemme Heinrich, who was interested in its origin, is cited to say (July 1998) that “Martin Gardner didn’t know” [its origin]; cf. <http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/crossing-bridge>.

He leaves the others in the past!
 The second takes a bit more time
 [Note to myself: think up a rhyme]
 The third's a somewhat pokey man
 He strolls along, sees what he can.
 The last one is so very slow
 You'd think he had no place to go!

So now they come upon a bridge
 And on the other side – a fridge!
 Well – you know men – they've gotta see
 What's inside the fridge to eat!
 One flashlight is the light they've got
 To guide them to the eating spot.
 The batteries will only last
 Twenty minutes – that's a fact.
 The bridge, alas, – and here's the trap –
 Is apparently a piece of crap.

So only two men at a time
 can cross the bridge – or they'll sink in brine!
 How can they all then make the trip?
 And use the light so no one slips?
 Send the fast guys first across
 The fastest returns with little loss.
 The pokey ones are next to go
 While Fast Guy waits (they sure are slow)
 Then send the other fast guy back
 To get his friend and complete the pack.

4 A Mathematical Analysis

A few questions naturally arise: a) is the classical solution unique?, b) if all men wish to cross the bridge as quickly as possible, what is the minimal time necessary for all of them to cross? c) is the total time recorded by the classical solution minimal?, d) what is the “idea” behind the classical solution? e) is it general? f) what about the case when $N \geq 4$ wish to cross the bridge? In what follows we will respond to these questions.

First, let us review the problem. Four persons have to cross a bridge. The times each needs to cross it are: 1, 2, 5, 10 minutes, respectively. It is dark and they only have one flashlight. The following constraints apply:

- the bridge can only be crossed if one has a flashlight;
- two persons at most are allowed to be on the bridge at the same time;
- the speed of a pair is determined by the speed of the slower person;
- the flashlight cannot be thrown.

A *solution* of the problem is a procedure which respects the above constraints and can be used by the four persons to cross the bridge. Of course, people wish to cross the bridge as quickly as possible; this leads to the notion of *minimal solution*, which is a solution for which the sum of crossing times is minimal.

It is convenient to adopt the following notation. We write $\xrightarrow{A,B}$ for “the men A and B walk across the bridge” and \xleftarrow{A} for “the man A walks back with the flashlight”. The above (classical) solution can now be presented as follows:

Left side	Move	Right side	Time
A, B, C, D			
C, D	$\xrightarrow{A, B}$	A, B	2 minutes
A, C, D	\xleftarrow{A}	B	1 minute
A	$\xrightarrow{C, D}$	B, C, D	10 minutes
A, B	\xleftarrow{B}	C, D	2 minutes
	$\xrightarrow{A, B}$	A, B, C, D	2 minutes
			Total: 17 minutes

Classical solution.

The answer to the first question is *negative*. Indeed, the same result can be obtained using the following slightly different procedure:

Left side	Move	Right side	Time
A, B, C, D			
C, D	$\xrightarrow{A, B}$	A, B	2 minutes
B, C, D	\xleftarrow{B}	A	2 minutes
B	$\xrightarrow{C, D}$	A, C, D	10 minutes
A, B	\xleftarrow{A}	C, D	1 minute
	$\xrightarrow{A, B}$	A, B, C, D	2 minutes
			Total: 17 minutes

Classical solution: second variant.

We continue by observing that the total time recorded by the classical solution *does not depend* upon the time required by C to cross the bridge!³ So, if the individual times are 1, 2, 5, 10 or 1, 2, 4, 10 or 1, 2, 9, 10, then the classical solution will require always 17 minutes. What about the situation when the times are 1, 4, 5, 10? The classical solution produces the total time $4 + 1 + 10 + 4 + 4 = 23$ minutes, which is *not minimal* because it is beaten by the following solution:

Left side	Move	Right side	Time
A, B, C, D			
B, C	$\xrightarrow{A, D}$	A, D	10 minutes
A, B, C	\xleftarrow{A}	D	1 minute
C	$\xrightarrow{A, B}$	A, B, D	4 minutes
A, C	\xleftarrow{A}	B, D	1 minute
	$\xrightarrow{A, C}$	A, B, C, D	5 minutes
			Total: 21 minutes

A different solution.

To present our argument we will consider a slightly more general setting, namely we still have four men A, B, C, D , but the times needed to cross the bridge are $a < b < c < d$, respectively. The classical solution leads to the time $a + 3b + d$. It's main idea is to "offset" the time required by C by "shadowing" it with the longer time needed by D . This idea produces the minimal time in case $2b < a + c$:

³Unless larger than 10.

Left side	Move	Right side	Time
A, B, C, D			
C, D	$\xrightarrow{A, B}$	A, B	b minutes
A, C, D	\xleftarrow{A}	B	a minutes
A	$\xrightarrow{C, D}$	B, C, D	d minutes
A, B	\xleftarrow{B}	C, D	b minutes
	$\xrightarrow{A, B}$	A, B, C, D	b minutes
			Total: $a + 3b + d$ minutes

Minimal solution when $2b < a + c$.

However, it fails when $2b > a + c$, when the minimum is $2a + b + c + d$ and can be obtained by the following solution:⁴

Left side	Move	Right side	Time
A, B, C, D			
B, C	$\xrightarrow{A, D}$	A, D	d minutes
A, B, C	\xleftarrow{A}	D	a minutes
C	$\xrightarrow{A, B}$	A, B, D	b minutes
A, C	\xleftarrow{A}	B, D	a minutes
	$\xrightarrow{A, C}$	A, B, C, D	c minutes
			Total: $2a + b + c + d$ minutes

Minimal solution when $2b > a + c$.

The argument supporting the above statements is not difficult to present. To obtain a solution we need to have three “double” crossings and two “crossings back”. The “theoretical minimum” is thus attained for the time $2a + 2b + d$ (because d cannot be offset, a is the minimum time and could be realised at most twice, and the next minimal time is b). The “theoretical minimum” time *cannot* be realised by any solution! Indeed, let us assume that the moves are \xleftarrow{A} (two times), $\xrightarrow{A, \alpha}$, $\xrightarrow{A, \beta}$, $\xrightarrow{\gamma, D}$, where $\alpha > a, \beta > a$ and $\gamma < d$. It is impossible to find $\alpha, \beta \in \{a, b, c, d\}$ such that $2a + \alpha + \beta + d = 2a + 2b + d$.

Hence, we need to relax the “theoretical minimum” $2a + 2b + d$ to get minimal solutions. Various possibilities occur:

- replacing a by b we get $a + 3b + d$, which is minimum only if it is smaller than $2a + b + c + d$, i.e., when $2b < a + c$;
- replacing b by c we get $2a + b + c + d$, which is minimum only if it is smaller than $a + 3b + d$, i.e., when $2b \geq a + c$;
- replacing a by c we get $a + 2b + c + d$ which is not minimum because it is greater than $a + 3b + d$,
- replacing a by d we get $a + 2b + 2d$ which is not minimum because it is greater than $a + 3b + d$,
- replacing b by d we get $2a + b + 2d$ which is not minimum because it is greater than $2a + b + c + d$,
- replacing (a by b and b by c) or (a by c and b by c) or (a by b and b by d) or (a by c and b by d) or (a by d and b by d) leads to a total time which is greater than $a + 3b + d$.

⁴If $2b = a + c$, then $2a + b + c + d = a + 3b + d$.

To conclude, the *minimum total time is*: $a + b + d + \min\{2b, a + c\}$. In the original problem, $\min\{2b, a + c\} = \min\{2 \cdot 2, 1 + 5\} = 4$, so the minimum is $1 + 2 + 10 + 4 = 17$.

Note that the “obvious solution”, in which the fastest person goes back and forth with all other people, *is minimal* in case $2b \geq a + c$, but it is not minimal in the other case.

Left side	Move	Right side	Time
A, B, C, D			
C, D	$\xrightarrow{A, B}$	A, B	b minutes
A, C, D	\xleftarrow{A}	B	a minutes
D	$\xrightarrow{A, C}$	A, B, C	c minutes
A, D	\xleftarrow{A}	B, C	a minutes
	$\xrightarrow{A, D}$	A, B, C, D	d minutes
			Total: $2a + b + c + d$ minutes

The “obvious solution” is minimal when $2b \geq a + c$.

5 A Computational Analysis

More insight into the Crossing Bridge Problem can be obtained by computer simulation. One possible way is to represent any possible move by a suitable Boolean matrix.

$$\begin{array}{c}
 \xrightarrow{A, B} \\
 \xleftarrow{A} \\
 \xrightarrow{C, D} \\
 \xleftarrow{B} \\
 \xrightarrow{A, B}
 \end{array}
 \begin{bmatrix}
 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 1 & 1 & 1 \\
 0 & 0 & 1 & 1 \\
 1 & 1 & 1 & 1
 \end{bmatrix}$$

Boolean codification of a solution.

On the position (i, j) we put a 1 if after the i th crossing the person j has crossed the bridge, and 0, otherwise. For example, the classical solution can be represented by the following 6×4 Boolean matrix. The first and last rows are always the same, so the 4×4 Boolean sub-matrix which is written in bold actually completely describes the classical solution.

If we consider the generalization of the Bridge Crossing Problem where N persons with different speeds wish to cross the bridge (all rules remain unchanged), then any solution needs $2N - 3$ crossings ($N - 1$ double and $N - 2$ single), so a solution is completely determined by a $(2N - 5) \times N$ Boolean matrix. Of course, not any Boolean matrix corresponds to a solution of the Bridge Crossing Problem with N people (not necessarily minimal). In a matrix corresponding to a solution every two adjacent rows are the same, except for one/two positions of 1s which depend upon the parities of indices of rows.⁵ For example, for $N = 4$, there are 65,536 4×4 Boolean matrices, but only 108 correspond to solutions (0.164%). For $N = 5$, there are 1,073,741,824 6×5 Boolean matrices and only 4,320 correspond to solutions (0.000402%). The results reported here have been obtained with a program written in C. The program uses the following C subroutine for checking whether a given $(2N - 5) \times N$ Boolean matrix corresponds to a solution of the Bridge Crossing Problem with N people.

⁵This is only a necessary condition; a characterization of matrices corresponding to solutions will be given later in the form of a C program.

```

/*
The function checks whether X is a matrix corresponding to
a solution of the Bridge Crossing Problem with N persons.
*/
int XOK(int X[M][N]){
    int j, i, sum1 = 0, sum2;
    for (j = 0; j < N; j++){
        sum1 += X[0][j];
    }
    if (sum1 != 2){//first line is not OK
        return 0;
    }
    for (j = N-1; j > 0; j--){
        if (X[0][j] == 1) {
            break;
        }
    }
    for (i=1; i < 2*N-4; i++){
        if (i%2 == 0){ //two persons cross the bridge
            sum1 = sum2 = 0;
            for(j = 0; j < N; j++){
                sum2 += X[i][j];
                sum1 += X[i-1][j];
            }
            if (sum2-sum1 != 2){//line i is not OK
                return 0 ;
            }
            for (j=0; j < N; j++){
                if (X[i-1][j]==1 && X[i][j] ==0){//line i is not OK
                    return 0;
                }
            }
        }
        if (i%2 == 1){ //one person goes back
            sum1 = sum2 = 0;
            for( j= 0; j < N; j++){
                sum2 += X[i][j];
                sum1 += X[i-1][j];
            }
            if (sum2-sum1 != -1){//line i is not OK
                return 0;
            }
            for (j = 0; j < N; j++){
                if (X[i][j] == 1 && X[i-1][j] == 0 ){//line i is not OK
                    return 0;
                }
            }
        }
    }
    sum1 = 0;
    for(j = 0; j < N; j++){
        sum1 += X[2*N-5][j];
    }
    if (sum1 != N-2){//last significant line is not OK
        return 0;
    }
    return 1;
}

```

}

An exhaustive analysis of all possible 108 solutions of the original problem (i.e. when the crossing times are 1, 2, 5, 10) shows that there are *only two* minimal solutions, namely those presented in the previous section. So, minimality is rare, it can be found only in 1.85% of all solutions.⁶ The average crossing time required is 30, which can be realised by 6 different solutions. The following matrix is an example:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

The worst solutions score time 50. There are 6 such solutions, and here is an example:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

A different picture is offered by the problem in which we have 4 people, but the times are 1, 4, 5, 10. As we have shown in Section 4, the optimum is realised for the time 21. Actually, there are 6 different minimal solutions:

1	2	3	4	5	6
1001	1001	1010	1010	1100	1100
0001	0001	0010	0010	0100	0100
1011	1101	1011	1110	1101	1110
0011	0101	0011	0110	0101	0110

Minimal solutions for 1, 4, 5, 10.

The average crossing time is in this case 32 (but it is not realised by any solution). Finally, there are again 6 solutions scoring the longest time, 50, among them the solution which scores 50 in the classical case. The reader can easily verify that our condition of minimality (see the end of Section 4) is consistent with the results of the reported experiments.

The above analysis shows that for $N = 4$ there are 8 minimal solutions (7.40%) given by the following matrices:

1	2	3	4	5	6	7	8
1100	1100	1001	1001	1010	1010	1100	1100
0100	1000	0001	0001	0010	0010	0100	0100
0111	1011	1011	1101	1011	1110	1101	1110
0011	1011	0011	0101	0011	0110	0101	0110

All minimal solutions for $N = 4$.

The first two matrices correspond to the condition $2b < a + c$, while the last six correspond to $2b > a + c$.

Out of 4,320 solutions for $N = 5$, 32 are minimal (0.74%), 24 for the case $2b > a + d$ (the “obvious solution” belongs to this class), and 8 for $2b < a + d$.

⁶Which may explain the difficulty of the problem noted by various authors.

6 The General Case

In case $N = 3$, the minimal crossing time is obtained by summing up all three individual crossing times. Assume that we have a group of $N \geq 4$ people $A[1], A[2], \dots, A[N]$ trying to cross the bridge and the individual crossing times are respectively $t[1] < t[2] < \dots < t[N]$. Every solution has to contain $2N - 3$ crossings, $N - 2$ single and $N - 1$ double. From a “theoretical” point of view the minimal time for $N - 2$ single crossings is $(N - 2)t[1]$ and the minimal time for $N - 1$ double crossings is $(N - 1)t[2]$. However, we need to move $A[N]$ which requires a crossing time $t[N]$, therefore the “theoretical minimum” is $(N - 2)t[1] + (N - 2)t[2] + t[N]$. As in the case with four people, the “theoretical minimum” cannot be scored by any solution. Indeed, to be able to realise all single crossings with the minimum cost $t[1]$ all double crossings have to be carried on by $A[1]$ and $A[2]$, which means that at the end only three people cross the bridge ($A[2], A[N]$ and somebody else).

To achieve minimality we notice that all single crossings should take at most time $t[2]$. One way to obtain times corresponding to solutions is to increase repeatedly the “theoretical minimum” by replacing the time $t[2]$ with longer crossing times, e.g. $t[3], t[4]$, a.s.o., till we get a total crossing time corresponding to a solution. The second way is to change first one single $t[1]$ into $t[2]$ and then continue by replacing $t[2]$ with longer crossing times till we get a total crossing time corresponding to a solution.

In the first case the total times obtained are

$$\begin{aligned} & (N - 2)t[1] + (N - 3)t[2] + t[3] + t[N], \\ & (N - 2)t[1] + (N - 4)t[2] + t[3] + t[4] + t[N], \\ & \vdots \\ & (N - 2)t[1] + 2t[2] + t[3] + \dots + t[N - 2] + t[N] \end{aligned}$$

which do not correspond to solutions, but if we continue the process one step further we get

$$(N - 2)t[1] + t[2] + t[3] + \dots + t[N - 1] + t[N]$$

which is the first crossing time of a solution.

In the second case the total times obtained

$$\begin{aligned} & (N - 3)t[1] + (N - 1)t[2] + t[N], \\ & (N - 3)t[1] + (N - 2)t[2] + t[3] + t[N], \\ & \vdots \\ & (N - 3)t[1] + 4t[2] + t[3] + \dots + t[N - 3] + t[N] \end{aligned}$$

do not correspond to solutions, but the next iteration

$$(N - 3)t[1] + 3t[2] + t[3] + \dots + t[N - 2] + t[N]$$

corresponds to a solution.

To find examples for solutions realising the above times we first start with the following common part:

Left side	Move	Right side	Time
$A[1], \dots, A[N]$			
$A[3], \dots, A[N]$	$\xrightarrow{A[1], A[2]}$	$A[1], A[2]$	$t[2]$ minutes
$A[1], A[3], \dots, A[N]$	$\xleftarrow{A[1]}$	$A[2]$	$t[1]$ minutes
$A[4], \dots, A[N]$	$\xrightarrow{A[1], A[3]}$	$A[1], A[2], A[3]$	$t[3]$ minutes
$A[1], A[4], \dots, A[N]$	$\xleftarrow{A[1]}$	$A[2], A[3]$	$t[1]$ minutes
\vdots	\vdots	\vdots	\vdots
$A[N - 1], A[N]$	$\xrightarrow{A[1], A[N - 2]}$	$A[1], \dots, A[N - 2]$	$t[N - 2]$ minutes
$A[1], A[N - 1], A[N]$	$\xleftarrow{A[1]}$	$A[2], \dots, A[N - 2]$	$t[1]$ minutes
			Total: $(N - 3)t[1] + t[2] + \dots + t[N - 2]$ minutes

Common part of a minimal solution for $N \geq 4$.

and then continue with

Left side	Move	Right side	Time
$A[N]$	$\xrightarrow{A[1], A[N-1]}$	$A[1], \dots, A[N-1]$	$t[N-1]$ minutes
$A[1], A[N]$	$\xleftarrow{A[1]}$	$A[2], \dots, A[N-1]$	$t[1]$ minutes
	$\xrightarrow{A[1], A[N]}$	$A[1], \dots, A[N]$	$t[N]$ minutes
			Total: $t[1] + t[N-1] + t[N]$ minutes

Final part of a minimal solution when $t[1] + t[N-1] < 2t[2]$.

or

Left side	Move	Right side	Time
$A[1]$	$\xrightarrow{A[N-1], A[N]}$	$A[2], \dots, A[N]$	$t[N]$ minutes
$A[1], A[2]$	$\xleftarrow{A[2]}$	$A[3], \dots, A[N]$	$t[2]$ minutes
	$\xrightarrow{A[1], A[2]}$	$A[1], \dots, A[N]$	$t[2]$ minutes
			Total: $2t[2] + t[N]$ minutes

Final part of a minimal solution when $t[1] + t[N-1] \geq 2t[2]$.

If we further increase the total time, then we loose minimality. Hence, the *minimum total time is*:

$$(N-3)t[1] + t[2] + \dots + t[N-2] + \min\{2t[2], t[1] + t[N-1]\} + t[N].$$

All minimal solutions can be obtained from the above solutions by “acceptable permutations” between crossings. For example, if $t[1] + t[N-1] < 2t[2]$ there are exactly $(N-1)!$ different minimal solutions.

7 Generalizations

The problem treated in this note can be generalised in various ways. For example, we can consider the Bridge Crossing Problem where N persons with different speeds wish to cross the bridge as quickly as possible, and up to K men are allowed to cross simultaneously the bridge (all other rules remain the same).

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