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Randomness and Reducibility

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1 Introduction. How random is a real? Given two reals, which is more random? If we partition reals into equivalence classes of reals of the "same degrees of randomness", what does the resulting structure look like? The goal of this paper is to look at questions like these, specifically by studying the properties of reducibilities that act as measures of relative randomness. One such reducibility, called domination or Solovay reducibility, was introduced by Solovay [34], and has been studied by Calude, Hertling, Khoussainov, and Wang [8], Calude [3], Slaman [29], and Downey, Hirschfeldt, and Nies [15], among others. Solovay reducibility has proved to be a powerful tool in the study of randomness of effectively presented reals. Motivated by certain shortcomings of Solovay reducibility, which we will discuss below, we introduce two new reducibilities and study, among other things, the relationships between these various measures of relative randomness.

We work with reals between 0 and 1, identifying a real with its binary expansion, and hence with the set of natural numbers whose characteristic function is the same as that expansion. (We also identify finite binary strings with rationals.) Our main concern will be reals that are limits of computable increasing sequences of rationals. We call such reals *computably enumerable* (c.e.), though they have also been called *recursively enumerable*, *left computable* (by Ambos-Spies, Weihrauch, and Zheng [2]), and *left semicomputable*. If, in addition to the existence of a computable increasing sequence $q_0, q_1, \ldots$ of rationals with limit α, there is a total computable function f such that $\alpha - q_{f(n)} < 2^{-n}$ for all n, then α is called *computable*. These and related concepts have been widely studied. In addition to the papers and books mentioned elsewhere in this introduction, we may cite, among others, early work of Rice [27], Lachlan [22], Soare [30], and Ceĭtin [10], and more recent papers by Ko [19, 20], Calude, Coles, Hertling, and Khoussainov [7], Ho [18], and Downey and LaForte [16].

An alternate definition of c.e. reals can be given by using the following definition.

1.1. Definition. A set $A \subseteq \mathbb{N}$ is *nearly computably enumerable* if there is a computable approximation $\{A_s\}_{s\in\omega}$ such that $A(x) = \lim_s A_s(x)$ for all x and $A_s(x) > A_{s+1}(x) \Rightarrow \exists y < x(A_s(y) < A_{s+1}(y))$.

As shown by Calude, Coles, Hertling, and Khoussainov [7], a real $0.\chi_A$ is c.e. if and only if A is nearly c.e.. An interesting subclass of the class of c.e. reals is the class of strongly c.e. reals. A real $0.\chi_A$ is said to be *strongly c.e.* if A is c.e.. Soare [31] noted that there are c.e. reals that are not strongly c.e..

A computer M is *self-delimiting* if, for all finite binary strings σ and $\tau \subsetneq \tau'$, we have $M^\sigma(\tau)\downarrow \Rightarrow M^\sigma(\tau')\uparrow$, where M^σ means that M uses σ as an oracle. It is *universal* if for each self-delimiting computer N there is a constant c such that, for all binary strings σ and τ, if $N^\sigma(\tau)\downarrow$ then $M^\sigma(\mu)\downarrow = N^\sigma(\tau)$ for some μ with $|\mu| \leqslant |\tau| + c$. We call c the *coding constant* of N.

Fix a self-delimiting universal computer M. We can define Chaitin's number $\Omega = \Omega_M$ via $\Omega = \sum_{M(\sigma)\downarrow} 2^{-|\sigma|}$, which is the *halting probability* of the computer M. The properties of Ω relevant to this paper are independent of the choice of M. A c.e. real is an Ω-*number* if it is Ω_M for some self-delimiting universal computer M.

The c.e. real Ω is random in the canonical Martin-Löf sense [26] of c.e. randomness. There are many equivalent formulations of c.e. randomness. The one that is most relevant to us here is Chaitin randomness, which we define below. The history of effective randomness is quite rich; references include van Lambalgen [36], Calude [4], Li and Vitanyi [25], and Ambos-Spies and Kučera [1].

Recall that the prefix-free complexity $H(\tau)$ of a binary string τ is the length of the shortest binary string σ such that $M(\sigma)\downarrow= \tau$. (The choice of self-delimiting universal computer M does not affect the prefix-free complexity, up to a constant additive factor.) Most of the statements about $H(\tau)$ made below also hold for the standard Kolmogorov complexity $K(\tau)$. For more on the the definitions and basic properties of $H(\tau)$ and $K(\tau)$, see Chaitin [14], Calude [4], Li and Vitanyi [25], and Fortnow [17]. Among the many works dealing with these and related topics, and in addition to those mentioned elsewhere in this paper, we may cite Solomonoff [32, 33], Kolmogorov [21], Levin [23, 24], Schnorr [28], Chaitin [11], and the expository article Calude and Chaitin [5].

A real α is *random*, or more precisely, 1-random, if there is a constant c such that $\forall n(H(\alpha \upharpoonright n) \geqslant n - c)$.

Many authors have studied Ω and its properties, notably Chaitin [12, 13, 14] and Martin-Löf [26]. In the very long and widely circulated manuscript [34] (a fragment of which appeared in [35]), Solovay carefully investigated relationships between Martin-Löf-Chaitin prefix-free complexity, Kolmogorov complexity, and properties of random languages and reals. See Chaitin [12] for an account of some of the results in this manuscript.

Solovay discovered that several important properties of Ω (whose definition is model-dependent) are shared by another class of reals he called Ω-like, whose definition is model-independent. To define this class, he introduced the following reducibility relation between c.e. reals.

1.2. Definition. Let α and β be c.e. reals. We say that α *dominates* β and that β is *Solovay reducible* (S-reducible) to α, and write $\beta \leqslant_{\mathrm{S}} \alpha$, if there are a constant c and a partial computable function $\varphi : \mathbb{Q} \to \mathbb{Q}$ such that for each rational $q < \alpha$ we have $\varphi(q)\downarrow< \beta$ and $\beta - \varphi(q) \leqslant c(\alpha - q)$. We write $\alpha \equiv_{\mathrm{S}} \beta$ if $\alpha \leqslant_{\mathrm{S}} \beta$ and $\beta \leqslant_{\mathrm{S}} \alpha$.

Solovay reducibility is reflexive and transitive, and hence $\equiv_{\mathrm{S}}$ is an equivalence relation on the c.e. reals. Thus we can define the *Solovay degree* $\deg_{\mathrm{S}}(\alpha)$ of a c.e. real α to be its $\equiv_{\mathrm{S}}$ equivalence class.

Solovay reducibility is naturally associated with randomness due to the following fact.

1.3. Theorem (Solovay [34]). *Let $\beta \leqslant_{S} \alpha$ be c.e. reals. There is a constant $O(1)$ such that $H(\beta \upharpoonright n) \leqslant H(\alpha \upharpoonright n) + O(1)$ for all n.*

It is this property of Solovay reducibility (which we will call the *Solovay property*), which makes it a measure of relative randomness. This is in contrast with Turing reducibility, for example, which does not have the Solovay property, since the complete c.e. Turing degree contains both random and nonrandom reals.

Solovay observed that Ω dominates all c.e. reals, and Theorem 1.3 implies that if a c.e. real dominates all c.e. reals then it must be random. This led him to define a c.e. real to be *Ω-like* if it dominates all c.e. reals (that is, if it is S-complete). The point is that the definition of Ω-like seems quite model-independent (in the sense that it does not require a choice of self-delimiting universal computer), as opposed to the model-dependent definition of Ω. However, Calude, Hertling, Khoussainov, and Wang [8] showed that the two notions coincide. This circle of ideas was completed recently by Slaman [29], who showed that all random c.e. reals are Ω-like.

For more on c.e. reals and S-reducibility, see for instance Chaitin [12, 13, 14], Calude, Hertling, Khoussainov, and Wang [8], Calude and Nies [9], Calude [3], Slaman [29], and Downey, Hirschfeldt, and Nies [15].

Solovay reducibility is an excellent tool in the study of the relative randomness of reals, but it has several shortcomings. One such shortcoming is that S-reducibility is quite unnatural outside the c.e. reals. It is not very hard to construct a noncomputable real that is not S-above the computable reals (in fact, this real can be chosen to be d.c.e., that is, of the form $\alpha - \beta$ where α and β are c.e.). This and similar facts show that S-reducibility is very unnatural when applied to non-c.e. reals. Another problem with S-reducibility is that it is uniform in a way that relative initial-segment complexity is not. This makes it too strong, in a sense, and appears to preclude its having a natural characterization in terms of initial-segment complexity. In particular, Calude and Coles [6] answered a question of Solovay by showing that the converse of Theorem 1.3 does not hold (see below for an easy proof of this fact). Thus, if our goal is to study relative initial segment complexity of reals, it behooves us to look beyond S-reducibility.

In this paper, we introduce two new measures of relative randomness that provide additional tools for the study of the relative randomness of reals, and study their properties and the relationships between them and S-reducibility. We begin with sw-reducibility, which has some nice features but also some shortcomings, and then move on to the more interesting rH-reducibility, which shares many of the best features of S-reducibility while not being restricted to the c.e. reals, and has a very nice characterization in terms of relative initial-segment complexity, which can be seen as a partial converse of the Solovay property. (Indeed, rH stands for "relative H".)

2. Strong Weak Truth Table Reducibility. Solovay reducibility has many attractive features, but it is not the only interesting measure of relative randomness. In this section, we introduce another such measure, sw-reducibility, which is more explicitly derived from the idea of initial segment complexity, and which is in some ways nicer than S-reducibility. In particular, sw-reducibility is much better adapted to dealing with non-c.e. reals.

Despite its several engaging aspects, we will see that sw-reducibility also has its problems, in particular the lack of a join operation, even on the c.e. reals. This will lead us to study rH-reducibility, a common weakening of S-reducibility and sw-reducibility, in the next section.

Recall that a Turing reduction $\Gamma^A = B$ is called a weak truth table (wtt) reduction if there is a computable function φ such that the use function $\gamma(x)$ is bounded by $\varphi(x)$.

2.4. Definition. Let $A, B \subseteq \mathbb{N}$. We say that B is *strongly weak truth table reducible* (sw-reducible) to A, and write $B \leqslant_{\text{sw}} A$, if there is a constant c and a wtt reduction Γ such that $B = \Gamma^A$ and $\forall x(\gamma(x) \leqslant x + c)$. For reals $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$, we say that β is sw-reducible to α, and write $\beta \leqslant_{\text{sw}} \alpha$, if $B \leqslant_{\text{sw}} A$.

Since sw-reducibility is reflexive and transitive, we can define the *sw-degree* $\deg_{\text{sw}}(\alpha)$ of a real α to be its sw-equivalence class.

Solovay [34] noted that for each c there is a constant $O(1)$ such that for all $n \geqslant 1$ and all binary strings σ, τ of length n, if $|0.\sigma - 0.\tau| < c2^{-n}$ then $|H(\tau) - H(\sigma)| \leqslant O(1)$. Using this result, it is easy to check that sw-reducibility has the Solovay property.

2.5. Theorem. *Let $\beta \leqslant_{sw} \alpha$ be c.e. reals. There is a constant $O(1)$ such that $H(\beta \upharpoonright n) \leqslant H(\alpha \upharpoonright n) + O(1)$ for all $n \in \omega$.*

Theorem 2.9 below shows that the converse of Theorem 2.5 does not hold even for c.e. reals. But, in fact, this converse is not that far from holding.

2.6. Theorem. *Let α and β be c.e. reals such that $\liminf_n H(\alpha \upharpoonright n) - H(\beta \upharpoonright n) = \infty$. Then $\beta <_{sw} \alpha$.*

Proof. Let $c_\alpha(n)$ be the least s such that $\alpha_s \upharpoonright n = \alpha \upharpoonright n$, and define $c_\beta(n)$ analogously. Let M be a universal self-delimiting computer and define the self-delimiting computer N as follows. For each n, s, and σ, if $M(\sigma)[s]\downarrow = \beta_s \upharpoonright n$ and $N(\sigma)$ has not been defined before stage s then let $N(\sigma)\downarrow = \alpha_s \upharpoonright n$. Let e be the coding constant of N. For each n, if $c_\beta(n) \geqslant c_\alpha(n)$ then $\forall\sigma(M(\sigma)\downarrow = \beta \upharpoonright n \Rightarrow N(\sigma)\downarrow = \alpha \upharpoonright n)$, which implies that $H(\alpha \upharpoonright n) \leqslant H(\beta \upharpoonright n) + e$. Thus our hypothesis implies that $c_\beta(n) < c_\alpha(n)$ for almost all n, which clearly implies that $\beta \leqslant_{\text{sw}} \alpha$. By Theorem 2.5, $\alpha \nleqslant_{\text{sw}} \beta$, so $\beta <_{\text{sw}} \alpha$ $\square$

We now explore the relationship between S-reducibility and sw-reducibility on the c.e. and strongly c.e. reals. We begin by noting the following lemma, implicit in Solovay [34].

2.7. Lemma. *Let α and β be c.e. reals, and let $\alpha_0, \alpha_1, \ldots$ and $\beta_0, \beta_1, \ldots$ be computable increasing sequences of rationals converging to α and β, respectively. Then $\alpha \leqslant_S \beta$ if and only if there are a constant c and a total computable function f such that for all $n \in \omega$ we have $\alpha - \alpha_{f(n)} \leqslant c(\beta - \beta_n)$.*

Proof. First suppose that $\alpha \leqslant_{\text{S}} \beta$ and let c and φ be as in Definition 1.2. For each n let $f(n)$ be the least s such that $\alpha_s \geqslant \varphi(\beta_n)$. Then $\alpha - \alpha_{f(n)} \leqslant \alpha - \varphi(\beta_n) \leqslant c(\beta - \beta_s)$.

For the converse, suppose that c and f are as above. For each rational q, if there is a stage s_q such that $\beta_{s_q} \geqslant q$ then let $\varphi(q) = \alpha_{f(s_q)}$, and otherwise let $\varphi(q)\uparrow$. Then φ is defined on all rationals less than β, and for any such rational q we have $\alpha - \varphi(q) = \alpha - \alpha_{f(s_q)} \leqslant c(\beta - \beta_{s_q}) \leqslant c(\beta - q)$. Thus $\alpha \leqslant_{\text{S}} \beta$. $\square$

Whenever we mention a c.e. real α, we assume that we have chosen a computable increasing sequence $\alpha_0, \alpha_1, \ldots$ converging to α. The previous lemma guarantees that, in determining whether one c.e. real dominates another, the particular choice of such sequences is irrelevant.

In general, neither of the reducibilities under consideration implies the other.

2.8. Theorem. *There exist c.e. reals $\alpha \leqslant_{sw} \beta$ such that $\alpha \nleqslant_S \beta$. Moreover, α can be chosen to be strongly c.e..*

Proof. We must meet the following requirements.

$$\mathcal{R}_{e,c} : \exists q \in \mathbb{Q}(c(\beta - q) \ngeqslant \alpha - \Phi_e(q)),$$

where Φ_e is the eth partial computable function. We do this with a straightforward finite injury argument.

We discuss the strategy for a single requirement $\mathcal{R}_{e,c}$. Let k be such that $c \leqslant 2^k$. We must make the difference between β and some rational q quite small while making the difference between α and $\Phi_e(q)$ relatively large. At a stage t we pick a new big number d. For the sake of $\mathcal{R}_{e,c}$, we will control the first $d+k+3$ places of (the binary expansion of) β_s and α_s for $s \geqslant t$. We set $\beta_t(x) = 1$ for all x with $d \leqslant x \leqslant d+k+2$, while at the same time keeping $\alpha_s(x) = 0$ for all such x. We let $q = \beta_t$. Note that, since we are restraining the first $d+k+3$ places of β_s, we know that, unless this restraint is lifted, β_s can only change on positions greater than or equal to $d+k+3$, and hence $\beta - q \leqslant 2^{-(d+k+3)}$. This means that, unless we lift the restraint, $c(\beta - q) \leqslant 2^k 2^{-(d+k+3)} = 2^{-(d+3)}$.

We now need do nothing until we come to a stage $s \geqslant t$ such that $\Phi_{e,s}(q)\downarrow$ *and* $0 < \alpha_s - \Phi_{e,s}(q) \leqslant 2^{-(d+3)}$. Our action then is the following. First we add $2^{-(d+k+2)}$ to β_s.

Then we restrain β_u for $u > s+1$ on its first $d+k+3$ places. Assuming that this restraint is successful, it follows that $c(\beta - q) \leqslant 2^{-(d+3)} + 2^{-(d+2)} < 2^{-(d+1)}$.

Finally we win by our second action, which is to add 2^{-d} to α_{s+1}. Then $\alpha - \alpha_s \geqslant 2^{-d}$, so $\alpha - \Phi_e(q) \geqslant 2^{-d} > c(\beta - q)$, as required.

The theorem now follows by a simple application of the finite injury priority method.

It is easy to see that $\alpha \leqslant_{\mathrm{sw}} \beta$. When we add $2^{-(d+k+2)}$ to β_s, since $\beta_t(x) = 1$ for all x with $d \leqslant x \leqslant d+k+2$, the effect is to make position $d-1$ of β change from 0 to 1. On the α side, the only change is that position $d-1$ changes from 0 to 1. Hence we keep $A \leqslant_{\mathrm{sw}} B$ (with constant 0). It is also clear that α is strongly c.e.. □

We note that, since sw-reducibility has the Solovay property, the previous result gives a quick proof of the theorem, due to Calude and Coles [6], that the converse of Theorem 1.3 does not hold.

2.9. Theorem. *There exist c.e. reals $\alpha \leqslant_S \beta$ such that $\alpha \nleqslant_{sw} \beta$. Moreover, β can be chosen to be strongly c.e..*

Proof. The proof is a straightforward diagonalization argument, similar to the previous proof, but even easier. The strategy is described below. We build sets A and B and let $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$. We must meet the following requirements.

$$\mathcal{R}_{e,c} : \text{If } \Gamma_e \text{ has use } x+c \text{ then } \Gamma_e^B \neq A.$$

The idea is quite simple. We need only make B "sparse" and A "sometimes thick". That is, for the sake of $\mathcal{R}_{e,c}$, we set aside a block of $c+2$ positions of the binary expansion of β, say $n, n+1, \ldots, n+c+1$. Initially we have *none* of these numbers in B, but we put *all* of $n+1, \ldots, n+c+1$ into A. If we ever see a stage s where $\Gamma_{e,s}^{B_s}(n)\downarrow = 0$ with use $n+c$, we can satisfy the requirement by adding $2^{-(n+c+1)}$ to both α_s and β_s, the effect being that $B_s(n+c+1)$ changes from 0 to 1, $A_s(n+i)$ for $1 \leqslant i \leqslant c+1$ changes from 1 to 0, and $A_s(n)$ changes from 0 to 1.

It is easy to check that $\alpha \leqslant_{\mathrm{S}} \beta$ and that β is strongly c.e.. □

Note that it is easy to modify the above proof to ensure that $\alpha \nleqslant_{\mathrm{wtt}} \beta$.

The counterexamples above can be jazzed up with relatively standard degree control techniques to prove the following result.

2.10. Theorem. *Let* **a** *be a nonzero c.e. Turing degree. There exist c.e. reals α and β of degree* **a** *such that α is strongly c.e., $\alpha \leqslant_{sw} \beta$, and $\alpha \nleqslant_S \beta$. There also exist c.e. reals γ and δ of degree* **a** *such that δ is strongly c.e., $\gamma \leqslant_S \delta$, and $\gamma \nleqslant_{sw} \delta$.*

On the strongly c.e. reals, however, S-reducibility and sw-reducibility coincide.

2.11. Theorem. *If β is strongly c.e. and α is c.e. then $\alpha \leqslant_{sw} \beta$ implies $\alpha \leqslant_S \beta$.*

Proof. Let A and B be such that $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$, and suppose that $\Gamma^B = A$ with use $x+c$. We may assume that we have the approximations of A and B sped up so that every stage is expansionary. That is, for all stages s and all $z \leqslant s$, we have $\Gamma_s^{B_s}(z) = A_s(z)$. We may also assume that if z enters A at stage s then $s \geqslant z$. Now if z enters A at stage s then some number less than or equal to $z+c$ must enter B at stage s. Since B is c.e., this means that $\beta_s - \beta_{s-1} \geqslant 2^{-(z+c)}$. But z entering A corresponds to a change of at most 2^{-z} in the value of α, so $\beta_s - \beta_{s-1} \geqslant 2^{-c}(\alpha_s - \alpha_{s-1})$. Thus for all s we have $\alpha - \alpha_s \leqslant 2^c(\beta - \beta_s)$, and hence, by Lemma 2.7, $\alpha \leqslant_{\mathrm{S}} \beta$. □

2.12. Theorem. *If α is strongly c.e. and β is c.e. then $\alpha \leqslant_S \beta$ implies $\alpha \leqslant_{sw} \beta$.*

Proof. Let A and B be such that $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$. Note that, since α is strongly c.e., for all k and s we have $A \upharpoonright k = A_s \upharpoonright k$ if and only if $\alpha - \alpha_s \leqslant 2^{-(k+1)}$. Let f and c be as in Lemma 2.7 and let k be such that $c \leqslant 2^{k-2}$. To decide whether $x \in A$ using the first $x+k$ bits of B, find the least stage s such that $B_s \upharpoonright x+k = B \upharpoonright x+k$. We claim that $x \in A$ if and only if $x \in A_{f(s)}$. To verify this claim, first note that $\beta - \beta_s < 2^{-(x+k)}$, since otherwise β_s would have to change on one of its first $x+k$ places after stage s. Thus $\alpha - \alpha_{f(s)} \leqslant 2^{k-2}2^{-(x+k)} = 2^{-(x+2)}$, and hence, as noted above, A has stopped changing on the numbers $0, \ldots, x$ by stage $f(x)$. □

2.13. Corollary. *If α and β are strongly c.e. then $\alpha \leqslant_S \beta$ if and only if $\alpha \leqslant_{sw} \beta$.*

Some structural properties are much easier to prove for sw-reducibility than for S-reducibility. One example is the fact, which we will now show, that there are no minimal sw-degrees of c.e. reals, that is, that for any noncomputable c.e. real α there is a c.e. real strictly sw-between α and the computable reals. The analogous property for S-reducibility was proved by Downey, Hirschfeldt, and Nies [15] with a fairly involved priority argument.

2.14. Definition. Let A be a nearly c.e. set. The *sw-canonical c.e. set* A^* associated with A is defined as follows. Begin with $A^*_0 = \emptyset$. For all x and s, if either $x \notin A_s$ and $x \in A_{s+1}$, or $x \in A_s$ and $x \notin A_{s+1}$, then for the least j with $\langle x, j\rangle \notin A^*_s$, put $\langle x, j\rangle$ into A^*_{s+1}.

2.15. Lemma. *$A^* \leqslant_{sw} A$ and $A \leqslant_{tt} A^*$.*

Proof. Since A is nearly c.e., $\langle x, j\rangle$ enters A^* at a given stage only if some $y \leqslant x$ enters A at that stage. Such a y will also be below $\langle x, j\rangle$. Hence $A^* \leqslant_{\mathrm{sw}} A$ with use x. Clearly, $x \in A$ if and only if A^* has an odd number of entries in row x, and furthermore, since A is nearly c.e., the number of entries in this row is bounded by x. Hence $A \leqslant_{\mathrm{tt}} A^*$. □

2.16. Corollary. *If A is nearly c.e. and noncomputable then there is a noncomputable c.e. set $A^* \leqslant_{sw} A$.*

2.17. Corollary. *There are no minimal sw-degrees of c.e. reals.*

Proof. Let A be nearly c.e. and noncomputable. Then $A^* \leqslant_{\mathrm{sw}} A$ is noncomputable, and we can c.e. Sacks split A^* into two disjoint c.e. sets A^*_1 and A^*_2 of incomparable Turing degree. Note that $A^*_i \leqslant_{sw} A^*$. (To decide whether $x \in A^*_i$, ask whether $x \in A^*$ and, if the answer is yes, then run the enumerations of A^*_1 and A^*_2 to see which set x enters.) So $\emptyset <_{\mathrm{sw}} A^*_1 <_{\mathrm{sw}} A^* \leqslant_{\mathrm{sw}} A$. □

Actually, while the above proof yields more than just nonminimality, there is an easier proof that the sw-degrees of c.e. reals have no minimal members. Given a c.e. real $A = 0.a_1a_2\ldots$, consider the c.e. real $B = 0.a_10a_200a_3000a_4\ldots$. It is easy to prove that if A is noncomputable then so is B. But it is also easy to see that $B \leqslant_{\mathrm{sw}} A$, and that if it were the case that $A \leqslant_{\mathrm{sw}} B$ then A would be computable. Hence $\emptyset <_{\mathrm{sw}} B <_{\mathrm{sw}} A$.

There is a greatest S-degree of c.e. reals, namely that of Ω, but the situation is different for strongly c.e. reals.

2.18. Theorem. *Let α be strongly c.e.. There is a strongly c.e. real that is neither S-below nor sw-below α.*

Proof. The argument is nonuniform, but is still finite injury. Since sw-reducibility and S-reducibility coincide for strongly c.e. reals, it is enough to build a strongly c.e. real that is not sw-below α. Let A be such that $\alpha = 0.\chi_A$. We build c.e. sets B and C to satisfy the following requirements.

$$\mathcal{R}_{e,i} : \Gamma_e^A \neq B \vee \Gamma_i^A \neq C,$$

where Γ_e is the eth wtt reduction with use less than $x + e$. It will then follow that either $0.\chi_B \nleqslant_{\mathrm{sw}} \alpha$ or $0.\chi_C \nleqslant_{\mathrm{sw}} \alpha$.

The idea for satisfying a single requirement $\mathcal{R}_{e,i}$ is simple. Let $l(e,i,s) = \max\{x \mid \forall y \leqslant x(\Gamma_{e,s}^{A_s}(y) = B_s(y) \wedge \Gamma_{i,s}^{A_s} = C_s(y))\}$. Pick a large number $k >> e, i$ and let $\mathcal{R}_{e,i}$ assert control over the interval $[k, 3k]$ in both B and C, waiting until a stage s such that $l(e,i,s) > 3k$.

First work with C. Put $3k$ into C, and wait for the next stage s' where $l(e,i,s') > 3k$. Note that some number must enter $A_{s'} - A_s$ below $3k + i$. Now repeat with $3k - 1$, then $3k - 2, \ldots, k$. In this way, $2k$ numbers are made to enter A below $3k + i$. Now we can win using B, by repeating the process and noticing that, by the choice of the parameter k, A cannot respond another $2k$ times below $3k + e$.

The theorem now follows by a standard application of the finite injury method. $\square$

One thing we get out of the proof of Corollary 2.17 is that every c.e. real has a noncomputable strongly c.e. real sw-below it. The same is not true for S-reducibility.

2.19. Theorem. *There is a noncomputable c.e. real α such that all strongly c.e. reals dominated by α are computable.*

The proof of this theorem uses a result of Downey and LaForte [16]. A c.e. set $A \subseteq \{0,1\}^*$ *presents* a c.e. real α if A is prefix-free and $\alpha = \sum_{\sigma \in A} 2^{-|\sigma|}$. Downey and LaForte constructed a noncomputable c.e. real α such that if A presents α then A is computable. Given a strongly c.e. real $\beta \leqslant_{\mathrm{S}} \alpha$, it is possible to build a presentation A of α that "encodes" β, in the sense that, for some constant k, by knowing how many strings of length $n + k$ are in A, we can tell whether the nth bit of β is 1. Since A must be computable, this allows us to compute β.

As we have seen, in some ways the sw-degrees are nicer than the S-degrees. Unfortunately, the theorem below shows that this is not always the case. There is a simple join operator, arithmetic addition, which induces a join operation on the S-degrees. No such operation exists for the sw-degrees.

2.20. Theorem. *There exist nearly c.e. sets A and B such that for all nearly c.e. $W \geqslant_{sw} A, B$ there is a nearly c.e. Q with $A, B \leqslant_{sw} Q$ but $W \nleqslant_{sw} Q$. Thus the sw-degrees of c.e. reals do not form an uppersemilattice.*

The idea of the proof of this theorem is that if $W \geqslant_{\mathrm{sw}} B$ then, by changing B very often, we can cause W to change very often, and hence force W to contain large blocks of 1's. We can then use A to force a change in W somewhere within such a block, which, because W is nearly c.e., forces W to change below the block. But if the block is large enough then we can use this W change to destroy a potential sw-reduction from W to Q, while still allowing Q to be sw-above A and B. The full details involve a finite injury priority argument which is nonuniform in the sense that we prevent a given $W \geqslant_{\mathrm{sw}} A, B$ from being a join by constructing infinitely many c.e. reals $Q_i \geqslant_{\mathrm{sw}} A, B$ and using the argument outlined above to show that $W \nleqslant_{\mathrm{sw}} Q_i$ for at least one of the Q_i.

The lack of a join operation leads to difficulties in exploring the structure of the sw-degrees beyond what is done here, and is one of the motivations for the introduction of rH-reducibility in the following section. The following questions about the sw degrees seem particularly interesting.

2.21. Question. Are the sw-degrees of c.e. reals dense?

2.22. Question. Is there an sw-complete c.e. real? If so, then are all random c.e. reals sw-complete among the c.e. reals?

3. Relative H Reducibility. Both S-reducibility and sw-reducibility are uniform in a way that relative initial-segment complexity is not. This makes them too strong, in a sense, and it is natural to wish to investigate nonuniform versions of these reducibilities. Motivated by this consideration, as well as by the problems with sw-reducibility, we introduce another measure of relative randomness, called relative H reducibility, which can be seen as a nonuniform version of both S-reducibility and sw-reducibility, and which combines many of the best features of these reducibilities. Its name derives from a characterization, discussed below, which shows that there is a very natural sense in which it is an *exact* measure of relative randomness.

3.23. Definition. Let α and β be reals. We say that β is *relative H reducible* (rH-reducible) to α, and write $\beta \leqslant_{\mathrm{rH}} \alpha$, if there exist a partial computable binary function f and a constant k such that for each n there is a $j \leqslant k$ for which $f(\alpha \upharpoonright n, j)\downarrow = \beta \upharpoonright n$.

Since rH-reducibility is reflexive and transitive, we can define the *rH-degree* $\deg_{\mathrm{rH}}(\alpha)$ of a real α to be its rH-equivalence class.

There are several characterizations of rH-reducibility, each revealing a different facet of the concept. We mention three, beginning with a "relative entropy" characterization whose proof is quite straightforward. For a c.e. real β and a fixed computable approximation $\beta_0, \beta_1, \ldots$ of β, we will let the mind-change function $m(\beta, n, s, t)$ be the cardinality of $\{u \in [s,t] \mid \beta_u \upharpoonright n \neq \beta_{u+1} \upharpoonright n\}$.

3.24. Proposition. *Let α and β be c.e. reals. The following condition holds if and only if $\beta \leqslant_{rH} \alpha$. There are a constant k and computable approximations $\alpha_0, \alpha_1, \ldots$ and $\beta_0, \beta_1, \ldots$ of α and β, respectively, such that for all n and $t > s$, if $\alpha_t \upharpoonright n = \alpha_s \upharpoonright n$ then $m(\beta, n, s, t) \leqslant k$.*

The following is a more analytic characterization of rH-reducibility, which clarifies its nature as a nonuniform version of both S-reducibility and sw-reducibility.

3.25. Proposition. *For any reals α and β, the following condition holds if and only if $\beta \leqslant_{rH} \alpha$. There are a constant c and a partial computable function φ such that for each n there is a τ of length $n + c$ with $|\alpha - \tau| \leqslant 2^{-n}$ for which $\varphi(\tau)\downarrow$ and $|\beta - \varphi(\tau)| \leqslant 2^{-n}$.*

Proof. First suppose that $\beta \leqslant_{\mathrm{rH}} \alpha$ and let f and k be as in definition 3.23. Let c be such that $2^c \geqslant k$ and define the partial computable function φ as follows. Given a string σ of length n, whenever $f(\sigma, j)\downarrow$ for some new $j \leqslant k$, choose a new $\tau \supseteq \sigma$ of length $n + c$ and define $\varphi(\tau) = f(\sigma, j)$. Then for each n there is a $\tau \supseteq \alpha \upharpoonright n$ such that $\varphi(\tau)\downarrow = \beta \upharpoonright n$. Since $|\alpha - \tau| \leqslant |\alpha - \alpha \upharpoonright n| \leqslant 2^{-n}$ and $|\beta - \beta \upharpoonright n| \leqslant 2^{-n}$, the condition holds.

Now suppose that the condition holds. For a string σ of length n, let S_σ be the set of all μ for which there is a τ of length $n + c$ with $|\sigma - \tau| \leqslant 2^{-n+1}$ and $|\mu - \varphi(\tau)| \leqslant 2^{-n+1}$. It is easy to check that there is a k such that $|S_\sigma| \leqslant k$ for all σ. So there is a partial computable binary function f such that for each σ and each $\mu \in S_\sigma$ there is a $j \leqslant k$ with $f(\sigma, j)\downarrow= \mu$. But, since for any real γ and any n we have $|\gamma - \gamma \upharpoonright n| \leqslant 2^{-n}$, it follows that for each n we have $\beta \upharpoonright n \in S_{\alpha\upharpoonright n}$. Thus f and k witness the fact that $\beta \leqslant_{\mathrm{rH}} \alpha$. $\square$

The most interesting characterization of rH-reducibility (and the reason for its name) is given by the following result, which shows that there is a very natural sense in which rH-reducibility is an exact measure of relative randomness. Recall that the prefix-free complexity $H(\tau \mid \sigma)$ of τ relative to σ is the length of the shortest string μ such that $M^\sigma(\mu)\downarrow= \tau$, where M is a fixed self-delimiting universal computer.

3.26. Theorem. *Let α and β be reals. Then $\beta \leqslant_{rH} \alpha$ if and only if there is a constant c such that $H(\beta \upharpoonright n \mid \alpha \upharpoonright n) \leqslant c$ for all n.*

Proof. First suppose that $\beta \leqslant_{\mathrm{rH}} \alpha$ and let f and k be as in definition 3.23. Let m be such that $2^m \geqslant k$ and let $\tau_0, \dots, \tau_{2^m-1}$ be the strings of length m. Define the prefix-free machine N to act as follows with σ as an oracle. For all strings μ of length not equal to m, let $N^\sigma(\mu)\uparrow$. For each $i < 2^m$, if $f(\sigma, i)\downarrow$ then let $N^\sigma(\tau_i)\downarrow= f(\sigma, i)$, and otherwise let $N^\sigma(\tau_i)\uparrow$. Let e be the coding constant of N and let $c = e + m$. Given n, there exists a $j \leqslant k$ for which $f(\alpha \upharpoonright n, j)\downarrow= \beta \upharpoonright n$. For this j we have $N^{\alpha\upharpoonright n}(\tau_j)\downarrow= \beta \upharpoonright n$, which implies that $H(\beta \upharpoonright n \mid \alpha \upharpoonright n) \leqslant |\tau_j| + e \leqslant c$.

Now suppose that $H(\beta \upharpoonright n \mid \alpha \upharpoonright n) \leqslant c$ for all n. Let $\tau_0, \dots, \tau_k$ be a list of all strings of length less than or equal to c and define f as follows. For a string σ and a $j \leqslant k$, if $M^\sigma(\tau_j)\downarrow$ then $f(\sigma, j)\downarrow= M^\sigma(\tau_j)$, and otherwise $f(\sigma, j)\uparrow$. Given n, since $H(\beta \upharpoonright n \mid \alpha \upharpoonright n) \leqslant c$, it must be the case that $M^{\alpha\upharpoonright n}(\tau_j)\downarrow= \beta \upharpoonright n$ for some $j \leqslant k$. For this j we have $f(\alpha \upharpoonright n, j)\downarrow= \beta \upharpoonright n$. Thus $\beta \leqslant_{\mathrm{rH}} \alpha$. $\square$

An immediate consequence of this result is that rH-reducibility satisfies the Solovay property.

3.27. Corollary. *If $\beta \leqslant_{rH} \alpha$ then there is a constant c such that $H(\beta \upharpoonright n) \leqslant H(\alpha \upharpoonright n) + c$ for all n.*

It is not hard to check that the converse of this corollary is not true in general, but the following question is open.

3.28. Question. Let α and β be c.e. reals such that, for some constant c, we have $H(\beta \upharpoonright n) \leqslant H(\alpha \upharpoonright n) + c$ for all n. Does it follow that $\beta \leqslant_{\mathrm{rH}} \alpha$?

Although it might seem at first that the answer to this question should obviously be negative, Theorem 2.6 indicates that any counterexample would probably have to be quite complicated, and gives us hope for a positive answer.

The next two theorems, which show that rH-reducibility is a common weakening of S-reducibility and sw-reducibility, follow easily from Proposition 3.25.

3.29. Theorem. *Let α and β be c.e. reals. If $\beta \leqslant_S \alpha$ then $\beta \leqslant_{rH} \alpha$.*

3.30. Corollary. *A c.e. real α is rH-complete if and only if it is random.*

3.31. Theorem. *If $\beta \leqslant_{sw} \alpha$ then $\beta \leqslant_{rH} \alpha$.*

Theorems 2.8 and 2.9 show that the converses of Theorems 3.29 and 3.31 do not hold, but even among strongly c.e. reals, where S-reducibility and sw-reducibility agree, rH-reducibility is not equivalent to its stronger counterparts.

3.32. Theorem. *There exist strongly c.e. reals α and β such that $\beta \leqslant_{rH} \alpha$ but $\beta \nleqslant_{sw} \alpha$ (equivalently, $\beta \nleqslant_S \alpha$).*

The proof of this theorem is a straightforward finite injury argument.

It is interesting to note that, despite the nonuniform nature of its definition, rH-reducibility implies Turing reducibility.

3.33. Theorem. *If $\beta \leqslant_{rH} \alpha$ then $\beta \leqslant_T \alpha$.*

Proof. Let k be the least number for which there exists a partial computable binary function f such that for each n there is a $j \leqslant k$ with $f(\alpha \restriction n, j)\downarrow = \beta \restriction n$. There must be infinitely many n for which $f(\alpha \restriction n, j)\downarrow$ for all $j \leqslant k$, since otherwise we could change finitely much of f to contradict the minimality of k. Let $n_0 < n_1 < \cdots$ be an α-computable sequence of such n. Let T be the α-computable subtree of 2^ω obtained by pruning, for each i, all the strings of length n_i except for the values of $f(\alpha \restriction n_i, j)$ for $j \leqslant k$.

If γ is a path through T then for all i there is a $j \leqslant k$ such that γ extends $f(\alpha \restriction n_i, j)$. Thus there are at most k many paths through T, and hence each path through T is α-computable. But β is a path through T, so $\beta \leqslant_{\mathrm{T}} \alpha$. □

On the other hand, as remarked after the proof of Theorem 2.9, sw-reducibility does not imply wtt-reducibility, even among c.e. reals, and hence rH-reducibility does not imply wtt-reducibility.

Notice that, since any computable real is obviously rH-reducible to any other real, the above theorem shows that the computable reals form the least rH-degree.

Structurally, the rH-degrees of c.e. reals are nicer than the sw-degrees of c.e. reals.

3.34. Theorem. *The rH-degrees of c.e. reals form an uppersemilattice with least degree that of the computable sets and highest degree that of Ω. The join of the rH-degrees of the c.e. reals α and β is the rH-degree of $\alpha + \beta$.*

Proof. All that is left to show is that addition is a join. Since $\alpha, \beta \leqslant_{\mathrm{S}} \alpha + \beta$, it follows that $\alpha, \beta \leqslant_{\mathrm{rH}} \alpha + \beta$. Let γ be a c.e. real such that $\alpha, \beta \leqslant_{\mathrm{rH}} \gamma$. Then Proposition 3.24 implies that $\alpha + \beta \leqslant_{\mathrm{rH}} \gamma$, since for any n and $s < t$ we have $m(\alpha + \beta, n, s, t) \leqslant 2(m(\alpha, n, s, t) + m(\beta, n, s, t)) + 1$. □

In [15], Downey, Hirschfeldt, and Nies studied the structure of the S-degrees of c.e. reals. They showed that the S-degrees of c.e. reals are dense. They also showed that every incomplete S-degree splits over any lesser degree, while the complete S-degree does not split at all. The methods of that paper can be adapted to prove the analogous results for rH-degrees of c.e. reals.

3.35. Theorem. *For any rH-degrees $\mathbf{a} < \mathbf{b}$ of c.e. reals there is an rH-degree $\mathbf{c}$ of c.e. reals such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$.*

3.36. Theorem. *For any rH-degrees $\mathbf{a} < \mathbf{b} < \deg_{rH}(\Omega)$ of c.e. reals, there are rH-degrees $\mathbf{c_0}$ and $\mathbf{c_1}$ of c.e. reals such that $\mathbf{a} < \mathbf{c_0}, \mathbf{c_1} < \mathbf{b}$ and $\mathbf{c_0} \vee \mathbf{c_1} = \mathbf{b}$.*

3.37. Theorem. *For any rH-degrees $\mathbf{a}, \mathbf{b} < \deg_{rH}(\Omega)$ of c.e. reals, $\mathbf{a} \vee \mathbf{b} < \deg_{rH}(\Omega)$.*

Thus we see that rH-reducibility shares many of the nice structural properties of S-reducibility on the c.e. reals, while still being a reasonable reducibility on non-c.e. reals. Together with its various characterizations, especially the one in terms of relative H-complexity of initial segments, this makes rH-reducibility a tool with great potential in the study of the relative randomness of reals.

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A Technical Appendices

In these appendices we present some of the proofs that were left out of the body of the abstract for reasons of space. The theorems we prove here are Theorems 2.19, 2.20, 3.32, and 3.35–3.37.

A.1 Proof of Theorem 2.19

In this section, we show that there is a noncomputable c.e. real α such that all strongly c.e. reals dominated by α are computable.

We begin by noting the following lemma, proved in [15].

A.1. Lemma. *Let $\beta \leqslant_S \alpha$ be c.e. reals. There are a c.e. real γ and a positive $c \in \mathbb{Q}$ such that $\alpha = c\beta + \gamma$.*

A c.e. set $A \subseteq \{0,1\}^*$ *presents* a c.e. real α if A is prefix-free and

$$\alpha = \sum_{\sigma \in A} 2^{-|\sigma|}.$$

In [16], Downey and LaForte constructed a noncomputable c.e. real α such that if A presents α then A is computable. We claim that, for this α, if $\beta \leqslant_{\mathrm{S}} \alpha$ is strongly c.e. then β is computable.

To verify this claim, let $\beta \leqslant_{\mathrm{S}} \alpha$ be strongly c.e.. By Lemma A.1, there is a positive $c \in \mathbb{Q}$ such that $\alpha = c\beta + \gamma$. Let $k \in \omega$ be such that $2^{-k} \leqslant c$ and let $\delta = \gamma + (c - 2^{-k})\beta$. Then δ is a c.e. real such that $\alpha = 2^{-k}\beta + \delta$.

It is easy to see that there exist computable sequences of natural numbers $b_0, b_1, \dots$ and $d_0, d_1, \dots$ such that $2^{-k}\beta = \sum_{i\in\omega} 2^{-b_i}$ and $\delta = \sum_{i\in\omega} 2^{-d_i}$. Furthermore, since β is strongly c.e., so is $2^{-k}\beta$, and hence we can choose $b_0, b_1, \dots$ to be pairwise distinct, so that the nth bit of the binary expansion of $2^{-k}\beta$ is 1 if and only if $n = b_i$ for some i.

Since $\sum_{i\in\omega} 2^{-b_i} + \sum_{i\in\omega} 2^{-d_i} = 2^{-k}\beta + \delta = \alpha < 1$, Kraft's inequality tells us that there is a prefix-free c.e. set $A = \{\sigma_0, \sigma_1, \dots\}$ such that $|\sigma_0| = b_0$, $|\sigma_1| = d_0$, $|\sigma_2| = b_1$, $|\sigma_3| = d_1$, etc.. Now $\sum_{\sigma\in A} 2^{-|\sigma|} = \sum_{i\in\omega} 2^{-b_i} + \sum_{i\in\omega} 2^{-d_i} = \alpha$, and thus A presents α.

By our choice of α, this means that A is computable. But now we can compute the binary expansion of $2^{-k}\beta$ as follows. Given n, compute the number m of strings of length n in A. If $m = 0$ then $b_i \neq n$ for all i, and hence the nth bit of binary expansion of $2^{-k}\beta$ is 0. Otherwise, run through the b_i and d_i until either $b_i = n$ for some i or $d_{j_1} = \cdots = d_{j_m} = n$ for some $j_1 < \cdots < j_m$. By the definition of A, one of the two cases must happen. In the first case, the nth bit of the binary expansion of $2^{-k}\beta$ is 1. In the second case, $b_i \neq n$ for all i, and hence the nth bit of the binary expansion of $2^{-k}\beta$ is 0. Thus $2^{-k}\beta$ is computable, and hence so is β.

A.2 Proof of Theorem 2.20

In this section, we show that there exist nearly c.e. sets A and B such that for all nearly c.e. $W \geqslant_{\mathrm{sw}} A, B$ there is a nearly c.e. Q with $A, B \leqslant_{\mathrm{sw}} Q$ but $W \nleqslant_{\mathrm{sw}} Q$. (This implies that the sw-degrees of c.e. reals do not form an uppersemilattice.)

We build A, B, and W in stages, to meet the following requirements.

$$\mathcal{R}_e : (\Gamma_e^{W_e} = A \wedge \Delta_e^{W_e} = B) \Rightarrow \exists Q_e(A, B \leqslant_{\mathrm{sw}} Q_e \wedge W_e \nleqslant_{\mathrm{sw}} Q_e).$$

Here we assume that each Γ_e and Δ_e is an sw procedure with use bounded by $x+e$, and that the triples $\langle \Gamma_e, \Delta_e, W_e \rangle$ run through all triples consisting of a pair of such procedures together with a nearly c.e. set W_e. The above requirements are broken into subrequirements

$$\mathcal{R}_{e,i} : (\Gamma_e^{W_e} = A \wedge \Delta_e^{W_e} = B) \Rightarrow \exists Q_e (A, B \leqslant_{\mathrm{sw}} Q_e \wedge \Phi_i^{Q_e} \neq W_e),$$

where each Φ_i is an sw procedure with use bounded by $x+i$ and the Φ_i run over all such procedures.

Actually, the argument is *nonuniform*. We *really* construct sets Q_e together with backup sets $Q_{e,i}$ and meet the requirements

$$\mathcal{R}_{e,i} : (\Gamma_e^{W_e} = A \wedge \Delta_e^{W_e} = B) \Rightarrow \\ (A, B \leqslant_{\mathrm{sw}} Q_e \wedge A, B \leqslant_{\mathrm{sw}} Q_{e,i} \wedge (\Phi_i^{Q_e} = W_e \Rightarrow \Phi_j^{Q_{e,i}} \neq W_e)).$$

These naturally have subrequirements $\mathcal{R}_{e,i,j}$ trying to make $\Phi_i^{Q_e} \neq W_e$ or $\Phi_j^{Q_{e,i}} \neq W_e$.

The argument is a finite injury one, and hence it suffices to give the strategy for a single $\mathcal{R}_{e,i,j}$. The idea is the following. For a single $\mathcal{R}_{e,i,j}$, one picks a killing point n, which is large and fresh. If this happens at stage s then choosing $n = s$ would suffice with the standard use conventions. We may assume that $e, i, j << n$ and $e < i < j$.

Now the idea is that $\mathcal{R}_{e,i,j}$ will control the region $[n, (2j+1)n^2]$ of both A and B. We assume by priorities that the regions below n have ceased changing.

The key observation is the following. Suppose that we wish to kill $\Phi_i^{Q_e} = W_e$ or $\Phi_j^{Q_{e,i}} = W_e$. We need to have a situation where, through our changing A or B, we cause W_e to have to change on some m, while Q_e or $Q_{e,i}$ changes only on $k > m+i$ or $k > m+j$, respectively. However, W_e is not really under our control. But suppose that using *only B changes* we can get to a situation where W_e has a block of $2j+1$ consecutive 1's. That is, at stage s, we have $(\Phi_i^{Q_e}(z) = W_e(z))[s]$ and $(\Phi_j^{Q_{e,i}}(z) = W_e(z))[s]$ for all $z \leqslant m+j+1$, where $[m-j, m+j+1] \subseteq W_{e,s}$. (Here, m is the central number in the interval.) Further assume that the stage is e-expansionary, that is, $l(e,s) > \max\{l(e,t) : t < s\}$ and $l(e,s) > m+j+1$, where

$$l(e,s) = \max\{z : \forall y \leqslant z((\Gamma_e^{W_e}(y) = A(y) \wedge \Delta_e^{W_e}(y) = B(y))[s]\}.$$

Then we can win as follows.

Step 1. First we put some small number $p << m-i$ into $Q_e[s+1]$ and take all the numbers bigger than p (including, in particular, the interval $[m-i, m+i+1]$) out of $Q_e[s+1]$. We do *not*, however, change $Q_{e,i}$.

Step 2. Then we wait for the length of agreement to recover. That is, we wait for an e-expansionary stage $t > s$ such that $(\Phi_i^{Q_e}(z) = W_e(z))[t]$ and $(\Phi_j^{Q_{e,i}}(z) = W_e(z))[t]$ for all $z \leqslant m+j+1$. Since we have not changed $Q_{e,i}$ between stages s and t, we have $W_e[s] \upharpoonright m+j+1 = W_e[t] \upharpoonright m+j+1$.

We can now win by putting m into A, Q_e, and $Q_{e,i}$. Since W_e is supposedly above both A and B via Γ_e and Δ_e, respectively, W_e must change below $m+e < m+j$. Because W_e is nearly c.e. and contains the whole interval $[m-j, m+j+1]$, such a change can only occur *below* $m-j$. Thus some $p < m-j$ must enter W_e. But supposedly $\Phi^{Q_e}(p) = W_e(p)$. Therefore Q_e should have changed in the region below $p+j$, which it did not.

The conclusion is that one of the equalities is wrong.

Thus if we ever see a situation where, at some e, i, j expansionary stage, W_e contains a full interval $[j-m, m+j+1]$ with the end points between n and $(2j+1)n^2$ then we are done.

We must now deal with the case in which such a good block never occurs. We think of the argument to follow as an *entropy* one. The idea is that if W_e never contains a block of the appropriate size then it cannot change as often as we can change B, and hence we can ensure that W_e is not sw-above B.

We cycle through B configurations as follows, using the B changes to induce changes in W_e. At an e-expansionary stage s, we put $b_1 = (2j+1)n^2 - j$ into B. We wait until the next e-expansionary stage $s_1 > s$. Note that W_e must have changed between stages s and s_1, and indeed a number must have entered W_e below $(2j+1)n^2 - j + e$, and hence below $(2j+1)n^2$. Now we can repeat. We put $b_1 - 1$ into B, take b_1 out of B, and wait for the next e-expansionary stage $s_2 > s_1$, at which point there will have been another change in W_e below $(2j+1)n^2$. We keep repeating this: we next put b_1 into B again; at the next e-expansionary stage, we put $b_1 - 2$ into B and take out $b_1 - 1$ and b_1. We continue until we have put the whole block $[n+j, (2j+1)n^2 - j]$ into B. Our assumption is that, throughout this entire procedure, we never get a large block of consecutive 1's in W_e.

To keep $A, B \leqslant_{\mathrm{sw}} Q_e, Q_{e,i}$, we copy what we do to B into Q_e and $Q_{e,i}$. These will be the only changes to these sets below $(2j+1)n^2$, unless we see the desired block of 1's in W_e. Notice also that W_e will not change below n throughout this procedure, since otherwise the e, i, j computations could not recover. (Any $p < n$ entering W_e would require a change in the Q sets below $p + j < n + j$.)

The above procedure allows us to make $2^{(2j+1)n^2 - n - 2j}$ changes to B between $n + j$ and $(2j+1)n^2 - j$. If W_e is sw-above B then it must change in response to each of these changes. We compute an upper bound on how many times W_e can change in the interval $[n, (2j+1)n^2]$, assuming that it has no block of $2j+1$ many 1's in that interval.

We can split $[n, (2j+1)n^2]$ into less than n^2 consecutive blocks of size $2j+1$. For each W_e configuration at an e-expansionary stage, each of these intervals must contain at least one 0. For each such interval, it follows that there are only 2^{2j} possible configurations of that interval that can be realized. This gives W_e a maximum of $(2^{2j})^{n^2} = 2^{2n^2 j}$ possible configurations in the interval $[n, (2j+1)n^2]$. But since $n >> j$, which implies that $n^2 > n - 2j$, we have $2n^2 j < (2j+1)n^2 - n - 2j$. This means that W_e cannot change as often in the interval $[n, (2j+1)n^2]$ as we can change B in the interval $[n+j, (2j+1)n^2 - j]$, and hence we can force it to be the case that $B \nleqslant_{\mathrm{sw}} W_e$.

A standard application of the finite injury priority method completes the proof.

A.3 Proof of Theorem 3.32

We show that there exist strongly c.e. reals α and β such that $\beta \leqslant_{\mathrm{rH}} \alpha$ but $\beta \nleqslant_{\mathrm{sw}} \alpha$ (equivalently, $\beta \nleqslant_{\mathrm{S}} \alpha$). We build c.e. sets A and B to satisfy the following requirements.

$$\mathcal{R}_e : \Gamma_e^A \neq B,$$

where Γ_e is the eth wtt reduction with use less than $x + e$. We think of α and β as $0.\chi_A$ and $0.\chi_B$, respectively, and we build A and B in such as way as to enable us to apply Proposition 3.24 to conclude that $\beta \leqslant_{\mathrm{rH}} \alpha$.

The construction is a standard finite injury argument. We discuss the satisfaction of a single requirement $\mathcal{R}_e$. For the sake of this requirement, we choose a large n, restrain n

from entering B, and restrain $n+e+1$ from entering A. If we find a stage s such that $\Gamma_{e,s}^{A_s}(n)\downarrow = 0$ then we put n into B, put $n+e+1$ into A, and restrain the initial segment of A of length $n+e$. Unless a higher priority strategy acts at a later stage, this guarantees that $\Gamma_e^A(n) \neq B(n)$.

Furthermore, it is not hard to check that, because of the numbers that we put into A, for each n and $t > s$, if $\alpha_t \upharpoonright n = \alpha_s \upharpoonright n$ then $m(\beta, n, s, t) \leqslant 2$ (where $m(\beta, n, s, t)$ is as defined before Proposition 3.24). Thus, by Proposition 3.24, $\beta \leqslant_{\mathrm{rH}} \alpha$.

A.4 Proofs of Theorems 3.35–3.37

In this section, we show that every incomplete rH-degree of c.e. reals splits over any lesser rH-degree of c.e. reals, and that the rH-degrees of c.e. reals are upwards dense. Together, these two results show that the rH-degrees of c.e. reals are dense. The proofs are modified versions of the proofs of the analogous results for S-degrees, which are due to Downey, Hirschfeldt, and Nies [15]. (In [15] it is shown that the sum of two nonrandom c.e. reals is nonrandom, which implies that the complete rH-degree does not split.)

A.2. Lemma. *Let $\alpha \nleqslant_S \beta$ be c.e. reals and let $k \in \omega$. There are infinitely many $s \in \omega$ for which there is a $t \in \omega$ such that $\alpha_u - \alpha_s > k(\beta_u - \beta_s)$ for all $u > t$.*

Proof. If there are infinitely many $u \in \omega$ such that $\alpha_u - \alpha_s \leqslant k(\beta_u - \beta_s)$ then

$$\alpha - \alpha_s = \lim_u \alpha_u - \alpha_s \leqslant \lim_u k(\beta_u - \beta_s) = k(\beta - \beta_s).$$

So if this happens for all but finitely many s then $\alpha \leqslant_{\mathrm{S}} \beta$. (The finitely many s for which $\alpha - \alpha_s > k(\beta - \beta_s)$ can be brought into line by increasing the constant k.) $\square$

A.3. Theorem. *Let $\gamma <_{rH} \alpha <_{rH} \Omega$ be c.e. reals. There are c.e. reals β^0 and β^1 such that $\gamma + \beta^0, \gamma + \beta^1 <_{rH} \alpha$ and $\beta^0 + \beta^1 = \alpha$.*

Proof. We want to build β^0 and β^1 so that $\beta^0 + \beta^1 = \alpha$, and the following requirement is satisfied for each $e, k \in \omega$ and $i < 2$:

$$R_{i,e,k} : \exists n \forall j \leqslant k (\Phi_e((\gamma + \beta^i) \upharpoonright n, j)\downarrow \Rightarrow \Phi_e((\gamma + \beta^i) \upharpoonright n, j) \neq \alpha \upharpoonright n).$$

Most of the essential features of our construction are already present in the case of two requirements $R_{i,e,k}$ and $R_{1-i,e',k'}$, which we now discuss. We assume that $R_{i,e,k}$ has priority over $R_{1-i,e',k'}$. We will think of the β^j as being built by adding amounts to them in stages. Thus β_s^j will be the total amount added to β^j by the end of stage s.

We will say that $R_{i,e,k}$ is satisfied through n at stage s if $\forall j \leqslant k(\Phi_e((\gamma_s + \beta_s^i) \upharpoonright n, j)[s]\downarrow \Rightarrow \Phi_e((\gamma_s + \beta_s^i) \upharpoonright n, j) \neq \alpha_s \upharpoonright n)$. The strategy for $R_{i,e,k}$ is to act whenever either it is not currently satisfied or the least number through which it is satisfied changes. Whenever this happens, $R_{i,e,k}$ initializes $R_{1-i,e',k'}$, which means that the amount of α that $R_{1-i,e',k'}$ is allowed to funnel into β^i is reduced. More specifically, once $R_{1-i,e',k'}$ has been initialized for the mth time, the total amount that it is thenceforth allowed to put into β^i is reduced to 2^{-m}.

The above strategy guarantees that if $R_{1-i,e',k'}$ is initialized infinitely often then the amount put into β^i by $R_{1-i,e',k'}$ (which in this case is all that is put into β^i) adds up to a computable real. In other words, $\gamma + \beta^i \equiv_{\mathrm{rH}} \gamma <_{\mathrm{rH}} \alpha$. But this means that there is a stage s

after which $R_{i,e,k}$ is always satisfied and the least number through which it is satisfied does not change. So we conclude that $R_{1-i,e',k'}$ is initialized only finitely often, and that $R_{i,e,k}$ is eventually permanently satisfied.

This leaves us with the problem of designing a strategy for $R_{1-i,e',k'}$ that respects the strategy for $R_{i,e,k}$. The problem is one of timing. Since $R_{1-i,e',k'}$ is initialized only finitely often, there is a certain amount 2^{-m} that it is allowed to put into β^i after the last time it is initialized. Thus if $R_{1-i,e',k'}$ waits until a stage s such that $\alpha - \alpha_s < 2^{-m}$, adding nothing to β^i until such a stage is reached, then from that point on it can put all of $\alpha - \alpha_s$ into β^i, which of course guarantees its success. The problem is that, in the general construction, a strategy working with a quota 2^{-m} cannot effectively find an s such that $\alpha - \alpha_s < 2^{-m}$. If it uses up its quota too soon, it may find itself unsatisfied and unable to do anything about it.

The key to solving this problem (and the reason for the hypothesis that $\alpha <_{\mathrm{rH}} \Omega$) is the observation that, since the sequence $\Omega_0, \Omega_1, \ldots$ converges much more slowly than the sequence $\alpha_0, \alpha_1, \ldots$, Ω can be used to modulate the amount that $R_{1-i,e',k'}$ puts into β^i. More specifically, at a stage s, if $R_{1-i,e',k'}$'s current quota is 2^{-m} then it puts into β^i as much of $\alpha_s - \alpha_{s-1}$ as possible, subject to the constraint that the total amount put into β^i by $R_{1-i,e',k'}$ since the last stage before stage s at which $R_{1-i,e',k'}$ was initialized must not exceed $2^{-m}\Omega_s$. As we will see below, the fact that $\Omega \nleqslant_{\mathrm{rH}} \alpha$ implies that there is a stage v after which $R_{1-i,e',k'}$ is allowed to put in all of $\alpha - \alpha_v$ into β^i.

In general, at a given stage s there will be several requirements, each with a certain amount that it wants (and is allowed) to direct into one of the β^j. We will work backwards, starting with the weakest priority requirement that we are currently considering. This requirement will be allowed to direct as much of $\alpha_s - \alpha_{s-1}$ as it wants (subject to its current quota, of course). If any of $\alpha_s - \alpha_{s-1}$ is left then the next weakest priority strategy will be allowed to act, and so on up the line.

We now proceed with the full construction. We say that $R_{i,e,k}$ has stronger priority than $R_{i',e',k'}$ if $2\langle e,k\rangle + i < 2\langle e',k'\rangle + i'$.

We say that $R_{i,e,k}$ is satisfied through n at stage s if

$$\forall j \leqslant k(\Phi_e((\gamma_s + \beta^i_s) \upharpoonright n, j)[s]\downarrow \Rightarrow \Phi_e((\gamma_s + \beta^i_s) \upharpoonright n, j) \neq \alpha_s \upharpoonright n)$$

Let $n^{i,e,k}_s$ be the least n through which $R_{i,e,k}$ is satisfied at stage s, if such an n exists, and let $n^{i,e,k}_s = \infty$ otherwise.

We say that $R_{i,e,k}$ requires attention at stage s if either $n^{i,e,k}_s = \infty$ or $n^{i,e,k}_s \neq n^{i,e,k}_{s-1}$. If $R_{i,e,k}$ requires attention at stage s then we say that each requirement of weaker priority than $R_{i,e,k}$ is initialized at stage s.

Each requirement $R_{i,e,k}$ has associated with it a c.e. real $\tau^{i,e,k}$, which records the amount put into β^{1-i} for the sake of $R_{i,e,k}$.

We decide how to distribute $\delta = \alpha_s - \alpha_{s-1}$ between β^0 and β^1 at stage s as follows.

1. Let $j = s$ and $\varepsilon = 0$.

2. Let $i < 2$ and $e, k \in \omega$ be such that $2\langle e,k\rangle + i = j$. Let m be the number of times $R_{i,e,k}$ has been initialized and let t be the last stage at which $R_{i,e,k}$ was initialized. Let
$$\zeta = \min(\delta - \varepsilon, 2^{-(j+m)}\Omega_s - (\tau^{i,e,k}_{s-1} - \tau^{i,e,k}_t)).$$

(It is not hard to check that ζ is non-negative.) Add ζ to ε and to the current values of $\tau^{i,e,k}$ and β^{1-i}.

3. If $\varepsilon = \delta$ or $j = 0$ then add $\delta - \varepsilon$ to the current value of β^0 and end the stage. Otherwise, decrease j by one and go to step 2.

This completes the construction. Clearly, $\beta^0 + \beta^1 = \alpha$.

We now show by induction that each requirement initializes requirements of weaker priority only finitely often and is eventually satisfied. Assume by induction that $R_{i,e,k}$ is initialized only finitely often. Let $j = 2\langle e,k\rangle + i$, let m be the number of times $R_{i,e,k}$ is initialized, and let t be the last stage at which $R_{i,e,k}$ is initialized. The following are clearly equivalent:

1. $R_{i,e,k}$ is satisfied,
2. $\lim_s n_s^{i,e,k}$ exists and is finite, and
3. $R_{i,e,k}$ eventually stops requiring attention.

Assume for a contradiction that $R_{i,e,k}$ requires attention infinitely often. Since $\Omega \not\leqslant_{\mathrm{rH}} \alpha$, which implies that $\Omega \not\leqslant_{\mathrm{S}} \alpha$, it follows from Lemma A.2 that there are $v > u > t$ such that for all $w > v$ we have $2^{-(j+m)}(\Omega_w - \Omega_u) > \alpha_w - \alpha_u$. Furthermore, by the way the amount ζ added to $\tau^{i,e,k}$ at a given stage is defined in step 2 of the construction, $\tau_u^{i,e,k} - \tau_t^{i,e,k} \leqslant 2^{-(j+m)}\Omega_u$ and $\tau_{w-1}^{i,e,k} - \tau_u^{i,e,k} \leqslant \alpha_{w-1} - \alpha_u$. Thus for all $w > v$,

$$\begin{aligned}
&\alpha_w - \alpha_{w-1} = \alpha_w - \alpha_u - (\alpha_{w-1} - \alpha_u) < \\
&\quad 2^{-(j+m)}(\Omega_w - \Omega_u) - (\alpha_{w-1} - \alpha_u) = 2^{-(j+m)}\Omega_w - (2^{-(j+m)}\Omega_u + \alpha_{w-1} - \alpha_u) \leqslant \\
&\qquad 2^{-(j+m)}\Omega_w - (\tau_u^{i,e,k} - \tau_t^{i,e,k} + \tau_{w-1}^{i,e,k} - \tau_u^{i,e,k}) = 2^{-(j+m)}\Omega_w - (\tau_{w-1}^{i,e,k} - \tau_t^{i,e,k}).
\end{aligned}$$

From this we conclude that, after stage v, the reverse recursion performed at each stage never gets past j, and hence everything put into β^i after stage v is put in for the sake of requirements of weaker priority than $R_{i,e,k}$.

Let τ be the sum of all $\tau^{1-i,e',k'}$ such that $R_{1-i,e',k'}$ has weaker priority than $R_{i,e,k}$. Let $s_l > t$ be the lth stage at which $R_{i,e,k}$ requires attention. If $R_{1-i,e',k'}$ is the pth requirement on the priority list and $p > j$ then $\tau^{i',e',k'} - \tau_{s_l}^{i',e',k'} \leqslant 2^{-(p+l)}\Omega$. Thus

$$\tau - \tau_{s_l} \leqslant \sum_{p \geqslant 1} 2^{-(p+l)}\Omega = 2^{-l}\Omega \leqslant 2^{-l},$$

and hence τ is computable.

Putting together the results of the previous two paragraphs, we see that $\gamma + \beta^i \equiv_{\mathrm{rH}} \gamma <_{\mathrm{rH}} \alpha$. It follows that there is an $n \in \omega$ such that $R_{i,e,k}$ is eventually permanently satisfied through n, and such that $R_{i,e,k}$ is eventually never satisfied through any $n' < n$. Thus $\lim_s n_s^{i,e,k}$ exists and is finite, and hence $R_{i,e,k}$ is satisfied and eventually stops requiring attention. $\square$

A.4. Theorem. *Let $\gamma <_{rH} \Omega$ be a c.e. real. There is a c.e. real β such that $\gamma <_{rH} \beta <_{rH} \Omega$.*

Proof. We want to build $\beta \geqslant_{\mathrm{rH}} \gamma$ to satisfy the following requirements for each $e, k \in \omega$:

$$R_{e,k} : \exists n \forall j \leqslant k (\Phi_e(\gamma \upharpoonright n, j)\downarrow \Rightarrow \Phi_e(\gamma \upharpoonright n, j) \neq \beta \upharpoonright n).$$

and

$$S_{e,k} : \exists n \forall j \leqslant k (\Phi_e(\beta \upharpoonright n, j)\downarrow \Rightarrow \Phi_e(\beta \upharpoonright n, j) \neq \Omega \upharpoonright n).$$

(We will in fact make $\beta \geqslant_{\mathrm{S}} \gamma$.)

As in the previous proof, the analysis of an appropriate two-strategy case will be enough to outline the essentials of the full construction. Let us consider the strategies $S_{e,k}$ and $R_{e',k'}$, the former having priority over the latter.

The strategy for $S_{e,k}$ is basically to make β look like γ. At each point of the construction, $R_{e',k'}$ has a certain fraction of Ω that it is allowed to put into β. (This is in addition to the coding of γ into β, of course.) We will say that $S_{e,k}$ is satisfied through n at stage s if $\forall j \leqslant k(\Phi_e(\beta_s \upharpoonright n, j)[s]\downarrow \Rightarrow \Phi_e(\beta_s \upharpoonright n, j) \neq \Omega_s \upharpoonright n)$. Whenever either it is not currently satisfied or the least number through which it is satisfied changes, $S_{e,k}$ initializes $R_{e',k'}$, which means that the fraction of Ω that $R_{e',k'}$ is allowed to put into β is reduced.

As in the previous proof, if $S_{e,k}$ is not eventually permanently satisfied through some n then the amount put into β by $R_{e',k'}$ is computable, and hence $\beta \equiv_{\mathrm{rH}} \gamma$. But, as before, this implies that there is a stage after which $S_{e,k}$ is permanently satisfied through some n and never again satisfied through any $n' < n$. Once this stage has been reached, $R_{e',k'}$ is free to code a fixed fraction of Ω into β, and hence it too succeeds.

We now proceed with the full construction. We say that a requirement $X_{e,k}$ has stronger priority than a requirement $Y_{e',k'}$ if either $\langle e, k\rangle < \langle e', k'\rangle$ or $\langle e, k\rangle = \langle e', k'\rangle$, $X = R$, and $Y = S$.

We say that $R_{e,k}$ is satisfied through n at stage s if

$$\forall j \leqslant k(\Phi_e(\gamma_s \upharpoonright n, j)[s]\downarrow \Rightarrow \Phi_e(\gamma_s \upharpoonright n, j) \neq \beta_s \upharpoonright n).$$

We say that $S_{e,k}$ is satisfied through n at stage s if

$$\forall j \leqslant k(\Phi_e(\beta_s \upharpoonright n, j)[s]\downarrow \Rightarrow \Phi_e(\beta_s \upharpoonright n, j) \neq \Omega_s \upharpoonright n).$$

For a requirement $X_{e,k}$, let $n_s^{X_{e,k}}$ be the least n through which $X_{e,k}$ is satisfied at stage s, if such an n exists, and let $n_s^{X_{e,k}} = \infty$ otherwise.

We say that a requirement $X_{e,k}$ requires attention at stage s if either $n_s^{X_{e,k}} = \infty$ or $n_s^{X_{e,k}} \neq n_{s-1}^{X_{e,k}}$.

At stage s, proceed as follows. First add $\gamma_s - \gamma_{s-1}$ to the current value of β. If no requirement requires attention at stage s then end the stage. Otherwise, let $X_{e,k}$ be the strongest priority requirement requiring attention at stage s. We say that $X_{e,k}$ acts at stage s. If $X = S$ then initialize all weaker priority requirements and end the stage. If $X = R$ then let $j = \langle e, k\rangle$ and let m be the number of times that $R_{e,k}$ has been initialized. If s is the first stage at which $R_{e,k}$ acts after the last time it was initialized then let t be the last stage at which $R_{e,k}$ was initialized, and otherwise let t be the last stage at which $R_{e,k}$ acted. Add $2^{-(j+m)}(\Omega_s - \Omega_t)$ to the current value of β and end the stage.

This completes the construction. Since β is bounded by $\gamma + \sum_{i \geqslant 0} 2^{-i}\Omega = \gamma + 2\Omega$, it is a well-defined c.e. real. Furthermore, $\gamma \leqslant_{\mathrm{S}} \beta$.

We now show by induction that each requirement initializes requirements of weaker priority only finitely often and is eventually satisfied. Assume by induction that there is a stage u such that no requirement of stronger priority than $X_{e,k}$ requires attention after stage u. The following are clearly equivalent:

1. $X_{e,k}$ is satisfied,
2. $\lim_s n_s^{X_{e,k}}$ exists and is finite,
3. $X_{e,k}$ eventually stops requiring attention, and
4. $X_{e,k}$ acts only finitely often.

First suppose that $X = R$. Let $j = \langle e, k \rangle$ and let m be the number of times that $R_{e,k}$ is initialized. (Since $R_{e,k}$ is not initialized at any stage after stage u, this number is finite.) Suppose that $R_{e,k}$ acts infinitely often. Then the total amount added to β for the sake of $R_{e,k}$ is $2^{-(j+m)}\Omega$, and hence $\beta \equiv_{\mathrm{rH}} 2^{-(j+m)}\Omega \equiv_{\mathrm{rH}} \Omega \nleqslant_{\mathrm{rH}} \gamma$. It follows that there is an $n \in \omega$ such that $R_{e,k}$ is eventually permanently satisfied through n, and such that $R_{e,k}$ is eventually never satisfied through $n' < n$. Thus $\lim_s n_s^{R_{e,k}}$ exists and is finite, and hence $R_{e,k}$ is satisfied and eventually stops requiring attention.

Now suppose that $X = S$ and $S_{e,k}$ acts infinitely often. If $v > u$ is the mth stage at which $S_{e,k}$ acts then the total amount added to β after stage v for purposes other than coding γ is bounded by $\sum_{i \geqslant 0} 2^{-(i+m)}\Omega < 2^{-m+1}$. This means that $\beta \equiv_{\mathrm{rH}} \gamma \ngeqslant_{\mathrm{rH}} \Omega$. It follows that there is an $n \in \omega$ such that $S_{e,k}$ is eventually permanently satisfied through n, and such that $S_{e,k}$ is eventually never satisfied through $n' < n$. Thus $\lim_s n_s^{S_{e,k}}$ exists and is finite, and hence $S_{e,k}$ is satisfied and eventually stops requiring attention. $\square$