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The Poincaré–Hardy Inequality on the Complement of a Cantor Set

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The Poincaré–Hardy Inequality on the Complement of a Cantor Set

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Abstract

The Poincaré–Hardy inequality

$$\int \frac{|u|^2}{\mathrm{dist}^2(x,E)}\, dm \leq \mathcal{K}^2 \cdot \int |\bigtriangledown\ u|^2 dm$$

is derived in $\mathcal{R}_3$ on the complement of a Cantor set E. We use a special self-similar tiling and a natural metric on the space of trajectories generated by a Mauldin–Williams graph which is homeomorphic with the space of tiles endowed with the Euclidean distance. A crude estimation of the constant $\mathcal{K}^2$ is 2,100. Two applications will be briefly discussed. In the last one, the constant $\mathcal{K}^{-1} \approx 0.021819$ plays the role of an estimate for the dimensionless Plank constant in the corresponding uncertainty principle.

1 Introduction

In Classical Analysis the Poincaré–Hardy inequality (see, for example, Hardy, Littlewood, Polya [8] or [6]) is one of most popular tools for comparing the generalized smoothness of a given function and its square integrability with a singular weight–function. The inequality is also used in Quantum Mechanics for deriving the uncertainty principle Schiff [20] and in Mathematical Hydrodynamics for proving the existence and uniqueness of solutions of Navier–Stokes equations, Ladyzhenskaja [13]. Combined with Garding inequality [7] it proves a surprisingly sharp instrument of qualitative spectral analysis of differential operators [2]; it even appears as a central point of the proof of semi-boundedness of solvable few–body Hamiltonians in Quantum Scattering [17]. A version of Poincaré–Hardy inequality on the complement of a uniformly δ-regular set was derived in [1] in connection with the question on the uniqueness of the solution of the Dirichlet problem for second order elliptic equations in a domain with a uniformly δ–regular boundary. The uniform δ–regularity is equivalent to the existence of the corresponding superharmonic strong barrier function (see Theorem 2 in [1]) and is invariant under conformal transformations of the space (an equivalent of uniform perfectness), [19, 10].

The necessity to have a convenient tool for analysis of Dirichlet forms in Hilbert spaces of square integrable functions with singular weights requires Poincaré–Hardy inequalities in multidimensional spaces on complements of perfect zero–measure sets (fractals) with *sharp estimates of corresponding constants.*

In the present note we derive the simplest version of the Poincaré–Hardy inequality on the complement of a Cantor set in $\mathcal{R}_3$. We have chosen the Cantor set because of its simplicity and usefulness (Cantor sets are highly useful mathematical models for physical phenomena which include, for example, the distribution of galaxies in the universe and the fractal structure of the rings of Saturn, see Pickover [18], or [9]). We reduce the proof of Poincaré–Hardy inequality to the estimation of a discretized integral which appears from the analysis of an analog of the strong barrier function, see Theorem 3.1 below. This estimation is based on the generating Mauldin–Williams graph of the Cantor set together with a proper measure constructed on all cylinder sets of trajectories generated by the generating finite automaton, see

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Calude [4].[1] The above measure leads to a metric space homeomorphic with the space of tiles endowed with the Euclidean distance.

This paper describes a simple case study of a connection between the analysis of smooth functions on the complement of a uniformly δ-regular set (or just a zero–measure perfect set), on one hand, and Symbolic Dynamics (see, for example, Schuster [21], Lind and Marcus [14]), on the other hand. Although the phenomenon studied is analytically trivial, still the characteristic features of a possible general construction can be already seen here:

• a special self-similar tiling of a neighborhood of a singular set, parameterized by trajectories generated by some Mauldin–Williams graph which defines the authomorphisms of the set.

• a homeomorphism between an Euclidean metric structure on the tiling and the metric space of trajectories.

It is obvious that the above structures contain more information on the underlying set than just the uniform δ–regularity, so they may be used for a more precise estimation of the constant in the Poincaré–Hardy inequality (or even for deriving new versions of it).

In what follows we will also compute an estimation of the constant $\mathcal{K}^2$ appearing in Poincaré–Hardy inequality. Our constant is certainly not the best; sharper estimates need more accurate operations with integrals on tiles.

2 Prerequisites

We denote by Σ the binary alphabet $\{0,1\}$ and by Σ^* the set of all non-empty binary strings, i.e., $\Sigma^* = \{0,1,00,01,10,11,000,\ldots\}$. If $a = a_1a_2\ldots a_n$ is a string of n digits, then its length is denoted by $|a| = n$. By Σ^l we denote the set of strings of length l. The *concatenation* of two strings a, c is denoted by ac. A string a is a prefix of a string b (we write $a \subset b$) in case $b = ac$, for some $c \in \Sigma^*$. The *negation* of a string $a \in \{0,1\}$ is denoted by $\bar{a} = a - 1$, so that $\bar{0} = 1$, $\bar{1} = 0$. For $a, d \in \Sigma^*$ we denote by $a \cap d$ the maximum common prefix of the strings a, d. Clearly, $|a \cap b| \le \min\{|a|, |b|\}$, and $|a \cap d| = |a|$ if and only if $a \subset d$. Let Σ^ω be the set of all infinite binary sequences. In analogy with the case of strings, if σ and τ are two distinct sequences, then $\sigma \cap \tau$ denotes the maximum common prefix of σ and τ; of course, $\sigma \cap \tau$ is a string. If σ and τ are two distinct sequences in Σ^ω and r is a real number in the unit interval $(0,1)$, then $\delta_r(\sigma,\tau) = r^{|\sigma\cap\tau|}$ is an *ultrametric* on Σ^ω. The space $(\Sigma^\omega, \delta_r)$ is complete, compact and separable. For different r, s in $(0,1)$, the spaces $(\Sigma^\omega, \delta_r)$ and $(\Sigma^\omega, \delta_s)$ are homeomorphic. For more information see Edgar [5].

A *middle third Cantor set* is constructed by removing successive open middle thirds from a sequence of closed intervals. In the traditional construction, the one we are going to use in this paper, we are starting from the interval $\Delta = [0,1]$ (the pre-Cantor set of zero order) from which we remove the "middle third" $(1/3, 2/3)$ on the first step, leaving the union of closed intervals $\Delta_0 = [0, \frac{1}{3}]$ and $\Delta_1 = [\frac{2}{3}, 1]$. The set $\Delta_0 \cup \Delta_1$ is called the *pre-Cantor set* of the first order. The endpoints of the closed intervals constitute its *skeleton.* In the second step we remove the middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ respectively from the intervals Δ_0, Δ_1, and thus obtain the closed intervals

$$\Delta_{00} = [0, \frac{1}{3^2}],\ \Delta_{01} = [\frac{2}{3^2}, \frac{1}{3}],\ \Delta_{10} = [\frac{2}{3}, \frac{7}{3^2}],\ \Delta_{11} = [\frac{8}{3^2}, 1],$$

which constitute the pre-Cantor set of the second order, and so on. For example, the skeleton of Δ is $\mathcal{E}_0 = \{0,1\}$, the skeleton of $\Delta_0 \cup \delta_1$ is $\mathcal{E}_1 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. This procedure continues indefinitely. The Cantor set E is defined as the intersection of the countable sequence of pre-Cantor sets E_a formed by all closed intervals enumerated by all binary strings a length $|a| = l$:

$$E = \bigcap_{l=0}^{\infty} E_l,\ E_l = \bigcup_{|a|=l} \Delta_a.$$

The endpoints of intervals constituting the pre-Cantor set E_l of order l, form the corresponding skeleton and are enumerated by all binary strings length $l+1$, that is, two strings $a0, a1$ correspond to each interval Δ_a. The first steps of this construction are pictured in Figure 1. The Cantor set is compact, perfect and has length zero.

[1] Trajectories may be represented as paths on a binary Bruhat–Tits tree, [3].

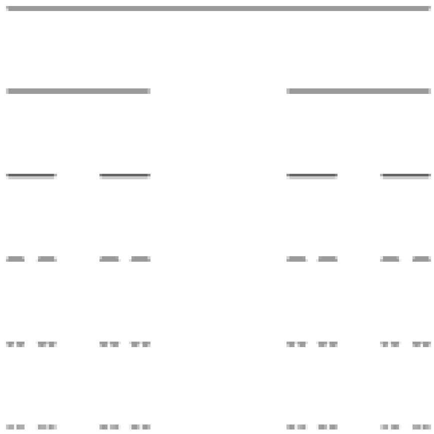

Figure 1: Cantor set

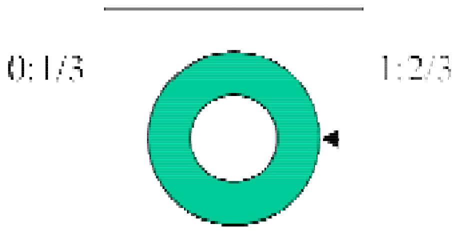

Figure 2: Mauldin–Williams graph for Cantor set

A convenient way to work with the Cantor set is to consider the Mauldin–Williams graph (see Edgar [5]; equivalently, we could use a non-deterministic automaton as in [12]) in Figure 2, the contracting ratio list $(r_0, r_1) = (1/3/, 1/3)$, and the functions $f_0, f_1 : [0,1] \to [0,1]$ defined by $f_0(x) = x/3, f_1(x) = (x+2)/3$. Let $r_\Lambda = 1$ (Λ is the empty string), $r_{\alpha i} = r_\alpha \cdot r_i$, for $\alpha \in \Sigma^*, i \in \Sigma$ and define $\delta(\sigma,\tau) = r_{\sigma\cap\tau}$. It is seen that δ is an ultrametric and, in fact, $\delta(\sigma,\tau) = 3^{-|\sigma\cap\tau|} = \rho_{1/3}(\sigma,\tau)$. According to Theorem 4.2.3 in [5] there exists a unique continuous function $h : \Sigma^\omega \to [0,1]$ satisfying the following two conditions:

1. $h(i\sigma) = f_i(h(\sigma))$, for all $i \in \Sigma, \sigma \in \Sigma^\omega$,
2. $h(\Sigma^\omega) = E$.

The function h can be defined inductively by the following equations:

$$h(0\sigma) = \frac{h(\sigma)}{3},\ h(1\sigma) = \frac{h(\sigma)+2}{3}, \tag{1}$$

for all $\sigma \in \Sigma^\omega$, see Edgar [5]. For example, $h(0101010101\ldots) = 1/4$ because of the equality $h(0101010101\ldots) = \frac{1}{9}(2 + h(0101010101\ldots))$.

The map h has a "bounded distortion" with respect to the ultrametric $\delta_{1/3}$, that is, for every $\sigma,\tau \in \Sigma^\omega$,

$$\frac{1}{3}\,\delta_{1/3}(\sigma,\tau) \le |\ h(\sigma) - h(\tau)\ | \le \delta_{1/3}(\sigma,\tau). \tag{2}$$

Re-phrased, the ultrametric $\delta_{1/3}(\sigma,\tau)$ on the set of binary sequences is equivalent to Euclidean distance between $h(\sigma), h(\tau)$. Note that h does not have the above property with respect to any other ultrametric δ_r with $r \neq 1/3$.

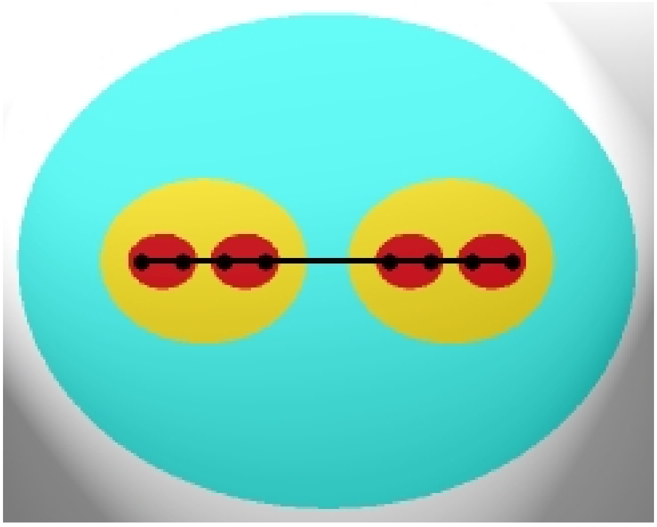

Figure 3: Cantor tiling

We now define a *special tiling* of a neighborhood of the Cantor set by extending the map h to the elliptic body Ω with focuses 0, 1

$$\Omega = \{x : |x| + |x-1| \leq 5/3\}\,,\ \text{diam}\ \Omega = 5/3.$$

We shall see below that the sum of all tiles enumerated by these sequences gives an elliptic body Ω_a, diam $\Omega_a = 5 \cdot 3^{-|a|-1}$, and the metric space of trajectories is homeomorphic to the space of tiles endowed with the Euclidean distance, see Lemma 4.1.

We denote by $W_2^1(\mathcal{R}_3)$ the Sobolev space of all square-integrable functions on $\mathcal{R}_3$ which have square-integrable derivatives of the first order. This is a complete Hilbert space endowed with the dot-product

$$< u \cdot v >_{W_2^1(\mathcal{R}_3)} = \int_{\mathcal{R}_3} \left(\langle \overline{\bigtriangledown u} \bigtriangledown u \rangle + \bar{u} v \right) dx^3,$$

and the corresponding norm

$$|u|_{W_2^1(\mathcal{R}_3)} = \sqrt{< u \cdot u >_{W_2^1(\mathcal{R}_3)}}\,.$$

For more details about the Sobolev classes which we will be used below see [15].

We denote by dist the Euclidean distance. A set E is bounded in case $\sup_{x \in E}$ dist$(x, 0) < \infty$. A *fractal* is a closed zero-measure set E in $\mathcal{R}_3$ such that the function $d_E(x) = \text{dist}(x, E)$ is Lip_1-continuous:

$$|d_E(x) - d_E(y)| \leq C(K) \cdot |x - y|,$$

on each compact subset K of the complement E' of E in $\mathcal{R}_3$.[2]

3 Poisson construction

The Lebesgue measure $\mu(\delta)$ of the δ–neighborhood $E_\delta = \{x \,|\, \text{dist}(x, E) < \delta\}$ of E in $\mathcal{R}_3$ is a "sufficiently smooth" function of δ and can be generally estimated, for small δ, as

$$\mu(\delta) = \int_{E_\delta} dm \leq C(\alpha) \delta^{3-\alpha},$$

[2]This condition is automatically fulfilled for any set which has a finite skeleton $\left\{x_{E,s}^K\right\} \in E$, $s = 1, 2, \ldots, N$ for each compact $K \in E'$ such that dist(x, E) can be bilaterally estimated by $\min_{s=1,2,\ldots,N} |x - x_{E,s}^K|$. Cantor set obviously has this property.

with finite non-negative $\alpha \leq 3$ and some positive $C(\alpha)$. The lower bound α_E of values of the parameter α for which this estimate holds is called the *Minkowski dimension* $\dim_E = \alpha_E$ of the set E, see for instance Edgar [5] and the literature quoted there. The Minkowski dimension of sets in $\mathcal{R}_n$ may be defined in a similar way; it does not depend on the dimension n and may take any nonnegative value less than n. In the most interesting cases the Minkowski dimension coincides with the *Hausdorff dimension* [5]. In particular, the Minkowski dimension of the above Cantor set E is equal to $\frac{\log 2}{\log 3}$.[3]

The following general statement serves as a base for our calculations in the next section.

Theorem 3.1 *For every function $u \in W_2^1(\mathcal{R}_3)$ and every bounded fractal $E \in \mathcal{R}_3$ with $dim_E < 1$ fulfilling the condition*

$$\mathcal{K}_E = \sup_{x\in E} \frac{d_x}{4\pi} \int \frac{1}{|x-s|^2} \frac{dm(s)}{d_E^2(x)} < \infty, \tag{3}$$

the Poincaré–Hardy inequality holds with the constant $\mathcal{K}_E^2$:

$$\int \frac{u^2}{d_E^2(x)}\, dm \leq \ \mathcal{K}_E^2 \cdot \int |\bigtriangledown u|^2\, dm. \tag{4}$$

Proof. It is sufficient to obtain the inequality (4) for any smooth function u with a compact support in the complement E' of E in $\mathcal{R}_3$.

If the Minkowski dimension α_E is less than 1, then the function $d_E^{-2}(x) = \text{dist}^{-2}(x, E)$ being Lip_1-continuous on any compact in $E' = \mathcal{R}_3 \setminus E$ is integrable on any bounded neighborhood E_δ. Indeed, we may rewrite the integral $\int_{E_1} d_E^{-2}(x)\, dm$ as $\int_0^1 \delta^{-2}\, d\mu(\delta)$ and then reduce it via integration by parts for any $\alpha \in (\alpha_E, 1)$ to the following form:

$$\lim_{\delta\to 0} \left(2\int_\delta^1 s^{-3}\mu(s)\, ds + \mu_(1) - \frac{\mu(\delta)}{\delta^2}\right) \leq \lim_{\delta\to 0}\left(\mu(1) + C(\alpha)\frac{1}{1-\alpha}(1-\delta^{1-\alpha})\right) < \infty.$$

Hence the function d_E^{-2} is integrable on any bounded domain in $\mathcal{R}_3$. Then we consider a Poisson equation

$$-\bigtriangleup v + \kappa^2 u = \frac{1}{d_E^2},\ \kappa^2 > 0, \tag{5}$$

and represent its generalized solution via the corresponding Green function

$$v(x) = \int_{\mathcal{R}_3} \frac{e^{-\kappa|x-s|}}{4\pi|x-s|} \frac{dm(s)}{d_E^2(s)}, \tag{6}$$

on any compact subdomain of the complement E' of E in $\mathcal{R}_3$. The generalized solution (6) is twice continuously differentiable, $v \in C^{2+\beta}(K)\beta > 0$, which permits the integration by parts for any smooth real function u with a compact support K_u in E':

$$\int \frac{u^2}{d_E^2(x)}\, dm = -\int u^2 \bigtriangleup v\, dm + \kappa^2 \int u^2 v\, dm = \int u < \bigtriangledown u, \bigtriangledown v > \, dm + \kappa^2 \int u^2 v\, dm,$$

so, the following estimate holds true for any positive κ:

$$\int \frac{u^2}{d_E^2(x)}\, dm \leq \left(\int \frac{u^2}{d_E^2(x)}\, dm\right)^{1/2} \cdot \left(4 \cdot \int |\bigtriangledown u|^2 \left(|\bigtriangledown v| d_E(x)\right)^2 dm\right)^{1/2} + \kappa^2 \cdot \int u^2 v\, dm. \tag{7}$$

We can estimate $\bigtriangledown v$ as

$$|\bigtriangledown v(x)| = |\int (1+\kappa|x-s|) \frac{e^{-\kappa|x-s|}}{4\pi|x-s|^2} \frac{x-s}{|x-s|} \frac{dm(s)}{d_E^2(x)}\ |$$

$$\leq \int \frac{e^{-\kappa|x-s|}}{4\pi|x-s|^2} \frac{dm(s)}{d_E^2(x)} + \kappa \cdot \int \frac{e^{-\kappa|x-s|}}{4\pi|x-s|} \frac{dm(s)}{d_E^2(x)}. \tag{8}$$

[3]In fact, the Minkowski dimension of the Cantor set embedded into any $\mathcal{R}_n$ does not depend on n and is equal to $\frac{\log 2}{\log 3}$.

Together with (7), for fixed u, (8) gives:

$$\int \frac{u^2}{d_E^2(x)}\, dm \le \lim_{\kappa \to 0} \Bigg\{ \left(4 \cdot \int \frac{u^2}{d_E^2(x)}\, dm\right)^{1/2} \left(\int |\bigtriangledown u(x)|^2 \cdot \left[\int \frac{e^{-\kappa|x-s|}}{4\pi|x-s|^2} \frac{dm(s)}{d_E^2(x)}\right.\right. \tag{9}$$

$$\left.\left. +\kappa \cdot \int \frac{e^{-\kappa|x-s|}}{4\pi|x-s|} \frac{dm(s)}{d_E^2(x)}\right]^2 dm(x)\right)^{1/2} + \kappa^2 \cdot \int u^2 v \, dm \Bigg\}.$$

which implies, after passing to the limit $\kappa \to 0$, the inequality:

$$\int \frac{u^2}{d_E^2(x)}\, dm \le \left[\sup_x \frac{d_x}{\pi} \int \frac{1}{|x-s|^2} \frac{dm(s)}{d_E^2(x)}\right]^2 \cdot \int |\bigtriangledown u(x)|^2 \, dm(x). \tag{10}$$

The final result can be obtained now by taking the closure of (10) in the Sobolev space $W_2^1(\mathcal{R}_3)$. $\square$

In the remaining part of this note we will derive a crude estimate for the constant $\mathcal{K}^2$ for the Cantor set E. Our estimate is not optimal; however, our analysis of the discretized integral representing $\mathcal{K}^2$ reveals that the main part of this constant appears from an estimate of some infinite sum over a special tiling described in the following section. This tiling appears from the extension $\mathcal{H}$ of the parameterizing map of the Cantor set onto some neighbourhood of it in $\mathcal{R}_3$, see the construction in the next section.

4 A special tiling

Consider the Cantor set E on x–axis in $\mathcal{R}_3$ and denote by e_1 the unit vector looking at the positive direction of the x–axis. Consider a tiling of the whole space $\mathcal{R}_3$ formed by the complement $\mathcal{R}_3 \setminus \Omega$ of the rotation-symmetric elliptic body Ω bordered by the ellipsoid Ω with focuses in $0, e_1$, that is on the skeleton of zero-order pre-Cantor set $\Delta = [0,1]$ on the x-axis:

$$\Omega = \left\{ x \,:\, |x| + |x - e_1| \le \frac{5}{3} \right\}.$$

Next we consider the map $\mathcal{H} : \Sigma \times \Omega \to \Omega$ defined for each $x \in \Omega$ as a splitting of one point x into two images:[4]

$$\mathcal{H}(0,x) = \frac{x}{3},\ \mathcal{H}(1,x) = \frac{2e_1 + x}{3}.$$

The function $\mathcal{H}$ can be extended in a natural way to a function, also denoted by $\mathcal{H}$, from $\Sigma^* \times \Omega$ into Ω by

$$\mathcal{H}(ia, x) = \mathcal{H}(i, \mathcal{H}(a,x)),$$

for all $i \in \Sigma, a \in \Sigma^*$ and $x \in \Omega$. Clearly, $\mathcal{H}(ab,x) = \mathcal{H}(a, \mathcal{H}(b,x))$, for all $a, b \in \Sigma^*$ and $x \in \Omega$.

The image $\mathcal{H}(\Sigma\times\Omega)$ consists of two components–two similar elliptic bodies $\Omega_0 = \mathcal{H}(0,\Omega) = \frac{1}{3}\Omega,\ \Omega_1 = \mathcal{H}(1,\Omega) = (\frac{2e_1}{3} + \frac{1}{3})\Omega$,

$$\Omega_0 = \left\{ x \,:\, |x| + |x - \frac{1}{3}e_1| \le \frac{5}{3^2} \right\},$$

$$\Omega_1 = \left\{ x \,:\, |x - \frac{2}{3}e_1| + |x - e_1| \le \frac{5}{3^2} \right\},$$

with focuses at the *skeleton* $\mathcal{E}_1$ of the first-order pre-Cantor set $E_1 = \Delta_0 \cup \Delta_1$, $\mathcal{E}_1 = \{r_{00}, r_{01}, r_{10}, r_{11}\}$:

$$r_{00} = 0,\ r_{01} = \frac{1}{3},\ r_{10} = \frac{2}{3},\ r_{11} = 1,$$

and the basic tile ω is formed as a complement $\Omega \setminus \mathcal{H}(\Sigma \times \Omega) = \Omega \setminus (\Omega_0 \cup \Omega_1)$.

On the next step we form two tiles ω_0, ω_1 of the first order which are similar to ω and are defined respectively as the complement

$$\omega_0 = \mathcal{H}(0,\Omega) \setminus (\mathcal{H}(00,\Omega) \cup \mathcal{H}(01,\Omega)) = \Omega_0 \setminus (\Omega_{00} \cup \Omega_{01})$$

[4] Note that $\mathcal{H}$ is the extension of h defined by (1).

of the bodies

$$\Omega_{00} = \left\{ x \,:\, |x - 0| + |x - \frac{1}{9}e_1| \le \frac{5}{3^3} \right\},$$

$$\Omega_{01} = \left\{ x \,:\, |x - \frac{2}{9}e_1| + |x - \frac{1}{3}e_1| \le \frac{5}{3^3} \right\}$$

in Ω_0 and the complement

$$\omega_1 = \mathcal{H}(1,\Omega) \setminus (\mathcal{H}(10,\Omega) \cup \mathcal{H}(11,\Omega)) = \Omega_1 \setminus (\Omega_{10} \cup \Omega_{11})$$

of the bodies

$$\Omega_{10} = \left\{ x \,:\, |x - \frac{6}{9}e_1| + |x - \frac{7}{9}e_1| \le \frac{5}{3^3} \right\},$$

$$\Omega_{11} = \left\{ x \,:\, |x - \frac{8}{9}e_1| + |x - e_1| \le \frac{5}{3^3} \right\}$$

in Ω_1. The focuses of ellipsoids bordering Ω_{00}, Ω_{01}, Ω_{10}, Ω_{11} form the skeleton $\mathcal{E}_2$ of the second-order pre-Cantor set $E_2 = \Delta_{00} \cup \Delta_{01} \cup \Delta_{10} \cup \Delta_{11}$ and are enumerated by all binary strings length 3: $r_{00} = r_{000} = 0$, $r_{00} + \frac{1}{3^2} = r_{001} = \frac{1}{3^2}$, $r_{01} = r_{010} = \frac{2}{3^2}$, $r_{01} + \frac{2}{3^2} = r_{011} = \frac{3}{3^2} = \frac{1}{3}$, $r_{10} = r_{100} = \frac{2}{3}$, $r_{10} + \frac{1}{3^2} = r_{101} = \frac{7}{3^2}$, $r_{10} + \frac{2}{3^2} = r_{110} = \frac{8}{3^2}$, $r_{10} + \frac{3}{3^2} = r_{111} = 1$.

The construction of the following tiles can be described by induction. On each step $l, |a| = l - 1$ we begin from the result of the previous step–the set of 2^{l-1}, non-intersecting elliptic bodies Ω_a bordered by the ellipsoids

$$\Omega_a = \left\{ x \,:\, |x - r_{a0}| + |x - r_{a1}| \le \frac{5}{3^l} \right\},$$

with focuses at the skeleton $\mathcal{E}_l$ of the pre-Cantor set $E_l = \cup_{|b|=l}\Delta_b$ enumerated by all possible binary strings $b = a0, a1$ of length l. Then we continue the construction by forming the tiles ω_a as complements $\omega_a = \Omega_a \setminus (\Omega_{a0} \cup \Omega_{a1})$ in Ω_a of the elliptic bodies, respectively bordered by the ellipsoids

$$\Omega_{a0} = \left\{ x \,:\, |x - r_{a00}| + |x - r_{a01}| = \frac{5}{3^{l+1}} \right\},$$

and

$$\Omega_{a1} = \left\{ x \,:\, |x - r_{a10}| + |x - r_{a11}| = \frac{5}{3^{l+1}} \right\},$$

and so on. Hence, for every $a \in \Sigma^*$, $\mathcal{H}(a,\Omega) = \Omega_a$ and

$$\omega_a = \mathcal{H}(a,\Omega) \setminus (\mathcal{H}(a0,\Omega) \cup \mathcal{H}(a1,\Omega)) = \Omega_a \setminus (\Omega_{a0} \cup \Omega_{a1}).$$

The following Lemma 4.1 will be used to derive bilateral estimates for the coefficient $\mathcal{K}^2$ in (4) in terms of the constructed tiling. We enumerate the tiles by binary strings b.[5]

Lemma 4.1 *The sets $\{\omega, \omega_1, \omega_2, \omega_c\}$, enumerated by all possible binary strings c, $|c| \ge 0$ form a tiling for the elliptic body Ω with the following properties:*

1. *The distance $d_E(x)$ from the point $x \in \omega_c$ to the Cantor set E may be bilaterally estimated by the distance $d_{|c|}(x)$ from x to the skeleton $\mathcal{E}_{|c|}$ of the pre-Cantor set $E_{|c|}$. In particular, the ratio $d_{|c|}(x)/d_E(x)$ takes the minimal and maximal values on the border $\partial\Omega_c$, $\partial\Omega_{c0}$, $\partial\Omega_{c1}$ of the tile ω_c and*

$$1 \le \frac{d_{|c|}(x)}{d_E(x)} \le 4, \ x \in \partial\Omega_{c0} \cup \partial\Omega_{c1},$$

$$1 \le \frac{d_{|c|}(x)}{d_E(x)} \le \frac{5}{\sqrt{17}}, \ x \in \partial\Omega_c. \tag{11}$$

The Euclidean volume of the tile ω_c is equal to $10^3\pi 3^{=3|c|-7}$ and the distance from the Cantor set E to $x \in \omega_c$ can be bilaterally estimated as

$$3^{-|c|-2} \le d_E(x)|_{x\in\omega_c} \le \frac{\sqrt{17}}{2} \cdot 3^{-|c|-1}.$$

[5]Recall that $a \cap b$ is the maximal common prefix of the strings a, b.

2. *The distance between the points $x_a \in \omega_a$ and $x_b \in \omega_c$ may be estimated from above as:*

$$|x_a - x_c| \leq 5 \cdot 3^{-|a \cap c|-1}. \tag{12}$$

If the the tiles ω_a, ω_c do not contact each other (that is, do not have a common piece of the boundary), then the distance between the points $x_a \in \omega_a$ and $x_c \in \omega_c$ may be estimated from below as

$$|x_a - x_c| \geq 3^{-|a \cap c|-2}. \tag{13}$$

Proof. Our tiling is self-similar, hence the estimate (11), if derived for the basic tile ω and the tiles ω_0, ω_1 of the first order, remains true, under proper scaling, for the whole tiling. Note, for instance, that the ratio $d_1(x)/d_E(x)$ takes the minimal and maximal values on the boundary of the tile ω and can be estimated as

$$1 \leq \frac{d_1(x)}{d_E(x)} \leq 4,\ x \in \partial\Omega_0 \cup \partial\Omega_1,$$

and

$$1 \leq \frac{d_1(x)}{d_E(x)} \leq \frac{5}{17},\ x \in \partial\Omega. \tag{14}$$

Similarly, the last part of the first statement follows from the estimate

$$\frac{1}{9} \leq d_E(x)|_{x \in \omega} \leq \frac{\sqrt{17}}{6}.$$

To prove the last part of the second statement we notice that from the condition $U_a \cap \omega_c = \emptyset$ follows that either $\omega_a \in \omega_{d0}$, $\omega_c \in \omega_{d1}$, for some string d of length k, or vice-versa. This implies the announced inequality:

$$|x_a - x_c| \geq \operatorname{dist}(\omega_{d0}, \omega_{d1}) = 3^{-|k|-2}.$$

□

Notice, that for every string a, the map $\mathcal{H}(a, \cdot)$ acts transitively on the constructed tiling, mapping each l-generation of tiles $\cup_{|a|=l}\omega_a$ into the following $l+1$-generation $\cup_{|a|=l+1}\omega_a$. The same function maps the l-generation of elliptic bodies $\mathcal{H}(\Sigma^l, \Omega) = \cup_{|a|=l}\Omega_a$ into itself. One can easily see that the Cantor set E is an invariant set of the map $\mathcal{H}$ similarly to the corresponding property of the parameterizing map $h : [0,1] \to [0,1]$. The restriction $\mathcal{H}$ onto $[0,1]$ coincides with h given by (1). This is the exact meaning of the statement at the end of the previous section, that the special tiling is formed by the continuation $\mathcal{H}$ of the parameterizing map h onto Ω. The transitive action of the map $\mathcal{H}$ on the tiling permits to define an analog of the unilateral shift on the orthogonal sum of Hilbert spaces $\oplus \sum_a \mathcal{L}_2(\omega_a)$.

5 Estimates for the discretized integral

We begin this section with a few preliminary results. For a given tile ω_b we consider a triple of its closest neighbours: its *mother* ω_a of ω_b such that $b = a0$ or $b = a1$ and two *daughters* ω_{b0}, ω_{b1} which form together with ω_b the *cut* of the corresponding Bruhat–Tits tree at the level b. We denote $\omega_b \cap \omega_a \cap \omega_{b0} \cap \omega_{b1}$ by U_b and consider its complement $\Omega \setminus U_b$ in Ω which is represented by joining all remaining tiles

$$\Omega \setminus U_b = \bigcup_c \omega_c,\ c \neq a, b, b0, b1. \tag{15}$$

First note that *for $x \in \omega_1$, the integral over $U_1 = \omega \cup \omega_1 \cup \omega_{10} \cup \omega_{11}$,*

$$\mathcal{J}_1(x) = \frac{1}{4\pi} \int_{U_1} \frac{1}{|x-s|^2} \frac{dm(s)}{d_E^2(x)},$$

is a uniformly continuous function of $x \in \omega_1$, and there exist two absolute constants A_1, B_1 such that

$$A_1 \leq \frac{d_E(x)}{4\pi} \int_{U_1} \frac{1}{|x-s|^2} \frac{dm(s)}{d_E^2(s)} \leq B_1. \tag{16}$$

An obviously crude but still reasonable numerical estimate is:

$$A_1 = \frac{1}{3},\ B_1 = 150. \tag{17}$$

Indeed, due to the first statement in Lemma 4.1, the distance $d_E(s)$ from the set E on the cut U_1 can be estimated by the distance $d_2(s)$ from the skeleton $\mathcal{E}_2$:

$$\frac{1}{d_2^2(s)} \le \frac{1}{d_E^2(s)} \le 16 \cdot \frac{1}{d_2^2(s)}.$$

Hence,

$$\begin{array}{ccc} \frac{d_E(x)}{4\pi} \int_{U_1} \frac{16}{|x-s|^2} \frac{dm(s)}{d_E^2(s)} & \le & \frac{d_E(x)}{4\pi} \int_{U_1} \frac{16}{|x-s|^2} \frac{dm(s)}{d_2^2(s)} \le \\ \frac{4 d_E(x)}{\pi} \int_{U_1} \frac{1}{|x-s|^2} & & \left[\sum_{s_{ik} \in \mathcal{E}_2} \frac{1}{|s - s_{ik}|^2} \right] dm(s). \end{array}$$

Using the following estimate for the standard integral

$$\frac{1}{3} \le \frac{1}{4\pi} \int_{\mathcal{R}_3} \frac{1}{|x-1|\,|x|}\, dm \le \frac{7}{3}, \tag{18}$$

we obtain, after the change of variables,

$$4 \frac{d_E(x)}{\pi} \int_{U_1} \frac{1}{|x-s|^2} \left[\sum_{s_{ik} \in \mathcal{E}_2} \frac{1}{|s - s_{ik}|^2} \right] dm(s) \le 150 \cdot \frac{d_E(x)}{d_2(x)} \le 150. \tag{19}$$

The estimate from below may be obtained as follows:

$$\begin{array}{ccc} \frac{d_E(x)}{4\pi} \int_{U_1} \frac{1}{|x-s|^2} \frac{dm(s)}{d_E^2(s)} & \ge & \frac{d_E(x)}{4\pi} \int_{U_1} \frac{1}{|x-s|^2} \frac{dm}{d_2^2(s)} \ge \\ \max_{s_{ik} \in \mathcal{E}_2} \frac{d_E(x)}{4\pi} \int_{U_1} \frac{1}{|x-s|^2} \frac{dm}{|s - s_{ik}|^2} & \ge & \frac{1}{3} \cdot \frac{d_E(x)}{d_2(s)} \ge \frac{1}{3}. \end{array}$$

From the self-similarity of the tiling it follows that the same estimate holds for the corresponding integral over any cut U_c, for every string c and $x \in \omega_c$:

$$A_1 \le \frac{d_E(x)}{4\pi} \int_{U_c} \frac{1}{|x-s|^2} \frac{dm(s)}{d_E^2(s)} \le B_1. \tag{20}$$

The integral $C_\omega = \frac{1}{4\pi} \int_\omega \frac{dm(s)}{d_E^2(s)}$ *can be estimated as* $3^{-2} \le C_\omega \le 16$. *Consequently, due to the self-similarity, all integrals* $\frac{1}{4\pi} \int_{\omega_c} \frac{dm(s)}{d_E^2(s)}$ *can be estimated uniformly:*

$$3^{-|s|-2} \le \frac{1}{4\pi} \int_{\omega_c} \frac{dm(s)}{d_E^2(s)} \le 16 \cdot 3^{-|c|}. \tag{21}$$

Indeed, due to the first statement in Lemma 4.1 the integral

$$\frac{1}{4\pi} \int_\omega \frac{dm(s)}{d_E^2(s)}$$

may be estimated as

$$\frac{25}{17} \cdot \frac{1}{4\pi} \int_\omega \frac{dm(s)}{d_2^2(s)} \le 4 \cdot \frac{25}{17} \int_0^{1/3} \int_0^{4/6} \frac{\rho d\rho dh}{h^2 + \rho^2} \le 6 \cdot \int_0^{1/3} \ln \frac{h^2 + (2/3)^2}{h^2}\, dh \le 16.$$

An estimate of the integral from below may be derived from the estimate Lemma 4.1 of d_E from above.

Next note that

$$\sup_{x\in\mathcal{R}_3\backslash(\Omega_0\cup\Omega_1)} \frac{d_E\,(x)}{4\pi}\int\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}$$

can be estimated from above by the sum

$$4\cdot\frac{d_E}{\pi}\int_{\mathcal{R}_3\backslash(\cup_{ik}\Omega_{ik})}\frac{1}{|x-s|^2}\frac{dm(s)}{d_2^2(s)}+\frac{d_E\,(x)}{4\pi(d_E\,(x)-2\,.\,3^{-3})^2}\int_{(\bigcup_{ik}\Omega_{ik})}\frac{1}{d_E^2}\,dm.$$

Due to (18) we have

$$\frac{448}{3}\cdot\frac{d_E(x)}{d_2(x)}<150\cdot\frac{d_E(x)}{d_2(x)}<150,$$

in view of (21) and self-similarity,

$$\frac{3^{-3}\cdot d_E(x)}{(d_E(x)-2\cdot 3^{-3})^2}\cdot\frac{8}{3}\leq 200,$$

hence *the integral* $\frac{d_E\,(x)}{4\pi}\int\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}$, *for* $x\in\mathcal{R}_3\backslash(\Omega_0\cup\Omega_1)$, *is estimated from above by* 350 .

We obtain further the *dominating* estimate for the integral $\frac{d_E\,(x)}{4\pi}\int\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}$ for $x\in\omega_a$, $|a|\geq 1$.

Lemma 5.1 *The integral coefficient*

$$\frac{d_E\,(x)}{4\pi}\int\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}$$

can be discretized and estimated for $x\in\omega_a$ *as follows:*

$$\frac{d_E(x)}{4\pi}\int_{\mathcal{R}_3}\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}=\frac{d_E(x)}{4\pi}\int_{\mathcal{R}_3\backslash\Omega_0\cup\Omega_1}\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}$$

$$+\frac{d_E(x)}{\pi}\int_{U_a}\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}+\sum_b\frac{d_E(x)}{\pi}\int_{\omega_b,\,|b|\geq 1, b\cap U_a=\emptyset}\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}$$

$$\leq 300+900\cdot\sum_{\omega_b,\,|b|\geq 1, b\cap U_a=\emptyset}3^{2\cdot|a\cap b|-|ab|}.$$

Proof. The proof of the first statement is based on (18):

$$\frac{d_E(x)}{4\pi}\int_{\mathcal{R}_3\backslash(\Omega_0\cup\Omega_1}\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}\leq\frac{112}{3}\cdot\frac{d_E(x)}{d_1(x)}\leq 150.$$

In view of (17) and (20) we get:

$$\sup_{\mathbf{x}\in U_{\omega_1}}\frac{d_E(x)}{4\pi}\int_{U_{\omega_1}}\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}\leq 150.$$

To get an upper bound for the third term we use the estimate (21) for integrals $\frac{d_E(x)}{4\pi}\int_{\omega_b,\,b\neq a}\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}$, $x\in\omega_a$ and the second statement of Lemma 4.1:

$$\frac{d_E(x)}{4\pi}\int_{\omega_b,\,|b|\geq 1, b\cap U_a=\emptyset}\frac{1}{|x-s|^2}\frac{dm(s)}{d_E^2(s)}\leq 900\cdot\sum_{\omega_b,\,|b|\geq 1, b\cap U_a=\emptyset}3^{2\cdot|a\cap b|-|ab|}.$$

□

The next statement, of algebraic nature, completes the estimation of the integral representing the constant $\mathcal{K}$.

Lemma 5.2 *The following inequality holds true for every string $b \in \Sigma^*$:*

$$\sum_{a\in S} 3^{2\cdot|a\cap b|-|ab|} \leq 4. \tag{22}$$

Proof. Assume that $|b| = m$ and $b = b_1 b_2 \ldots b_m$. First, decompose the series in the left-hand side of (22) into two disjoint series:

$$\begin{aligned}
\sum_{a\in S} 3^{2\cdot|a\cap b|-|ab|} &= \sum_{k=1}^{\infty}\sum_{|a|=k} 3^{2\cdot|a\cap b|-|ab|} \\
&= \sum_{k=1}^{m-1}\sum_{|a|=k} 3^{2\cdot|a\cap b|-|ab|} + \sum_{k=m}^{\infty}\sum_{|a|=k} 3^{2\cdot|a\cap b|-|ab|}.
\end{aligned}$$

A typical string $a = a_1 a_2 \ldots a_k$ of length $k \leq m-1$ will be of one of the following two forms: $b_1 b_2 \ldots y_i \bar{b}_{i+1} \ldots a_k$ (for some $0 \leq i \leq k-1$) or $b_1 b_2 \ldots b_{k-1} b_k$. We have 2^{k-i-1} different strings of the first form and exactly one string of the last form. Similarly, a typical string $a = a_1 a_2 \ldots a_k$ of length $k \geq m$ will be of one of the following two forms: $b_1 b_2 \ldots b_i \bar{y}_{i+1} \ldots a_m \ldots a_k$ (for some $0 \leq i \leq m-1$) or $b_1 b_2 \ldots b_{m-1} b_m \ldots a_{m+1} a_k$. We have 2^{k-i-1} different strings of the first form and 2^{k-m} strings of the last form. An elementary computation, based on the above combinatorial analysis, justifies the following two inequalities which combine to prove (22):

$$\begin{aligned}
\sum_{k=1}^{m-1}\sum_{|a|=k} 3^{2\cdot|a\cap b|-|ab|} &= 3^{-m-k}\left(\sum_{i=0}^{k-1} 3^{2i}\cdot 2^{k-i-1} + 3^{2k}\right) \\
&= \frac{1}{7\cdot 3^m}\cdot\sum_{k=1}^{m-1}\left(\frac{2}{3}\right)^k\cdot\left(\left(\frac{9}{2}\right)^m - 1\right) + \frac{1}{2}\left(1-\frac{1}{3^{m-1}}\right) \\
&\leq \frac{4}{7},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=m}^{\infty}\sum_{|a|=k} 3^{2\cdot|a\cap b|-|ab|} &= 3^{-m-k}\left(\sum_{i=0}^{m-1} 3^{2i}\cdot 2^{k-i-1} + 3^{2m}\cdot 2^{k-m}\right) \\
&= \frac{1}{7\cdot 3^{m-1}}\cdot\left(\left(\frac{9}{2}\right)^m - 1\right)\left(\frac{2}{3}\right)^m + 3 \\
&\leq \frac{24}{7}.
\end{aligned}$$

□

For a string $b \in \Sigma^*$ of length greater than one let b' be the prefix of b of length $|b|-1$. We can use now the inequality (22) to deduce the following upper bound:

$$\sum_{\omega_b,\,|b|\geq 1, b\cap U_a=\emptyset} 3^{2\cdot|a\cap b|-|ab|} \leq \sum_{a\in S\setminus\{b,b0,b1,b'\}} 3^{2\cdot|a\cap b|-|ab|} \leq 2,$$

which leads directly to

Theorem 5.3 *The Poincaré–Hardy inequality*

$$\int \frac{|u|^2}{dist^2(x,E)}\,dm \leq 2,100\cdot\int|\bigtriangledown u|^2 dm$$

holds true in $\mathcal{R}_3$ on the complement of the Cantor set E.

6 Two applications

The inequality (4) can be used to derive several useful facts. In what follows we will present two such applications.

A. *Consider a real measurable function q, locally bounded on each compact in the complement E' of the fractal E. Then the Dirichlet form $\int_{\mathcal{R}_3}\left(|\nabla u|^2 + q(x)|u|^2\right) dm$ is closed in $W_2^1(\mathcal{R}_3)$ if q satisfies the following additional condition:*

$$\lim_{d_E(x)\to 0} |q(x)|\; d_E^2(x) = 0. \tag{23}$$

To prove the above statement we have to check that the inequality (4) implies the strong subordination of the quadratic form of the potential $\int_{\mathcal{R}_3} q(x)|u|^2\, dm$ to the Dirichlet form $\int_{\mathcal{R}_3} |\nabla u|^2\, dm$ (see Reed and Simon[22]). Indeed, for any $\varepsilon > 0$ we can choose a positive constant C such that

$$|q(x)| \leq C + \frac{\varepsilon}{\mathcal{K}_E d_E^2(x)},$$

which implies the strong subordination:

$$\int_{\mathcal{R}_3} |q(x)||u|^2 dm \leq \varepsilon \cdot \int_{\mathcal{R}_3} |\nabla u|^2 dm + C \cdot \int_{\mathcal{R}_3} |u|^2 dm.$$

B. The constant $\mathcal{K}$ plays the role of an estimate for the dimensionless Plank constant in the corresponding uncertainty principle. To see this, let us consider the self-adjoint operator (unbounded in $\mathcal{L}_2(\mathcal{R}_3)$) of multiplication by the function $\varepsilon(x)d_E(x)$, where the factor $\varepsilon(x) = \pm 1$ is chosen such that for every given smooth function u with a compact support in E' the mean value of the "balanced distance" with respect to some unitary-valued sign-factor $\varepsilon(x)$, $\varepsilon(x)d_E(x)$ to the singular set E is equal to zero:[6]

$$\int_{\mathcal{R}_3} \varepsilon(x) d_E(x)|u|^2 dm = 0.$$

We assume that the mean value of momentum is also zero:

$$\int_{\mathcal{R}_3} \nabla u\; \bar{u}\; dm = 0.$$

Under the above hypotheses we may estimate from below the product of the mean quadratic errors of the balanced distance and the mean quadratic error of the momentum, the Dirichlet integral,

$$\frac{1}{\mathcal{K}} \cdot \int_{\mathcal{R}_3} |u|^2\, dm \;\leq\; \left[\int_{\mathcal{R}_3} |d_E(x) u|^2 dm\right]^{1/2} \cdot \left[\frac{1}{\mathcal{K}^2} \cdot \int_{\mathcal{R}_3} \frac{|u|^2}{d_E^2(x)} dm\right]^{1/2} \tag{24}$$

$$\leq\; \left[\int_{\mathcal{R}_3} |d_E(x)|^2\, |u|^2\, dm\right]^{1/2} \cdot \left[\int_{\mathcal{R}_3} |\nabla u|^2\, dm\right]^{1/2}, \tag{25}$$

to obtain an analog of the classical dimensionless Heisenberg's uncertainty relation:

$$\frac{1}{2} \cdot \int_{\mathcal{R}_3} |u|^2\, dm \leq \left[\int_{\mathcal{R}_3} |x|^2 |u|^2\, dm\right]^{1/2} \cdot \left[\int_{\mathcal{R}_3} |\nabla u|^2\, dm\right]^{1/2}.$$

The constant $\mathcal{K}^{-1} \approx 0.021819$ plays the role of estimate for analog of the classical "dimensionless Plank constant" 1/2. It defines the admissible precision of simultaneous measurements of deviation of the the coordinate of the quantum particle from the singular set and the deviation of its total momentum from zero.

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[6] This is automatically true for real "smooth" functions.

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