CDMTCS
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Report
Series

Effective Model Theory: The Number of Models and Their Complexity

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CDMTCS-131
March 2000

Centre for Discrete Mathematics and
Theoretical Computer Science

Effective Model Theory: The Number of Models and Their Complexity[1]

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Abstract

Effective model theory studies model theoretic notions with an eye towards issues of computability and effectiveness. We consider two possible starting points. If the basic objects are taken to be theories, then the appropriate effective version investigates decidable theories (the set of theorems is computable) and decidable structures (ones with decidable theories). If the objects of initial interest are typical mathematical structures, then the starting point is computable structures. We present an introduction to both of these aspects of effective model theory organized roughly around the themes of the number and types of models of theories with particular attention to categoricity (as either a hypothesis or a conclusion) and the analysis of various computability issues in families of models.

1. Basic Notions

The lectures on which this paper is based were intended to be a brief introduction to effective model theory centered around one set of issues: the number

[1] This paper is primarily based on the short course on efective model theory given by the second author at the ASL summer meeting, Logic Colloquium '97, at the University of Leeds. Much of the material was also presented by the authors in invited talks at a special session of the ASL annual meeting, University of California, Irvine, USA, 1995; the Discrete Mathematics, Theoretical Computer Science, and Logic Conference, Victoria University, Wellington, New Zealand, 1996; the Special Session on Feasible Mathematics of the AMS annual meeting, Orlando, USA, 1996; the annual meeting of the ASL, University of Wisconsin, Madison, 1996, USA; and the Special Session on Computable Mathematics and its Applications of the AMS annual meeting, Baltimore, January 1998.

[2] Partially supported by ARO through MSI, Cornell University, DAAL03-91-C0027.

[3] Partially supported by NSF Grants DMS-9204308, DMS-9503503, INT-9602579 and ARO through MSI, Cornell University, DAAL03-91-C0027.

of models of specified type and, in particular, the notion of categoricity. For more general introductions we refer the reader to *The Handbook of Recursive Algebra* (Ershov et al. [1998]), especially the articles by Harizanov [1998] and Ershov and Goncharov [1998]. This *Handbook* also contains other useful survey papers on aspects of effective model theory and algebra and an extensive bibliography. The one most closely related to the theme of this paper is Goncharov [1998]. Another interesting survey is Millar [1999] in *The Handbook of Computability Theory* (Griffor [1999]). Two books in progress on the subject are Ash and Knight [1999] and Harizanov [2000]. These are all good sources for material and references. An extensive and very useful bibliography prepared by I. Kalantari [1998] can also be found in Ershov et al. [1998].

One might well begin with the question of what effective model theory is about. Of course, it is about investigating the subjects of model theory with an eye to questions of effectiveness. What then is model theory about and what does one mean by effectiveness? As for model theory we simply quote from two standard texts (to which we also refer the reader for the terminology, notation and results of classical model theory). Chang and Keisler [1990] say "Model theory is the branch of mathematical logic which deals with the connection between a formal language and its interpretations, or models." Hodges [1993] says "Model theory is the study of the construction and classification of structures within specified classes of structures." We can take these two definitions as expressing two views of the proper subject of model theory. The first starts with formal languages and so we may say with theories. (We take a *theory* T to be simply a set of sentences in some (first-order) language L, called the *language of* T. We say that a theory T is *complete* if $T \vdash \sigma$ or $T \vdash \neg\sigma$ for every sentence σ of L.) The second starts with mathematical structures. One might think of these views as, respectively, logical and algebraic. They lead to a basic dichotomy in the approach to effective model theory. Should we "effectivize" theories or structures. Of course, the answer is that we should investigate both approaches and their interconnections. As for what one means by "effectiveness," there are many notions ranging from ones in computer science to ones of descriptive set theory that have some claim to being versions of effectiveness. Most, if not all, of them can be reasonably called in to analyze different model theoretic questions. In this paper, we limit ourselves to what we view as the primary notion of effectiveness: Turing computability (or, equivalently, recursiveness). Thus we are lead to formal definitions of the two basic notions of our subject, effective theories and structures.

Definition 1.1 A theory T is *decidable* if the theorems of T form a computable set. A structure $\mathcal{A}$ (for a language L) with *underlying set* (or *domain*) A is *decidable* if $Th(\mathcal{A}, a)_{a\in A}$, the *complete* (or *elementary*) *diagram* of $\mathcal{A}$, i.e. the set of all sentences (with constant symbols for each element of A) true in $\mathcal{A}$, is computable. $\mathcal{A}$ is *computable* if $D(\mathcal{A}, a)_{a\in A}$, the (*atomic*) *diagram* of $\mathcal{A}$, i.e. the set of all atomic sentences or their negations (again with constant symbols for each element of A) true in $\mathcal{A}$, is computable.

For those whose basic object of interest, or at least starting point, consists of theories, the decidable theories are the natural effective objects of study. In line with standard model theoretic usage a structure whose complete theory has some property P is often said to also have property P and so we have decidable structures. This is the “logical” point of view. On the other hand, the algebraist or general mathematician usually starts with structures. From this point of view, the effective objects are the computable structures. After all, when one thinks of what a computable group should be one thinks that it should be a group structure for which the group operation is computable and similarly for all other typical algebraic structures. One does certainly not assume that even the word problem, let alone the complete diagram, is computable.

Note that we are deliberately avoiding all issues of coding or Gödel numbering. There are two common approaches to this issue. The Eastern, and especially the Russian, school favors numerations. One starts with a classical structure $\mathcal{A}$ and provides a *numeration* (or *enumeration*), that is a map ν from the natural numbers $\mathbb{N}$ onto the underlying set A of the structure $\mathcal{A}$. The *numerated* (or *enumerated*) structure $\langle \mathcal{A}, \nu \rangle$ is called *constructive* if the (appropriately coded) atomic diagram of $\mathcal{A}$, with constant symbols i for $i \in \mathbb{N}$ interpreted as $\nu(i)$, is computable (recursive). $\langle \mathcal{A}, \nu \rangle$ is *strongly constructive* if the complete diagram of $\mathcal{A}$ with constant symbols i for $i \in \mathbb{N}$ interpreted as $\nu(i)$ is computable. These notions essentially correspond to what we call computable and decidable structures, respectively.

An established Western approach is to say that all elements are natural numbers, all sets are subsets of $\mathbb{N}$ and all functions are functions from $\mathbb{N}$ to $\mathbb{N}$. In this view, languages are Gödel numbered, structures consist of a set of numbers and relations and functions on that set. The formal definitions of computable or recursive for subsets of, and functions on, $\mathbb{N}$ are then simply applied directly to theories and structures. We adopt what might be viewed as a less formal version of the second approach along the lines followed in Shoenfield [1971] and now, we think, prevalent in thinking (if not always in writing) about computability. Given that we are not considering issues raised by the theory of enumerations, we see no reason to explicitly code objects as numbers. After all, we now “know” what effective and computable mean not only for numbers but for all kinds of data structures from strings to arrays on arbitrary finite alphabets. Thus we talk about a computable language without the formalities of Gödel numbering and so about computable theories, types, etc. Similarly, we have computable structures, lists of names for their elements, diagrams and theories. These may or may not “be” sets of, or functions on, $\mathbb{N}$. Any reader who prefers explicit Gödel numbering is certainly able to make the appropriate translations. (We may at times, however, resort to indices to clarify certain uniformity issues.) For those interested in the issues related specifically to numerations we refer the reader to Ershov [1977].

Of course, the notions of effectiveness associated with Turing computability only make sense in the countable setting.

- All languages, sets, structures and the like are assumed to be countable unless explicitly stated otherwise.

Even so, not all sets or structures are computable. Classically, one typically identifies isomorphic structures. Of course, this eliminates all issues of effectiveness and so is often not appropriate here. We will have to distinguish between classically isomorphic models. The following definitions of presentations and presentability help us make these distinctions.

Definition 1.2 A structure $\mathcal{A}$ is *computably (decidably) presentable* if $\mathcal{A}$ is isomorphic to a computable (decidable) structure $\mathcal{B}$ which we call a *computable (decidable) presentation* of $\mathcal{A}$.

Before launching into theorems and analyses, we present a few examples of decidable or computable theories and structures. These theories and structures will serve as examples for many of the notions and results we consider below. Proofs for many of the facts we cite about these structures can be found in Chang and Keisler [1990, 3.4].

Example 1.3 Our language here is that of (linear) orders with one binary predicate $\leq$. We consider two theories $DeLO$, dense linear orderings with no first or last element and $DiLO$, discrete linear orderings with first but no last element. $DeLO$ is axiomatizable, $\aleph_0$-*categorical* (i.e. all countable models are isomorphic) and so complete and decidable. $DiLO$ is axiomatizable and complete and so decidable but not $\aleph_0$-categorical. The standard structures associated with these theories are $\mathbb{Q}$ and $\mathbb{N}$, respectively, with their natural orderings. Both are decidable. As $DeLO$ is $\aleph_0$-categorical every model (remember we are considering only countable structures) is isomorphic to $\mathbb{Q}$ and so decidably presentable. $DeLO$ has effective quantifier elimination and so every computable model is actually decidable. On the other hand, not every model of $DiLO$ is even computably presentable nor is every computable model decidable as we shall see below (for example, in Proposition 6.1). (To see that not every model of $DiLO$ is computably presentable, note that at the cost of a couple of jumps we can form the quotient of a given $DiLO$ by the equivalence relation of being finitely far apart. This procedure can produce an arbitrary ordering with first element. If the quotient ordering is not arithmetic, the original model can't be computably presented.)

Example 1.4 The next theory we mention is ACF_0, algebraically closed fields of characteristic 0. The language is that of field theory with $0, 1, +$ and $\times$. ACF_0 is axiomatizable, $\aleph_1$-*categorical*, i.e. all models of cardinality $\aleph_1$ are isomorphic, and so complete and decidable. ACF_0 also has effective quantifier elimination and so here too every computable model is actually decidable. Even though ACF_0 is not $\aleph_0$-categorical, every model is decidably presentable and below we prove a general theorem establishing this fact (Theorem 5.2).

Example 1.5 Finally, we briefly discuss PA, Peano Arithmetic or if one prefers any suitable finitely axiomatized subtheory such as Robinson's Q [1950]. The language has $0, 1, +$ and $\times$ with the usual axioms. Of course PA is axiomatizable but, by Gödel's incompleteness theorem, it is neither complete nor decidable. It is not $\aleph_0$-categorical (the compactness theorem provides nonstandard models not isomorphic to $\mathbb{N}$). No model is decidable (again by the incompleteness theorem) and only the standard model $\mathbb{N}$ is computably presentable.

Proposition 1.6 *(Tennenbaum [1959]) No nonstandard model of PA or even of Robinson's Q is computably presentable.*

Proof sketch. We assume that one has developed the theory T in question enough to, say prove unique factorization into primes and that the standard universal partial computable function is representable in that there is a formula $F(e, x, s, i)$ such that, for each e, x, s, i in $\mathbb{N}$, $\phi_{e,s}(x) = i$ if and only if $T \vdash F(e, x, s, i)$. (We do not bother to differentiate between a number and the numeral representing it.) One now shows that T proves the simple fact that

$(*)\forall s \exists y \forall e([F(e, e, s, 0) \to p_e|y \wedge p_{2e+1} \nmid y] \wedge [\neg F(e, e, s, 0) \to p_e \nmid y \wedge p_{2e+1}|y])$

where $p_e|y$ is a formula saying that the e^{th} prime divides y.

Now let $\mathcal{A}$ be any nonstandard model of T, s any nonstandard element of $\mathcal{A}$ and y the element of $\mathcal{A}$ guaranteed by $(*)$. We define the function f on $\mathbb{N}$ by $f(e) = 1$ if $\mathcal{A} \models p_e|y$ and $f(e) = 0$ if $\mathcal{A} \models p_{2e+1}|y$. Clearly $f(e)$ is computable from the atomic diagram of $\mathcal{A}$ by searching for an element z such that $\mathcal{A} \models p_e \times z = y$ or $\mathcal{A} \models p_{2e+1} \times z = y$. (One must exist by $(*)$.) However, f is clearly not computable. Indeed, f is *diagonally noncomputable*: $\forall e(f(e) \neq \phi_e(e))$. Thus $\mathcal{A}$ is not computably presentable. $\square$

2. The Effective Completeness Theorem

A common theme in model theory is the investigation of questions about when given theories have models with specified properties. Typical examples include characterizing when theories have atomic, prime, universal, homogeneous or saturated models. Other questions involve models of various ranks or dimension, with or without indiscernibles or even more ambitiously attempts to characterize all the models of a given theory. In effective model theory one naturally wants to know when theories have decidable or computable models of each type or even to attempt to characterize the decidable or computable models of a given theory. We will investigate a few examples of such questions. We begin with the issue of when a theory has a model at all – Gödel's completeness theorem.

Theorem 2.1 *(Completeness Theorem) If a theory T is consistent it has a model.*

We present one effective analog of the completeness theorem for decidable theories with a proof modeled on Henkin's proof of the classical completeness theorem. This method of construction is simple but basic for many results in both classical and effective model theory and we will see several variants latter on.

Theorem 2.2 *(Effective Completeness Theorem) If a theory T is consistent and decidable then it has a decidable model.*

Proof. We assume that the classical Henkin construction is known and so provide only a sketch so that we can check its effective content. Let L_c be the language L of T extended by infinitely many new constants c_i and let σ_e be a (computable) list of the sentences of L_c. We construct an increasing sequence of finite sets Ψ_s of sentences of L_c (with $\wedge\Psi_s = \psi_s$) consistent with T with union Ψ as in the Henkin proof of the completeness theorem. We need to satisfy the requirements P_e for each $e \in \mathbb{N}$:

- $P_e : \sigma_e \in \Psi$ or $\neg\sigma_e \in \Psi$ and if σ_e is of the form $\exists x\theta(x)$ and in Ψ then $\theta(c_i) \in \Psi$ for some i.

Construction: At stage s ask if σ_s is consistent with $T \cup \Psi_s$. If so put σ_s into Ψ_{s+1} and, if σ_s is $\exists x\theta(x)$, also put $\theta(c_i)$ into Ψ_{s+1} for some as yet unmentioned c_i. If σ_s is not consistent with $T \cup \Psi_s$ put $\neg\sigma_s$ into Ψ_{s+1}.

Verifications: Obviously, Ψ is complete and the standard argument shows that it is consistent. As usual the elements of the desired model $\mathcal{M}$ are the equivalence classes of the c_i under the equivalence relation $\equiv$ given by $c_i \equiv c_j$ iff $(c_i = c_j) \in \Psi$ and the relations and functions on $\mathcal{M}$ are determined in the natural way by the formulas in Ψ.

The only issue for us now is the effectiveness of the construction. First we note that one can verify that if T is decidable then Ψ is computable. The only question we must answer at stage s is if σ_s is inconsistent with $T \cup \Psi_s$. This is equivalent to whether or not $\psi_s \to \neg\sigma_s$ (with new free variables z_i substituted in for the constants c_i appearing in Ψ_s or σ_s) is a theorem of T. As T is decidable the answer to these questions is a computable function of s. Thus the equivalence relation $c_i \equiv c_j$ is computable. (Just look at Ψ_{s+1} where $c_i = c_j$ is σ_s.) So the equivalence classes form a computable set (the domain of $\mathcal{M}$) and the relations and functions on $\mathcal{M}$ are determined by Ψ. Indeed, as usual, a sentence σ is true in $\mathcal{M}$ if and only if $\sigma \in \Psi$ and so $\mathcal{M}$ is decidable as required. $\square$

One theme in effective model theory that we will not pursue investigates the question of how hard it is (say in terms of Turing degree or levels of the (hyper)arithmetic hierarchy) to construct models of a given type when it is not possible to produce decidable or even computable ones. We consider the completeness theorem as our only example. In the construction above the only noneffective step was deciding if σ_s is consistent with $T \cup \Psi_s$. As one can always answer this question computably in T' (the Turing jump of T), every consistent theory T has a model computable, indeed decidable, in T'.

Corollary 2.3 *If T is consistent then there is a model $\mathcal{M}$ of T such that the elementary theory of $\mathcal{M}$, $Th(\mathcal{M}, m)_{m \in M}$, is computable in T' (and so Δ_2^0 in T). Indeed by the low basis theorem, there is always one with $T' \leq_T (Th(\mathcal{M}, m)_{m \in M})'$.*

Proof. The first assertion follows immediately from the construction and discussion above. For the second, instead of a single Ψ we build a binary tree (of choices of σ_s (and Henkin axioms as appropriate) or $\neg\sigma_s$). We terminate any path that becomes inconsistent when we find a proof of inconsistency from T. This produces an infinite binary tree computable in T (the particular Ψ constructed above is an infinite path through this tree). The low basis theorem (Jockusch and Soare [1972]) says that there is an infinite path P through the tree with $P' \leq_T T'$. As above we can construct the desired model (and its complete diagram) computably in P as required. □

- For the sake of convenience we assume from now on that all theories are consistent.

We can now say (in some sense) when a theory T has a decidable model.

Corollary 2.4 *A complete theory T has a decidable model if and only if it is decidable. An arbitrary theory T has a decidable model if and only if it has a decidable complete extension.*

Proof. If $\mathcal{M}$ is a model of T and T is complete then the set of theorems of T is simply the intersection of $Th(\mathcal{M}, m)_{m \in M}$ with the sentences of the language L of T and so T is decidable if $\mathcal{M}$ is decidable. Even if T is not complete, if $\mathcal{M}$ is a decidable model of T then this set is a decidable complete extension of T. The other (if) direction of both assertions in the Corollary follow from Theorem 2.2. □

We will not in general assume that theories are complete. However, finite models have little interest from the viewpoint of Turing computability.

- We assume from now on that all theories have only infinite models.

Now that we "know" when a theory T has a decidable model, we might well ask how many decidable models a theory can have. For now we identify models up to classical isomorphisms and so we might better ask how many decidably presentable models can a theory have. The issues of identifying computable models only when there is a computable isomorphism between them will be taken up in §6-7.

If T is incomplete then every decidable complete extension has a decidable model by Theorem 2.2 and, of course, models of distinct extensions are not isomorphic. Moreover, every decidable model of T is a model of some complete decidable extension of T. Thus if one is interested in the number of decidably presentable models of a theory, it suffices to consider only complete decidable theories. We begin with the possibility that there is only one as in our example $DeLO$ of a decidable $\aleph_0$-categorical theory.

Proposition 2.5 *If a theory T is $\aleph_0$-categorical then the following conditions are equivalent:*

1. T is decidable.
2. T has a decidable model.
3. All models of T are decidably presentable.

Proof. As $\aleph_0$-categoricity implies completeness, the equivalences all follow directly from the hypothesis, definitions and Theorem 2.2. □

Now, it is a remarkable classical theorem due to Vaught [1961] that no complete theory has exactly two (isomorphism types of) models. The effective analog for decidable models is, however, false.

Theorem 2.6 *(Millar [1979], Kudaibergenov [1979]) There is a decidable theory T with exactly two (isomorphism types of) decidably presentable models.*

Proof sketch. Let f be a partial computable function whose range is $\{0, 1\}$ and which does not have a total computable extension. Consider the (computably enumerable but computably inseparable) sets $M_0 = \{x | f(x) = 0\}$ and $M_1 = \{x | f(x) = 1\}$. Let $f_0 \subset f_1 \subset \ldots$ be an effective approximation to f such that $k \notin dom(f_s)$ for all $k > s$.

The language of T contains infinitely many unary and binary predicates P_i and R_i, respectively, where $i \in \omega$. Consider first the theory T_0 whose axioms are the following set of statements:

1. $\forall x P_0(x) \& \forall y (P_{i+1}(y) \rightarrow P_i(y))$, where $i \in \omega$.
2. If $R_k(x, y)$, then $x \neq y$ and $P_k(x) \& P_k(y)$.
3. If $x \neq y$, $P_s(x) \& P_s(y)$ and $f_s(k) = 0$, then $R_k(x, y)$.
4. If $x \neq y$, $P_s(x) \& P_s(y)$ and $f_s(k) = 1$, then $\neg R_k(x, y)$.

One can check that the following four properties hold of T_0:

1. T_0 has a decidable model completion T. Moreover T has a unique 1–type (Definition 3.1) p such that $P_k(z) \in p$ for all $k \in \omega$.
2. If a model $\mathcal{A}$ of T has at least two elements realizing p, then $\mathcal{A}$ is not decidably presentable.
3. If a model $\mathcal{A}$ of T has fewer than two elements realizing p, then $\mathcal{A}$ is decidably presentable.
4. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are models of T with the same finite number of elements realizing p, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic.

These properties show that T has exactly two decidably presentable models. □

The above proof can easily be generalized:

Corollary 2.7 *For each $n \leq \omega$, there exists a theory with exactly n nonisomorphic decidable models.* $\square$

As for our examples above, an analysis of the structure of models of $DiLO$ as in Chang and Keisler [1990, 3.4] easily implies that there are countably many distinct decidable models. The same is true for ACF_0 as we shall see in Theorem 5.2.

Although the natural effective version of Vaught's theorem fails, the proof (properly effectivized) can be used to give a similar result for decidable models (Theorem 4.4 below). We first need to study another aspect of the question of how many decidable models a theory T can have: When are each of the classically studied types of models such as prime, atomic or saturated models of a decidable theory decidably presentable?

3. Decidable Prime Models

We begin our study of specific types of models with prime and atomic models. They will play a crucial role in the next two sections.

Definition 3.1 An *n-type* Γ or $\Gamma(x_1, \ldots, x_n)$ of a theory T is a set of formulas with n free variables in the language of T which is consistent with T such that $\sigma(x_1, \ldots, x_n)$ or $\neg\sigma(x_1, \ldots, x_n)$ belongs to Γ for each such formula. An n-type $\Gamma(x_1, \ldots, x_n)$ of a theory T is *principal* if there is a formula $\theta(x_1, \ldots, x_n)$ such that $T \vdash \theta(x_1, \ldots, x_n) \rightarrow \sigma(x_1, \ldots, x_n)$ for every $\sigma \in \Gamma$. In this case we say that $\theta(x_1, \ldots, x_n)$ is a *complete formula* that *generates* Γ.

Definition 3.2 A model $\mathcal{A}$ of a theory T in the language L is a *prime* model of T if it can be elementarily embedded into every model of T. $\mathcal{A}$ is *atomic* if every n-tuple of elements from A satisfies a complete formula $\theta(x_1, \ldots, x_n)$ of L. (Each of these models is unique (up to isomorphism) if it exists.)

The notions of prime and atomic coincide for countable models and so we motivate our characterization of decidable prime models by two classical characterizations.

Theorem 3.3 *A complete theory T in a language L has a prime model if and only if every formula of L consistent with T is a member of a principal type over T.*

Theorem 3.4 *A complete theory T in a language L has an atomic model if and only if every formula of L consistent with T can be extended to a complete formula.*

As the notions of atomic and prime coincide (for countable models), each of these theorems provides a characterization of the theories with prime models. We now consider what might be the appropriate effective versions of these theorems. In one direction, note that every type realized in a prime model of T is principal and all principal types are realized in every model of T. Thus, if T has a decidable prime model, not only is every formula consistent with T a member of a principal type (and so completable) but there is a uniformly computable list of these principal types given by the ones realized in the decidable prime model.

The classical theorems at first glance suggest that this condition might be sufficient. We should use this list of computable types to construct the model. However, an additional possible uniformity is suggested by each classical characterization. The characterization of prime models suggests that we might need to be able to go uniformly effectively from formulas to (indices for) principal types containing them. The characterization of atomic models suggests that one might need to be able to go uniformly effectively from formulas to generating formulas for the principal types containing them. Although the two classical versions are equivalent these two effective versions are not. The first is clearly necessary as given a formula ψ consistent with T and a decidable prime model $\mathcal{A}$ we can computably find an n-tuple of elements of $\mathcal{A}$ satisfying ψ. The set of formulas satisfied by this n-tuple $\mathcal{A}$ is then a computable principal type containing ψ. It turns out that this condition is also sufficient. The second condition clearly implies the first and so is sufficient but not, as it turns out, necessary.

Theorem 3.5 *(Harrington [1974]; Goncharov and Nurtazin [1973]) A complete decidable theory T has a decidable prime model if and only if there is a computable function taking each formula to (an index for) a computable principal type containing it.*

Proof. We construct the desired model by a priority argument reminiscent of that for the Sacks splitting theorem for computably enumerable sets [1963] but instead producing a Henkin construction that restricts the types realized to the principal ones.

Let σ_e list the formulas of L_c the language of T extended by new constants c_i. We construct in stages a sequence of finite sets $\Psi_s(c_1, \ldots, c_{n_s})$ of sentences consistent with T with union Ψ as in the proof of Theorem 2.2. Again we let $\psi_s = \wedge\Psi_s$. At each stage s of the construction $\Gamma_{e,s}$ will be a principal e-type containing the formula $\exists y_{e+1}, \ldots, \exists y_{n_s}\psi_s(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_s})$. Our goal is to satisfy the requirements P_e of Theorem 2.2 as well as new ones Q_e that guarantee that the model constructed is prime by making sure that only principal types are realized. We satisfy Q_e by making sure that $\Gamma_{e,s}$ is eventually constant and so that $[c_1], \ldots [c_n]$ satisfies the principal type $\Gamma_e (= \lim_s \Gamma_{e,s})$. (We denote the equivalence class of c_i in the model built from the constants as in Theorem 2.2 by $[c_i]$.)

- $P_e : \sigma_e \in \Psi$ or $\neg\sigma_e \in \Psi$ and if σ_e is of the form $\exists x\theta(x)$ and in Ψ then $\theta(c_i) \in \Psi$ for some i.

- Q_e : $\langle [c_1], \ldots [c_e] \rangle$ realizes a principal type $\Gamma_e = \lim_s \Gamma_{e,s}$.

Construction: At stage s, if only one of σ_s and $\neg\sigma_s$ is consistent with $T \cup \Psi_s$ put it into Ψ_{s+1}. Suppose it is ρ that is put into Ψ_{s+1} and so $T \vdash \psi_s \to \rho$. As $\exists y_{e+1}, \ldots, \exists y_{n_s} \psi_s(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_s})$ is in $\Gamma_{e,s}$ which is a complete type over T, and $T \vdash \psi_s \to \rho$, $\exists y_{e+1}, \ldots, \exists y_{n_{s+1}} \psi_{s+1}(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_{s+1}})$ is also in $\Gamma_{e,s}$. So we can let $\Gamma_{e,s+1}$ be $\Gamma_{e,s}$ for all e. If both σ_s and $\neg\sigma_s$ are consistent with $T \cup \Psi_s$, the problem is that adding σ_s (or $\neg\sigma_s$) to Ψ_s to form Ψ_{s+1} may make $\exists y_{e+1}, \ldots, \exists y_{n_{s+1}} \psi_s(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_{s+1}})$ not be a member of $\Gamma_{e,s}$ for various numbers e. This would force us to change our choice of the type realized by $\langle [c_1], \ldots, [c_e] \rangle$ and so make $\Gamma_{e,s+1} \neq \Gamma_{e,s}$. We view this as an injury to requirement Q_e (which requires that $\Gamma_{e,s}$ eventually stabilize). As in the Sacks splitting theorem we act so as to minimize the priority of the first requirement injured.

More precisely, we let ψ^0_{s+1} be $\psi_s \wedge \sigma_s$ and ψ^1_{s+1} be $\psi_s \wedge \neg\sigma_s$. We let $e_{i,s}$ (for $i = 0, 1$) be the least $e \leq s$ such that $\exists y_{e+1}, \ldots, \exists y_{n_{s+1}} \psi^i_{s+1}(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_{s+1}})$ is not in $\Gamma_{e,s}$. (If none exists, $e_{i,s} = s$.) If $e_{0,s} \leq e_{1,s}$ let $\psi_{s+1} = \psi^1_{s+1}$ and otherwise let $\psi_{s+1} = \psi^0_{s+1}$. Let $e_s = \min\{e_{0,s}, e_{1,s}\}$. For $e \leq e_s$ we can let $\Gamma_{e,s+1} = \Gamma_{e,s}$ as for such e, $\exists y_{e+1}, \ldots, \exists y_{n_{s+1}} \psi_{s+1}(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_{s+1}}) \in \Gamma_{e,s}$. For $e > e_s$ we redefine $\Gamma_{e,s+1}$ as the first in our uniformly computable list of principal types which contains $\exists y_{e+1}, \ldots, \exists y_{n_s+1} \psi_{s+1}(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_s+1})$.

If we have put $\exists x \theta(x)$ into Ψ, we put $\theta(c_i)$ in as well for some unused c_i. This clearly does not require any change in the $\Gamma_{e,s+1}$ already defined.

Verifications: As T is decidable and the types on our list are uniformly computable, the construction is clearly computable. We clearly satisfy the P_e requirements and so construct a decidable model $\mathcal{M}$ as in Theorem 2.2. As all sentences σ_i involving only $c_1, \ldots, c_e$ that are put into Ψ_s at stage s belong to the principal type $\Gamma_{e,s}$, if we can show that $\lim_s \Gamma_{e,s}$ exists for each e (and is say Γ_e) then we will have shown that, in $\mathcal{M}$, $\langle [c_1], \ldots [c_e] \rangle$ realizes the principal type Γ_e as required to guarantee that $\mathcal{M}$ is a prime model of T.

We prove by induction on e that there is a stage t_e such that $e_s > e$ for all $s \geq t_e$ and so $\Gamma_{e,s} = \Gamma_{e,t_e}$ for all $s > t_e$. Suppose that t_{e-1} exists. We need to show that e_s is greater than e for all sufficiently large s. Now, by the definition of t_{e-1}, $e \leq e_s$ for every $s > t_{e-1}$ and so by the choice of Ψ_{s+1} in the construction, $\Gamma_{e,s} = \Gamma_{e,t_e} = \Gamma_e$ for all $s > t_e$. As Γ_e is principal, some $\sigma(x_1, \ldots, x_e)$ is a generator and so by some stage $t \geq t_{e-1}$ we have added σ to Ψ_t. We claim that $e_s > e$ for every $s > t$. Consider σ_s for any $s > t$. The only way e_s could be e is if both σ_s and $\neg\sigma_s$ are consistent with $T \cup \Psi_s$ but $\exists y_{e+1}, \ldots, \exists y_{n_{s+1}} \psi^i_{s+1}(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_{s+1}})$ is not in $\Gamma_{e,s}$ for $i = 0$ or 1. As $\exists y_{e+1}, \ldots, \exists y_{n_{s+1}} \psi_s(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_{s+1}}) \to \sigma$ and σ is complete this would mean that

$$\exists y_{e+1}, \ldots, \exists y_{n_{s+1}} \psi_s(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_{s+1}}) \to \neg\exists y_{e+1}, \ldots, \exists y_{n_{s+1}} \sigma_s(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_{s+1}})$$

or that

$$\exists y_{e+1}, \ldots, \exists y_{n_{s+1}} \psi_s(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_{s+1}}) \to \neg \exists y_{e+1}, \ldots, \exists y_{n_{s+1}} \neg \sigma_s(x_1, \ldots, x_e, y_{e+1}, \ldots, y_{n_{s+1}})$$

so that σ_s or $\neg\sigma_s$, respectively, would be inconsistent with Ψ_s contrary to our assumption. Thus t is the required stage t_e. □

We finish this section with an alternative version of Theorem 3.5 and some remarks about various uniformity conditions.

Corollary 3.6 *A complete decidable theory T has a decidable prime model if and only if T has a prime model and the set of all principal types of T is uniformly computable.*

Proof. The only if direction of this Corollary is clearly implied by the Theorem. Suppose then that T has a prime model and the set of principal types of T is uniformly computable. As T has a prime model, every formula ψ is a member of a principal type and so the search among those in the given set for one containing ψ terminates and provides the computable function required in the theorem. □

The effective uniformity in the listing of the computable principal types is necessary as an explicit hypothesis:

Theorem 3.7 *(Millar[1978]) There is a complete decidable theory T all of whose types are computable with a prime model but no decidable (or even computable) one.*

Finally, we show that the possible alternate version of Theorem 3.5 that asks for a computable way to go from a formula to a completion is false and so "uniformly atomic" is stronger than "uniformly prime" even for decidable $\aleph_1$-categorical theories.

Proposition 3.8 *There is a (complete) decidable $\aleph_1$-categorical theory T with a decidable prime model but with no computable function taking formulas to complete extensions.*

Proof. The language of T has infinitely many unary predicates R_i. The axioms of T say that the cardinality of each R_i is exactly 2 and that R_i and R_j are disjoint for distinct i and j except for some *designated* triples $\langle i, j, k\rangle$ such that R_k consists of one element from each of R_i and R_j. Moreover, no two distinct designated triples have any entry in common. The actual list of axioms for T is thus determined by the list of designated triples. This list will be defined recursively to diagonalize against each possible computable partial function θ_e which might be a candidate for a function taking formulas to complete extensions. Thus T will be axiomatizable. It is also $\aleph_1$-categorical. (The part of the model consisting of elements in any R_i is uniquely determined by the axioms. The rest just consists of $\aleph_1$ many elements not in any R_i.) Thus T is complete and decidable.

The list of designated triples is effectively enumerated in increasing order (and so is computable) by waiting to diagonalize each θ_i at the formula $R_{2i}(x)$. If $\theta_i(R_{2i}(x))$ converges at stage s, we choose j, k larger than any number mentioned already and designate the triple $\langle 2i, 2j+1, 2k+1\rangle$. In particular, if $\theta_i(R_{2i}(x))$ is the generating formula $\theta(x)$ (which implies $R_{2i}(x)$) then θ cannot mention R_{2k+1}. We claim that T can prove neither that $\theta(x)$ implies $R_{2k+1}(x)$ nor that it implies $\neg R_{2k+1}(x)$ and so θ_i is not a function taking formulas to complete extensions. To see that no information about $R_{2k+1}(x)$ can be implied by $\theta(x)$ consider the theory T' gotten by restricting T to the language L' which is L without the predicate R_{2k+1}. T' is clearly also $\aleph_1$-categorical and consistent with $\theta(x)$. Let $\mathcal{A}$ be a model of T' and a an element realizing $\theta(x)$. Let b be the other element of R_{2i} in $\mathcal{A}$ and c and d the elements of R_{2j+1}. (R_{2i} and R_{2j+1} are disjoint by construction.) We can easily expand $\mathcal{A}$ to a model of T by interpreting R_{2k+1} as either $\{a, c\}$ or $\{b, d\}$. Thus $\theta(x)$ cannot imply either $R_{2k+1}(x)$ or $\neg R_{2k+1}(x)$. $\Box$

4. Saturated Models and the Number of Decidable Models

Definition 4.1 A model $\mathcal{A}$ of a theory T in the language L is a *saturated* model of T if it realizes every type of T with finitely many parameters from A. (If it exists, the saturated model of T is unique.)

The characterization of decidable theories with decidable saturated models is somewhat easier than for prime ones.

Theorem 4.2 *(Morley [1976], Millar [1978], Goncharov [1978a]) A decidable theory T has a decidable saturated model if and only if the types of T are uniformly computable.*

Proof sketch. If T has a decidable saturated model $\mathcal{A}$ then the types of T are uniformly computable as we can simply list the n-tuples from A and, for each of them the set of formulas it satisfies. For the other direction, we can use the uniformly computable list of types to do an effective Henkin construction. As the construction proceeds, we designate new constants to realize each potential type over previously introduced constants. As all the potential types over new constants are given uniformly computably as restrictions to a subset of their free variable of ones on our given list this procedure can be effectively organized. Roughly speaking, the plan is to continue to make the designated constants realize the appropriate type until an inconsistency is reached. We can check for inconsistencies with previously assigned types since they are all uniformly computable. We use a priority ordering to guarantee that, despite the need to cancel attempts at realizing certain potential types, each actual type over the constants introduced is in fact realized. Thus the model constructed is saturated as required. $\Box$

By Millar [1978], the explicit assumption of uniformity is necessary even if one assumes that the decidable theory T has a saturated model and all its types are computable. Millar [1978, p. 63] suggests that the proof of this results can be modified to show that there is no connection between the decidability of the saturated and prime models (when both exist). We now show that, in fact, if there is a decidable saturated model then there is a decidable prime model.

Proposition 4.3 *(Ershov [1980, 381-382], see also Goncharov [1997, Theorem 3.4.4]) If a complete theory T has a decidable saturated model then it has a decidable prime model.*

Proof. As T has a decidable model it is itself decidable by Corollary 2.4. As it has a decidable saturated model, Theorem 4.2 gives us a uniformly computable list Γ_e of all the types of T. By Theorem 3.5, it suffices to prove that, given any formula ϕ consistent with T, we can go effectively to a principal type Γ containing ϕ. We begin with the first type Γ_{n_0} on our list containing $\phi = \phi_0$. We proceed recursively to extend ϕ to ϕ_i and define a type Γ_{n_i} containing ϕ_i. Given ϕ_i, Γ_{n_i} and σ_i (from the list of all formulas with the same number of free variables as ϕ), we ask if both σ_i and $\neg\sigma_i$ are consistent with $T \cup \{\phi_i\}$. If not, $\phi_{i+1} = \phi_i$ and $n_{i+1} = n_i$. If so, we find the first e_0 and e_1 such that $\phi_i \wedge \sigma_i \in \Gamma_{e_0}$ and $\phi_i \wedge \neg\sigma_i \in \Gamma_{e_1}$, respectively. We let n_{i+1} be the larger of e_0 and e_1 and let ϕ_{i+1} be $\phi_i \wedge \sigma_i$ or $\phi_i \wedge \neg\sigma_i$ accordingly. It is clear that the sequence n_i is nondecreasing as at step i of the construction if e_0 and e_1 are defined then one of them is n_i and we always take the larger. As this procedure is effective, $\{\phi_i | i \in \omega\}$ generates a computable type Γ containing ϕ. If n_i is not eventually constant, Γ would be a type of T not equal to any Γ_e for a contradiction. Once n_i has stabilized say at n we can define e_0 and e_1 at only finitely many stages s as each time we do so we extend ϕ_s and eliminate one possible Γ_j for $j < n$ from future consideration. Thus ϕ_i also eventually stabilizes say at ϕ_e. It is now clear that ϕ_e generates the type Γ_n which is therefore the required principal type containing ϕ. □

We now see what the proof of Vaught's theorem that a complete theory cannot have exactly two models gives us.

Corollary 4.4 *If a complete but not $\aleph_0$-categorical theory T has a decidable saturated model then it has at least three decidable models.*

Proof. Let $\mathcal{A}$ be a decidable saturated model of T. By Proposition 4.3, T has a decidable prime model $\mathcal{B}$. As T is not $\aleph_0$-categorical, the decidable saturated model $\mathcal{A}$ of T is not a prime model and so $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic. Thus $\mathcal{A}$ realizes a nonprincipal (but computable) type $\Gamma(\overline{x})$. $\mathcal{A}$ can clearly be expanded to a saturated model of $T \cup \Gamma(\overline{c})$ by properly interpreting the constants $\overline{c}$ and so $T \cup \Gamma(\overline{c})$ has a decidable saturated model and hence a decidable prime model $\mathcal{C}$ by Proposition 4.3. Of course, the restriction of $\mathcal{C}$ is a decidable model of T. As in the proof of Vaught's theorem (as in Chang and Keisler [1990, Theorem 2.3.15]), this model cannot be isomorphic to either $\mathcal{A}$ or $\mathcal{B}$. □

On the other hand, if a decidable theory T has no decidable prime model (and so no decidable saturated model) then it has infinitely many decidable prime models. To see this, we quote a simple case of Millar's effective omitting types theorem.

Theorem 4.5 *(Millar [1983]) If T is a decidable theory and $\{\Gamma_i | i < n\}$ a finite set of computable nonprincipal types of T then there is a decidable model of T omitting every (i.e. not realizing any) Γ_i.*

Corollary 4.6 *If a decidable theory T does not have a decidable prime model then T has infinitely many decidable models.*

Proof. By Theorem 2.2, T has a decidable model $\mathcal{A}$. As $\mathcal{A}$ is not a prime model it realizes some nonprincipal type Γ_1. By Theorem 4.5, there is a decidable model $\mathcal{A}_1$ of T omitting Γ_1. As $\mathcal{A}_1$is not prime, it realizes a nonprincipal type Γ_2 distinct from Γ_1 by construction. We now get a decidable $\mathcal{A}_2$ omitting both Γ_1 and Γ_2. Continuing in this way we get an infinite sequence Γ_i of computable nonprincipal types of T and decidable nonisomorphic models $\mathcal{A}_i$ of T as required. (Each $\mathcal{A}_i$ realizes Γ_{i+1} but not Γ_j for any $j \leq i$.) $\square$

Another variation on the question of how many decidable models a decidable theory can have asks when is every model of T decidably presentable. One obvious necessary condition is that all types in T are computable. (Every type is realized in some model and only computable types can be realized in a decidable model.) Thus, in particular, T can have only countably many types. This condition is not sufficient and the problem remains open in general. There are a couple of partial answers. The answer is simple for $\aleph_0$-categorical theories and is supplied by Proposition 2.5. The nicest result is for $\aleph_1$-categorical theories to which we now turn.

5. $\aleph_1$-Categorical Theories

If a theory T is $\aleph_1$-categorical (and so complete) but not $\aleph_0$-categorical then the Baldwin-Lachlan theorem [1971] supplies us with a full classification of the models of T in terms of a well defined notion of dimension. There are countably many models $\mathcal{A}_i$ of T and they are arranged in a liner order of type $\omega + 1$ with respect to elementary embedding ascending with increasing dimension:

$$\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \mathcal{A}_2 \preceq \ldots \preceq \mathcal{A}_n \preceq \ldots \preceq \mathcal{A}_\infty.$$

$\mathcal{A}_0$, the model of dimension zero is the prime model of T and $\mathcal{A}_\infty$, the unique model of infinite dimension, is the saturated model of T. The model $\mathcal{A}_i$ for $i > 0$ is the model of dimension i.

The classic example of an $\aleph_1$ but not $\aleph_0$- categorical theory is ACF_0. Here the dimension of a model is its transcendence degree over the prime field $\mathbb{Q}$. $\mathcal{A}_0$, the prime model, is the algebraic closure of $\mathbb{Q}$. $\mathcal{A}_\infty$, the saturated model, is the algebraic closure of the rationals extended by infinitely many transcendental elements. Each $\mathcal{A}_i$ for $i > 0$ is the algebraic closure of $\mathbb{Q}$ extended by i many transcendentals.

The general problem we wish to address is the following:

Question 5.1 If T is $\aleph_1$ but not $\aleph_0$-categorical theory when (and which of) its models are decidably or computably presentable?

5.1 Decidable Models of $\aleph_1$-Categorical Theories

Of course, if T is $\aleph_1$-categorical and so complete, it has a decidable model if and only if it is itself decidable (Theorem 2.4). Actually, the decidability of T is enough to guarantee that every model is decidably presentable:

Theorem 5.2 *(Harrington[1974], Khisamiev [1974]) If T is $\aleph_1$-categorical and decidable then every model of T is decidably presentable.*

Proof. We first use the results of Baldwin and Lachlan [1971] to show that we can reduce the problem to that of the existence of decidable prime models for a decidable theory T. (All the model theoretic facts we cite in this proof can be found in Baldwin and Lachlan [1971].)

As T is $\aleph_1$-categorical, there is a principal n-type $\Gamma(x_1, \ldots, x_n)$ such that $T' = T \cup \Gamma(c_1, \ldots, c_n)$ (with c_i new constants) has a *strongly minimal formula*, i.e. a formula $\phi(x)$ of L' (the language L of T expanded by new constants c_i) such that for every model $\mathcal{A}$ of T' and every formula $\psi(x)$ of L', exactly one of $\{a \in A | \mathcal{A} \models \phi(a) \wedge \psi(a)\}$ and $\{a \in A | \mathcal{A} \models \phi(a) \wedge \neg\psi(a)\}$ is finite. Of course, T' is $\aleph_1$-categorical. Note that as T is decidable and Γ is principal, T' is also decidable ($T' \vdash \phi \Leftrightarrow \phi \in \Gamma \Leftrightarrow T \vdash \theta \rightarrow \phi$ where θ is a generator of Γ). As all models of T can be extended to ones of T', we can assume for the proof of our theorem that T has a strongly minimal formula ϕ.

Now each model of an $\aleph_1$-categorical theory T with a strongly minimal formula ϕ is the prime model of an extension T' of T by constants d_i satisfying a type Δ which says that $\phi(d_i)$ holds for each i and that the d_i are algebraically independent, i.e. there is no formula $\psi(x, \overline{y}) \in \Delta$ such that for some n, $\exists^{\leq n} x(\phi(x) \wedge \psi(x, \overline{y})) \in \Delta$. (In fact, the cardinality of the set of d_i is the dimension of the model and uniquely determines it.) Again T' is clearly $\aleph_1$-categorical. We must verify that it is also decidable, i.e. Δ is computable. We prove by induction on the number n of d_i that the corresponding types Δ_n and theories $T_n = T \cup \Delta_n(d_1, \ldots, d_n)$ are uniformly decidable. (They are complete by definition.) For $n+1$, consider any formula $\psi(x, d_1, \ldots, d_n)$. In each model $\mathcal{A}$ of T_n exactly one of $\{a \in A | \mathcal{A} \models \phi(a) \wedge \psi(a, d_1, \ldots, d_n)\}$ and $\{a \in A | \mathcal{A} \models \phi(a) \wedge \neg\psi(a, d_1, \ldots, d_n)\}$ is finite by the strong minimality of ϕ. By compactness, there is then an $m \in \mathbb{N}$ such that

$T_n \vdash \exists^{\leq m} x(\phi(x) \wedge \psi(x, \overline{d}))$ or $T_n \vdash \exists^{\leq m} x(\phi(x) \wedge \neg\psi(x, \overline{d}))$. As T_n is decidable, we can search for and find such an m for ψ or $\neg\psi$. The other is in Δ, i.e. if $T_n \vdash \exists^{\leq m} x(\phi(x) \wedge \psi(x, \overline{d}))$ then $\neg\psi(x, \overline{d}) \in \Delta$ and if $T_n \vdash \exists^{\leq m} x(\phi(x) \wedge \neg\psi(x, \overline{d}))$ then $\psi(x, \overline{d}) \in \Delta$. Thus each T_n and $T_\infty = \cup T_n$ is decidable and the models of T are precisely the prime models of these theories. To prove our theorem it therefore suffices to show that each of these theories has a decidable prime model.

By Theorem 3.5, it suffices to show that if T is a decidable $\aleph_1$-categorical theory with a strongly minimal formula ψ then there is a computable function taking any formula $\sigma(\overline{x})$ to a computable principal type Γ_σ containing σ.

Given σ, we construct a computable type Γ in stages e by starting with σ and adding on each σ_e in turn if it is consistent with what we have put in Γ so far and, if σ_e is $\exists y(\psi(y) \wedge \theta(y, \overline{x}))$, we also add in $\exists y(\psi(y) \wedge \theta(y, \overline{x}) \wedge \phi(y))$ for some algebraic ϕ, i.e. one such that $T \vdash \exists^{\leq n} y(\psi(y) \wedge \phi(y))$ for some $n \in \omega$. Of course, if σ_e is not consistent with what we have so far we add on $\neg\sigma_e$. The point here is that if $\exists y(\psi(y) \wedge \theta(y, \overline{x}))$ is consistent with what we have so far then the formula gotten by adding it on is realized in the prime model of T say by $\overline{c}$. Now that model has only algebraic realizations of ψ and so whatever element witnessed $\exists y(\psi(y) \wedge \theta(y, \overline{c}))$ is algebraic and so also satisfies some algebraic formula ϕ. Thus $\exists y(\psi(y) \wedge \theta(y, \overline{x}) \wedge \phi(y))$ can be consistently added on as desired.

We claim that Γ is principal and so the required Γ_σ. Consider the prime model $\mathcal{A}$ of $T \cup \Gamma(\overline{c})$ and any $a \in A$ such that $\mathcal{A} \models \psi(a)$. As $\mathcal{A}$ is a prime model of $T \cup \Gamma(\overline{c})$, a realizes a principal type over $T \cup \Gamma(\overline{c})$ generated say by $\theta(y, \overline{c})$. If a is not algebraic then for every formula ϕ and every $n \in \omega$, $T \cup \Gamma(\overline{c}) \vdash \theta(y, \overline{c}) \to [\phi(y) \to \neg\exists^{\leq n} y(\psi(y) \wedge \phi(y))]$. On the other hand, as $\mathcal{A} \models \psi(a) \wedge \theta(a, \overline{c})$, $\exists y(\psi(y) \wedge \theta(y, \overline{x})) \in \Gamma$ and so by construction $\exists y(\psi(y) \wedge \theta(y, \overline{x}) \wedge \phi(y)) \in \Gamma$ for some ϕ such that $T \vdash \exists^{\leq n} y(\psi(y) \wedge \phi(y))$ for some n for a contradiction. Thus $\mathcal{A}$ has only algebraic solutions of ψ, i.e. it is the model of dimension 0, and so $\mathcal{A}$ is actually the prime model of T. As Γ is realized in $\mathcal{A}$, it must be principal over T as required.

(This last argument is attributed to Lachlan in Harrington [1974]. Harrington's own proof is also instructive. It begins with the observation that the function taking a formula σ to its rank as defined in Baldwin [1973] can be seen to be a computable map from formulas into $\mathbb{N}$ by the arguments presented in that paper. Thus, given a formula σ consistent with T, we may computably define a type $\Gamma = \cup\Gamma_e$ containing σ by putting in, for each e in turn, either σ_e or $\neg\sigma_e$ so as to always preserve consistency and to reduce the rank of $\bigwedge \Gamma_e$ if possible. Eventually, the rank must stabilize and so we produce a principal type Γ containing σ.) $\square$

5.2 Computable Models

We now turn to the question of which models of an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T are computably presentable if T is not decidable. It is easy to find such a theory with no computable models by coding a noncomputable set S

into every model. (For example, extend ACF_0 by adding on new unary predicates P_i and, for each $i \in \omega$, axioms $\forall x(P_i(x) \to x = 0)$ and $P_i(0)$ if $i \in S$ but $\neg P_i(0)$ if $i \notin S$.) Thus the question is, if T has a computable but no decidable model, which of the models $\mathcal{A}_i$ of T can or cannot be computable. Only a few facts are known.

Theorem 5.3 *(Goncharov [1978], Kudaibergenov [1980]) For every $n \in \mathbb{N}$ there is an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T such that $\mathcal{A}_0, \ldots, \mathcal{A}_n$ are all computably presentable but not $\mathcal{A}_i$ for $i > n$.*

Proof. Fix $n \in \mathbb{N}$. The language for the required theory T will consist of a unary predicate P_k and an n-ary predicate R_k for each $k \in \mathbb{N}$. The axioms for T will code a computably enumerable but not computable set $B = \cup B_s$ into each model of dimension greater than n while maintaining the possibility that the models of dimension less than or equal to n are computably presentable.

Axioms:

- The P_k are nested downward with respect to k and exactly one element drops out at each k, i.e. for each $k \in \mathbb{N}$ we have the following axioms:

  *$\forall x(P_{k+1}(x) \to P_k(x))$
  *$\exists! x(P_k(x) \wedge \neg P_{k+1}(x))$
- For each $k \in \mathbb{N}$ we wish to require that

  $$R_k(x_1, \ldots, x_n) \Leftrightarrow \bigwedge\{x_i \neq x_j | i \neq j\} \wedge \exists s(k \in B_s \wedge x_1, \ldots, x_n \in P_s).$$

  We enforce this requirement by the following axioms:

  *$R_k(x_1, \ldots, x_n) \to x_i \neq x_j$ for $i \neq j$.
  *For each $s \in \mathbb{N}$ and $k \in B_s$:
  $\bigwedge\{x_i \neq x_j | i \neq j\} \wedge x_1, \ldots, x_n \in P_s \to R_k(x_1, \ldots, x_n)$.
  *For each $s \in \mathbb{N}$ and $k \notin B_s$: $\bigvee\{x_i \notin P_s | i \leq n\} \to \neg R_k(x_1, \ldots, x_n)$.

Verifications: It is easy to see that the cardinality of $\cap P_s^{\mathcal{A}}$ uniquely determines the isomorphism type of any model $\mathcal{A}$ of T and that all models $\mathcal{A}$ of size $\aleph_1$ have $\aleph_1$ many elements in $\cap P_s^{\mathcal{A}}$. Thus T is $\aleph_1$-categorical. Indeed, the cardinality of $\cap P_s^{\mathcal{A}}$ is the dimension of $\mathcal{A}$.

We claim that a model $\mathcal{A}$ of T is computably presentable if and only if there are fewer than n distinct elements in $\cap P_s^{\mathcal{A}}$. For one direction, suppose that there are distinct $c_1, \ldots, c_n$ in $\cap P_s^{\mathcal{A}}$. In that case, $k \in B \Leftrightarrow \mathcal{A} \models R_k(c_1, \ldots, c_n)$ and so $\mathcal{A}$ cannot be computably presentable as B is not computable.

For the other direction, we wish to construct a computable model $\mathcal{A}$ of T with $m < n$ many elements $c_1, \ldots, c_m$, in $\cap P_s^{\mathcal{A}}$. We let the other elements of the desired model be the natural numbers and we put i in P_k if and only if $i \geq k$. We now only have to computably define the predicates R_k. Given distinct elements $a_1, \ldots, a_n$ from A, not all of them are from among the c_i and so we can effectively find an s and indeed the smallest s such that one of them is not in P_s. We then let

$R_k(a_1, \ldots, a_n)$ hold if and only if $k \in B_{s-1}$. This clearly defines a computable model $\mathcal{A}$ of T with $|\cap P_s^{\mathcal{A}}| = m$ as required. $\square$

Thus any initial segment of the models of T can be the computably presentable ones. The obvious questions arise as to what else is possible.

Question 5.4 Which subsets of $\omega + 1$ can be the set of computably presentable models of an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T with a computable model? In particular, must the prime model always be computably presentable? Must the saturated model be computably presentable if all the others are?

The following theorem answers the two specific questions asked. All other instances of the general question are open.

Theorem 5.5 *(Khoussainov, Nies and Shore [1997]) There are $\aleph_1$-categorical but not $\aleph_0$-categorical theories T_1 and T_2 such that*

i)All models of T_1 except the prime one are computably presentable.
ii)All models of T_2 except the saturated one are computably presentable.

Proof (For T_1). Given $S \subset \omega$ we construct a structure $\mathcal{A}_S$ of signature $L = (P_0, P_1, P_2, \ldots)$, where each P_i is a binary predicate symbol having the following properties:

- The theory T_S of the structure $\mathcal{A}_S$ is $\aleph_1$- but not $\aleph_0$–categorical and A_S is the prime model of T_S.
- Each nonprime model A of T_S has a computable presentation if and only if S is Σ^0_2.
- A computable prime model provides S with a certain recursion-theoretic property but there exists a Σ^0_2–set which does not have this property.

The building blocks of our structures $\mathcal{A}_S$ will be finite structures that we call n-cubes and now define by induction on n.

Definition 5.6 A 1–*cube* $\mathcal{C}_1$ is a structure $(\{a, b\}, P_0)$ such that $P_0(x, y)$ holds in $\mathcal{C}_1$ if and only if ($x = a$ and $y = b$) or ($y = a$ and $x = b$). Given two disjoint n-cubes we get an $n+1$-*cube* as an expansion of their union by letting P_n be an isomorphism between the n-cubes. An ω–*cube* is an increasing union of n–cubes, $n \in \omega$ with signature $(P_0, P_1, P_2, \ldots)$

Definition 5.7 If $S \subseteq \omega$, $\mathcal{A}_S$ is the disjoint union of n-cubes for $n \in S$ and $T_S = Th(\mathcal{A}_S)$.

Lemma 5.8 *If S is infinite, then T_S is $\aleph_1$- but not $\aleph_0$–categorical and the model with no ω-cubes is its prime model.*

Proof. It is easy to see that the model $\mathcal{A}_S$ satisfies the following conditions which are all expressible by a set of axioms in the language L:

1. $\forall x \exists y P_0(x, y)$ and for each n, P_n defines a partial one-to-one function. (We abuse notation by also denoting this partial function by P_n.)
2. For all $n \neq m$ and for all x, $P_n(x) \neq P_m(x)$.
3. For each n and for all x if $P_n(x)$ is defined, then $P_0(x)$, $P_1(x)$, ..., $P_{n-1}(x)$ are also defined.
4. For all n, m and for all x if $P_n(x)$ and $P_m(P_n(x))$ are defined, then $P_m(P_n(x)) = P_n(P_m(x))$.
5. For all k, $n > n_1 \geq n_2 \geq \ldots \geq n_{k-1} \geq n_k$, $\forall x(P_{n_1}(\ldots(P_{n_k}(x)\ldots) \neq P_n(x))$.
6. For each $n \in \omega$, $n \in S$ if and only if there exists exactly one n–cube which is not contained in an $n+1$–cube.

Let $\mathcal{M}$ be a model which satisfies all the above statements. For each $n \in S$, $\mathcal{M}$ must have an n–cube which is not contained in an $n+1$–cube. If an $x \in M$ does not belong to any n–cube for $n \in S$, then x is in an ω–cube. Thus any two models which satisfy this list of axioms are isomorphic if and only if they have the same number of ω–cubes. In particular, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are models of T_S of cardinality $\aleph_1$, each has $\aleph_1$ many ω–cubes (as each cube is countable). Thus $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic and T_S is an $\aleph_1$- but not $\aleph_0$–categorical theory. It is clear that the prime model is the one with no ω-cubes. $\square$

Lemma 5.9 *Each nonprime model of T_S is computably presentable if and only if S is Σ^0_2.*

Proof. If $\mathcal{M}$ is a model of T_S, $s \in S$ if and only if $\mathcal{M} \models \exists x \exists y \forall z(P_s(x, y) \& \neg P_{s+1}(x, z))$. Thus if $\mathcal{M}$ is computably presentable S is Σ^0_2. For the other direction, note that it suffices to construct a computable model $\mathcal{M}_1$ with one ω-cube when $S \in \Sigma^0_2$. (We can computably add on more ω-cubes as desired.) We build $\mathcal{M}$ by putting in an n-cube when, according to the Σ^0_2 representation of S as $\{n | \exists x \forall y H(x, y, n)\}$, we seem to have a witness x that $n \in S$. When the witness fails, we merge this n-cube into the ω-cube that we are building. More formally, at stage 0 we start to build a substructure $\mathcal{B}$ that will be an m-cube for some m at every stage s and will at the end of the construction be an ω-cube. At stage s, we first put into $\mathcal{M}$ an n-cube for each $n < s$ for which we do not have one and associate the cube with the first number x that has not yet been associated with n. Then, we merge $\mathcal{B}$ and the existing n-cubes for those $n < s$ for which there is a $y < s$ such that $H(x, y, n)$ fails for the x currently associated with n into an m-cube for some m larger than any number yet used in the construction. Clearly the substructure $\mathcal{B}$ becomes the only ω-cube of $\mathcal{M}$. Moreover, for $n \in \omega$, there is an n-cube in the final structure $\mathcal{M}$ if and only if $\exists x \forall y H(x, y, n)$, i.e. if and only if $n \in S$ as required. $\square$

We now provide the recursion theoretic property of S that is guaranteed by the existence of a computable prime model of T_S (but not by any of the other models being computably presentable).

Definition 5.10 A function f is *limitwise monotonic* if there exists a computable function $\phi(x,t)$ such that $\phi(x,t) \leq \phi(x,t+1)$ for all $x,t \in \omega$, $\lim_t \phi(x,t)$ exists for every $x \in \omega$ and $f(x) = \lim_t \phi(x,t)$.

Lemma 5.11 *If the prime model of T_S is computably presentable then S is the range of a limitwise monotonic function.*

Proof. Suppose $\mathcal{M}$ is a computable prime model of T_S. Define $\phi(x,s)$ for each $x \in M$ and $s \in \mathbb{N}$ as the largest $n < s$ such that $P_n(x,y)$ holds for some $y < s$. It is clear that $\phi(x,s)$ is monotonic in s. As every $x \in M$ is in an n-cube for some n, $\phi(x,s)$ is equal to this n for all sufficiently large s. $\square$

Lemma 5.12 *There exists a Δ^0_2 set A which is not the range of any limitwise monotonic function.*

Proof. Let $\phi_e(x,t)$ be a list of all candidates for representations of limitwise monotonic functions f_e. At stage s we define a finite set A_s so that $A(y) = lim_s A_s(y)$ exists for all y (and hence A is Δ^0_2). We also satisfy the following requirements to guarantee that A is not the range of a limitwise monotonic function.

R_e : If $f_e(x) = lim_t \phi_e(x,t) < \omega$ for all x, then $range(f_e) \neq A$.

The strategy to satisfy a single R_e works as follows: At stage s, pick a witness m_e, enumerate m_e into A (i.e. set $A_s(m_e) = 1$). Now R_e is satisfied (since m_e remains in A) unless at some later stage t_0 we find an x such that $\phi_e(x,t_0) = m_e$. If so, R_e ensures that $A(\phi_e(x,t)) = 0$ for all $t \geq t_0$. Thus, either $f_e(x)\uparrow$ or $f_e(x)\downarrow$ and $f_e(x) \notin A$.

Keeping $\phi_e(x,t)$ out of A for all $t \geq t_0$ can conflict with a lower priority $(i > e)$ requirement R_i since it maybe the case that $m_i = \phi_e(x,t')$ for some $t' > t_0$. However, if $f_e(x)\downarrow$, then from some point on there is only one number that R_e prevents from being a candidate for m_i. If $f_e(x)\uparrow$, then the restriction is transitory, i.e. as $\phi_e(x,t)$ is monotonic in t each candidate for m_i is eventually released and never prevented from being chosen as the final value of m_i.Thus each lower priority R_i will eventually be able to choose a witness m_i that it will never have to change because of the actions of R_e. In this way, every requirement can be satisfied by a typical finite injury priority argument. $\square$

Proof sketch (For T_2). We take a Π^0_2 set S defined by $k \in S \Leftrightarrow \forall n \exists m H(n,m,k)$ which is not Σ^0_2. (H is some computable predicate on $\mathbb{N}^3$.) We now code S into a computable structure $\mathcal{A}$ with unary predicates P_i and predicates $R_{k,s}$ of arity k for $i,k,s \in \mathbb{N}$. The relevant properties of $\mathcal{A}$ that can be guaranteed by axioms in this language are as follows:

- The $P_i^{\mathcal{A}}$ form a descending chain of sets with one element dropping out at each i.

- The $R_{k,s}^{\mathcal{A}}$ code the approximation $H(n,m,k)$ to $k \in S$ by requiring that if j is least such that $\forall n \leq s \exists m \leq j(H(n,m,k))$ and $x_1,\ldots,x_k \in P_j$ are distinct for $i \leq k$ then $R_{k,s}(x_1,\ldots,x_k)$ holds and not otherwise. (In particular, if $k \notin S$ then for some s_0 we have axioms saying that $R_{k,s}(x_1,\ldots,x_k)$ does not hold for any $s \geq s_0$ and any $x_1,\ldots,x_k$.)

The theory T_S of $\mathcal{A}_S$ is $\aleph_1$- but not $\aleph_0$-categorical with the dimension of a model $\mathcal{A}$ being once again determined by the cardinality of $\cap P_i^{\mathcal{A}}$. The intuition is that the more elements there are in $\cap P_i^{\mathcal{A}}$ for a model $\mathcal{A}$ of T_S, the more of the Π_2^0-approximation to S that we can "recover" from the diagram of $\mathcal{A}$. In particular, if $\mathcal{A}$ is the saturated model of T_S, $\cap P_i^{\mathcal{A}}$ is infinite and S is Σ_2 in $\mathcal{A}$: $k \in S \Leftrightarrow \exists x_1,\ldots x_k \in A[(\forall i)(\mathcal{A} \models P_i(x_1) \wedge \ldots P_i(x_k)\,) \wedge (\forall s)(\mathcal{A} \models R_{k,s}(x_1,\ldots x_k))]$. As S is not Σ_2^0, the saturated model of T_S is not computably presentable. For each $t < \omega$, however, we can (nonuniformly) build a computable model $\mathcal{A}_t$ of T_S with t many elements in $\cap P_i^{\mathcal{A}_t}$. The information needed is $S \cap (t+1)$ and, for each $k \leq t$ which is not in S the least n for which there is no m such that $H(n,m,k)$ holds. □

All the theorems in this subsection about computable models of $\aleph_1$-categorical theories use infinite signatures. Not too much is known about the existence of such structures and theories in finite signatures or for ones that are extensions of standard algebraic theories. One interesting example is Herwig, Lempp and Ziegler [1999] who have established Theorem 5.3 for $n = 0$ with T an extension of the theory of groups in the standard signature.

6. Computable Dimension and Categoricity

Until now we have taken the classical approach of identifying models up to classical isomorphism. However, it is not obvious that even two computable (or decidable) models that happen to be isomorphic should be identified when one is interested in effective procedures. There could well be (and indeed, as we shall see, there are) structures with presentations $\mathcal{A}$ and $\mathcal{B}$ such that the two presentations have different effective properties. For example, there are computable presentations of $\langle \mathbb{N}, \leq \rangle$ on which the successor function is not computable.

Proposition 6.1 *There is a computable presentation $\mathcal{A} = \langle A, \leq_A \rangle$ of $\langle \mathbb{N}, \leq \rangle$ such that the successor function on $\mathcal{A}$ is not computable.*

Proof. $\mathcal{A}$ will consist of the even numbers in their usual order plus an infinite set of odd numbers determined and placed in the ordering by a procedure designed to guarantee that no computable function ϕ_e is the successor function on $\mathcal{A}$. At stage s we check, for each $e < s$, if $\phi_e(2e)$ has converged at stage s and is equal to $2e+2$. If so we put $2s+1$ into A and place it between $2e$ and $2e+2$. It is obvious that A is computable and that $\phi_e(2e)$ is not the successor of $2e$ in $\mathcal{A}$ for any e. □

The natural approach to the issue raised by such examples is to identify structures or presentations only when there is a computable isomorphism between them. Of course, this only makes sense when the structures themselves are computable.

- Henceforth all structures will be computable.

Definition 6.2 $\mathcal{A}$ is *computably isomorphic* to $\mathcal{B}$, $\mathcal{A} \cong_c \mathcal{B}$, if there is a computable $f : A \to B$ which is an isomorphism. We also say then that $\mathcal{A}$ and $\mathcal{B}$ are *of the same computable isomorphism type*.

Definition 6.3 The *(computable) dimension* of a structure $\mathcal{A}$ is number of its computable isomorphism types. $\mathcal{A}$ is *computably categorical* if its computable dimension is 1, i.e. every $\mathcal{B}$ isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$.

Note that in a computably categorical structure $\mathcal{A}$ every definable relation that is computable in any presentation of $\mathcal{A}$ is computable in every presentation of $\mathcal{A}$ and so for such structures the effectiveness of definable properties is independent of the presentation.

Example 6.4 $\mathbb{Q}$ (the rationals) with its usual linear order is computably categorical: The standard back and forth argument showing that the theory of dense linear orderings without endpoints is countably categorical is effective and so produces computable isomorphisms between any two such orderings.

Example 6.5 $\mathbb{N}$ as a model or PA or indeed as a structure with only the successor function $s(x)$ (given as $x+1$ in the language of arithmetic) is computably categorical: Given any $\mathcal{B}$ isomorphic to $\mathbb{N}$, one defines the required computable $f : \mathbb{N} \to \mathcal{B}$ by recursion. $f(0)$ is the first element of $\mathcal{B}$ and if $f(n)$ is defined as $b \in B$ then $f(n+1) = s^{\mathcal{B}}(b)$. However, it is easy to see from Proposition 6.1 that $\langle \mathbb{N}, \leq \rangle$ is not computably categorical. (If s is the successor function on $\mathbb{N}$ and $f : \mathbb{N} \to A$ were a computable isomorphism into the $\mathcal{A}$ of Proposition 6.1, fsf^{-1} would be a computable successor function on $\mathcal{A}$.)

Example 6.6 Every finitely generated structure is computably categorical by the natural generalization of the preceding argument for $\langle \mathbb{N}, s \rangle$.

Example 6.7 $\overline{\mathbb{Q}}$, the algebraic closure of the rationals and so the prime model of ACF_0, is computably categorical but $\widetilde{\mathbb{Q}}$, the countable saturated model of ACF_0 (i.e. the algebraic closure of the rationals extended by infinitely many transcendentals) has computable dimension ω (Corollary 6.12).

All of these examples have dimension 1 or ω but, actually, every $n \leq \omega$ is possible.

Theorem 6.8 *(Goncharov [1980a]) For each n, $1 \leq n \leq \omega$ there is a structure of dimension n.*

Goncharov uses a priority argument to construct families of uniformly computably enumerable sets with (in a precise sense) exactly n many distinct enumerations and then codes them into structures so as preserve the dimension. We will see other approaches to these results in Theorem 6.22 and Corollary 7.16. Although there are interesting codings of these families into familiar types of mathematics structures such as groups and rings (see §9), we do not know of any "natural" structures with dimension n for $1 < n < \omega$. Indeed, for many classes of structures it is possible to prove that they are computably categorical or have dimension ω. In most of these cases it is actually possible to characterize the structures that are computably categorical.

Theorem 6.9 *(Goncharov [1973], LaRoche [1977], Remmel [1981], Goncharov and Dzgoev [1980]) A Boolean algebra is computably categorical if it has finitely many atoms. If not, it has dimension ω.*

Theorem 6.10 *(Remmel [1981a], Goncharov and Dzgoev [1980]) A linear order is computably categorical if it has only finitely many pairs of adjacent elements. If not, it has dimension ω.*

We can deduce a similar result on algebraically closed fields from a general theorem about computable categoricity among decidable presentations of a structure.

Theorem 6.11 *(Nurtazin [1974]) Suppose $\mathcal{A}$ is a decidable structure. If there are finitely many elements $\bar{c} \in A$ such that $(\mathcal{A}, \bar{c})$ is the prime model of the theory $Th(\mathcal{A}, \bar{c})$ and the set of complete formulas of this theory is computable, then any two decidable presentations of $\mathcal{A}$ are computably isomorphic. On the other hand, if there are no such $\bar{c}$, then there are infinitely many decidable presentations of $\mathcal{A}$ no two of which are computably isomorphic.*

Corollary 6.12 *(Nurtazin [1974]; Metakides and Nerode [1979]) An algebraically closed field of finite transcendence degree over its prime field is computably categorical. One of infinite transcendence degree has dimension ω.*

Proof. Let T be the theory of algebraically closed fields of characteristic 0. As T has quantifier elimination every computable model $\mathcal{A}$ of T is decidable. (Given a sentence with quantifiers (in the expanded language with constants for elements of A) find the quantifier free equivalent. Its truth can be decided by the computability of $\mathcal{A}$. As T is $\aleph_1$-categorical every model $\mathcal{A}$ is the prime model of $T' = T \cup \Gamma(\bar{c}) \cup \Delta(\bar{d})$ for a computable principal type Γ providing the theory with a strongly minimal formula and the type Δ of a sequence of transcendentals (independent elements) as described in the proof of Theorem 5.2. (Actually, for this particular T, Γ is not needed as it is already strongly minimal.) The sequence $\bar{d}$ is finite if and only if the transcendence degree of $\mathcal{A}$ over its prime field is finite. In particular if the transcendence degree is infinite, there is no finite sequence as required and so

$\mathcal{A}$ would have infinite computable dimension. On the other hand, if the sequence is actually finite, we can effectively decide if a given formula $\phi(\overline{d},\overline{x})$ is an atom. As in the proof of Theorem 5.2, we can go effectively to a computable principal type Γ of T' containing $\phi(\overline{d},\overline{x})$. For this particular theory, however, we can enumerate the complete formulas. (In characteristic 0, they just say that (for some ordering of the x's), each x in turn satisfies some irreducible polynomial over the previous ones.) We can thus find such a generating formula γ in Γ and then ask if $\phi \to \gamma$. If so ϕ is complete and not otherwise. (Metakides and Nerode [1979] give a direct proof of this Corollary.) $\square$

An important program is thus to characterize or at least classify computably categorical structures and theories whose models are computably categorical. One major success along these lines is the characterization by Goncharov [1975] of computably categorical structures whose two quantifier theory is decidable in terms of Scott families.

Definition 6.13 A **Scott family** for a structure $\mathcal{A}$ is a computable sequence

$$\phi_0(\bar{a}, x_1, \ldots, x_{n_0}), \phi_1(\bar{a}, x_1, \ldots, x_{n_1}), \ldots,$$

of $\exists$-formulas, i.e. prenex ones with only existential quantifiers, satisfiable in $\mathcal{A}$, where $\bar{a}$ is a finite tuple of elements from $\mathcal{A}$, such that every n-tuple of elements from $\mathcal{A}$ satisfies one these formulas and any two tuples satisfying the same formula from the above sequence can be interchanged by an automorphism of $\mathcal{A}$.

Definition 6.14 A structure $\mathcal{A}$ is n-*decidable* (for $n \in \mathbb{N}$) if the set of prenex sentences of $Th(\mathcal{A}, a)_{a\in A}$ with $n-1$ alternations of quantifiers is computable. So, for example, $\mathcal{A}$ is 1-*decidable* if the set of prenex sentences of $Th(\mathcal{A}, a)_{a\in A}$ with either only existential or only universal quantifiers is decidable.

Proposition 6.15 *If a structure $\mathcal{A}$ has a Scott family, then $\mathcal{A}$ is computably categorical.*

Proof. Let $\phi_0(\bar{a}, x_1, \ldots, x_{n_0}), \phi_1(\bar{a}, x_1, \ldots, x_{n_1}), \ldots$ be a Scott family for $\mathcal{A}$, where $\bar{a} = (a_0, \ldots, a_{m-1})$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be computable presentations of $\mathcal{A}$. We define a mapping $f : A_1 \to A_2$ by stages. We can assume that for each $j \in \{0, \ldots, m-1\}$, a^i_j is the element in A_i corresponding to the constant a_j. At even stages we define images of elements from $\mathcal{A}_1$, at odd stages we define preimages of elements from $\mathcal{A}_2$.

Stage 0. Set $f_0 = \{(a^1_0, a^2_0), \ldots, (a^1_{m-1}, a^2_{m-1})\}$.

Stage 2k>0. We can suppose that the function f_{2k-1} has been defined. Assume that $f_{2k-1} = \{(a^1_0, a^2_0), \ldots, (a^1_{m-1}, a^2_{m-1}), (b_1, d_1), \ldots, (b_s, d_s)\}$ and that f_{2k-1} can be extended to an isomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$. Let b be the first number in $\mathcal{A}_1$ not in the domain of f_{2k-1}. Consider the tuple $(b_1, \ldots, b_s, b)$. Find an i such that $\phi_i(\bar{a}, b_1, \ldots, b_s, b)$ holds in $\mathcal{A}_1$. Hence $\exists x \phi_i(\bar{a}, d_1, \ldots, d_s, x)$ holds in $\mathcal{A}_2$. Find the first $d \in A_2$ for which $\phi_i(\bar{a}, d_1, \ldots, d_s, d)$ holds. Extend f_{2k-1} by letting $f_{2k} = f_{2k-1} \bigcup \{(b, d)\}$.

Stage 2k+1. We define f_{2k+1} similarly so as to put the least element of A_2 not yet in the range of f_{2k} into that of f_{2k+1}.

Finally, let $f = \bigcup_{i\in\omega} f_i$. Clearly, f is a computable isomorphism. $\square$

Theorem 6.16 *(Goncharov [1975]) If $\mathcal{A}$ is 2-decidable then it is computably categorical if and only if it has a Scott family.*

Of course, the if direction of this Theorem follows from the preceding Proposition. For the other direction, one uses a priority argument to build a $\mathcal{B}$ and a Δ_2^0 isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Attempts are made to make sure that no ϕ_e is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. If one of the attempts fails, the construction builds a Scott family for $\mathcal{A}$. (See Ash and Knight [1999] for the details of an ingenious but relatively simple proof.)

Note that the definition of computable categoricity is on its face a Π_1^1 property. This theorem gives a Σ_1^1 equivalent (having a Scott family). Actually, the property of having a Scott family can easily be seen to be arithmetic as the requirement for an isomorphism can be replaced by the existence of a set of finite partial isomorphisms with the back and forth property. Thus, for 2-decidable structures, Theorem 6.16 gives a characterization that is significantly simpler than the underlying definition of computable categoricity.

We now turn to the specific issue of persistence of computable categoricity under expansions by constants that will turn out to be a route into various results and examples of the sorts listed above. In particular, it will lead us to a proof that the existence of a Scott family is not necessary for computable categoricity.

6.1 Persistence of Computable Categoricity

Classically, it is an easy consequence of the Ryll-Nardzewski Theorem that having a countably categorical theory is *persistent*, i.e. preserved under expansions by finitely many constants.

Theorem 6.17 *If $Th(\mathcal{A})$, the theory of a structure $\mathcal{A}$, is countably categorical then so is the theory of any expansion of $\mathcal{A}$ by finitely many constants.*

The natural question for computable categoricity has been considered by Millar, Goncharov and others. It is posed as the Millar-Goncharov problem in Ershov and Goncharov [1986]:

Question 6.18 (Millar,Goncharov) Is computable categoricity persistent, i.e. if $\mathcal{A}$ is computably categorical is also every expansion of $\mathcal{A}$ by finitely many constants?

It is not hard to see that if a structure $\mathcal{A}$ has a Scott family $\phi_i(\bar{a}, x_1, \ldots, x_{n_i})$ then every expansion by finitely many constants $c_1, \ldots c_m$ also has one. We simply slightly modify the original Scott family. (Essentially, one replaces each formula

$\phi_i(\bar{a}, x_1, \ldots, x_{n_i})$ by $\phi_i(\bar{a}, c_1, \ldots, c_m, x_1, \ldots x_{n_i-m})$ and then lists only the satisfied formulas. Then, one can easily check that the sequence $\psi_0, \psi_1, \ldots$ is a Scott family for the expanded structure $(\mathcal{A}, c_1, \ldots, c_m)$.) Thus Theorem 6.16 gives us an answer when $\mathcal{A}$ is 2-decidable.

Corollary 6.19 *(Goncharov [1975]) If $\mathcal{A}$ is 2-decidable then the expansion of $\mathcal{A}$ by finitely many constants is also computably categorical.*

Millar has improved this result by one quantifier by a quite different proof. So, roughly speaking, it suffices to be able to solve systems of equalities and inequalities.

Theorem 6.20 *(Millar [1986]) If $\mathcal{A}$ is 1-decidable then the expansion of $\mathcal{A}$ by finitely many constants is also computably categorical.*

Proof (Hirschfeldt)**.** Suppose we are given $\mathcal{A}$ and $\mathcal{B}$ isomorphic, computably categorical and 1-decidable with $\langle \mathcal{A}, a \rangle \cong \langle \mathcal{B}, b \rangle$. We will build $\mathcal{C}$ via a Henkin construction, a sequence g_s of partial isomorphisms from $\mathcal{C}$ to $\mathcal{B}$ and, for each potential isomorphism $\Phi_e : \mathcal{C} \to \mathcal{A}$, a partial map $h_e : \mathcal{C} \to \mathcal{B}$ such that

- either there is an e such that h_e is total and $h_e \Phi_e^{-1}$ is an isomorphism from $\langle \mathcal{A}, a \rangle$ to $\langle \mathcal{B}, b \rangle$,
- or $g = \lim_s g_s$ exists and is an isomorphism from $\mathcal{C}$ to $\mathcal{B}$ but no Φ_e is an isomorphism from $\mathcal{C}$ to $\mathcal{A}$.

As the second alternative contradicts the hypothesis that $\mathcal{A}$ is computably categorical, we will have the desired computable isomorphism between $\langle \mathcal{A}, a \rangle$ and $\langle \mathcal{B}, b \rangle$. In the construction we actually act, when we can, to guarantee that Φ_e is not an isomorphism from $\mathcal{C}$ to $\mathcal{A}$ (and so we do not have to worry about it). Thus we let R_e be the requirement that Φ_e is not an isomorphism from $\mathcal{C}$ to $\mathcal{A}$. As the construction proceeds, we say that R_e is satisfied (or not) depending on whether we have a certain type of witness to Φ_e's not being an isomorphism from $\mathcal{C}$ to $\mathcal{A}$.

For convenience, we assume that the domain of each model considered here is $\mathbb{N}$. Let $\{\theta_n\}_{n \in \omega}$ be an effective list of all atomic sentences in the language of $\mathcal{A}$ expanded by adding a constant $\mathbf{i}$ for each $i \in \omega$. By θ_n^0 and θ_n^1 we mean $\neg\theta_n$ and θ_n, respectively.

For any conjunction Γ of literals containing no constant $\mathbf{i}$ for $i > m$ and partial computable function Φ with computable domain, we let $f(k) = \mathbf{n}$ if $\Phi(k) \downarrow = n$, $f(k) = x_k$ if $\Phi(k) \uparrow$, and denote by $\Gamma[\Phi]$ the formula $\exists x_0 \cdots \exists x_m \Gamma(\mathbf{0}/f(0), \ldots \ldots, \mathbf{m}/f(m))$. So, for example, if θ_n is the sentence $P(\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3})$ and $\Phi = \{\langle 1, 7 \rangle, \langle 3, 5 \rangle\}$, then $\theta_n^1[\Phi] = \exists x_0 \exists x_1 \exists x_2 \exists x_3 P(x_0, \mathbf{7}, x_2, \mathbf{5})$, while, on the other hand, $\theta_n^0[\Phi] = \exists x_0 \exists x_1 \exists x_2 \exists x_3 \neg P(x_0, \mathbf{7}, x_2, \mathbf{5})$.

We note a few immediate consequences of this definition. In what follows, ε will always be either 0 or 1.

Proposition 6.21

1. If $\mathcal{M} \models \Gamma[\Phi]$ and Φ is an extension of Ψ then $\mathcal{M} \models \Gamma[\Psi]$.
2. If $\mathcal{M} \models \Gamma[\Phi]$ and $S \supset \operatorname{dom}\Phi$ then there is an extension Ψ of Φ with domain S such that $\mathcal{M} \models \Gamma[\Psi]$.
3. If $\mathcal{M} \models \Gamma[\Phi]$ and $\mathcal{M} \models \neg((\Gamma \wedge \theta_n^\varepsilon)[\Phi])$ then $\mathcal{M} \models (\Gamma \wedge \theta_n^{1-\varepsilon})[\Phi]$.
4. Let $a_0, \dots, a_m$ and $b_0, \dots, b_m$ be two sequences of natural numbers. If $\mathcal{M} \models \neg(\Gamma[\{\langle n, a_n\rangle \mid n \leq m\}])$ and $\mathcal{N} \models \Gamma[\{\langle n, b_n\rangle \mid n \leq m\}])$ then $\langle \mathcal{M}, a_0, \dots, a_n\rangle \not\cong \langle \mathcal{N}, b_0, \dots, b_n\rangle$.
5. Suppose that Φ is total and surjective, $\operatorname{dom}\Psi = \{0, \dots, r-1\}$, $\mathcal{M} \models \theta[\Phi \restriction r]$ and $\mathcal{N} \models \neg(\theta[\Psi])$ for some literal θ. Then there is a total computable f which is the identity on $\{0, \dots, r-1\}$ such that $\mathcal{M} \models \theta[\Phi \circ f]$. Let $\theta' = \neg\theta[f]$. Then $\mathcal{M} \models \neg(\theta'[\Phi])$ and $\mathcal{N} \models \neg((\neg\theta')[\Psi])$. $\square$

We now describe our construction.

Construction. At each stage s, we define partial computable functions g_s and $h_{i,s}$, $i \in \omega$.We also construct the atomic diagram $\Delta_{\mathcal{C}}$ of $\mathcal{C}$ by adding on one of the literals θ_s^0 or θ_s^1 at each stage s. We use the following notations: Γ_s is the conjunction of all the literals in $\Delta_{\mathcal{C}}$ at the end of stage s; $z_{e,s}$ is the least number such that $\Phi_{e,s}(z_{e,s}) = a$, if one exists, $z_{e,s} = 0$ otherwise; $r_{e,s} = \sup((\bigcup_{i<e} \operatorname{dom} h_{i,s}) \cup \{z_{i,s} \mid i \leq e\} \cup \{e\})$.

We say that a stage s is e-expansionary if $\Phi_{e,s}$ is injective, $\Phi_{e,s}(z_{e,s}) \downarrow= a$, $\{0, \dots, r_{e,s}\} \subseteq \operatorname{dom}\Phi_{e,s}$, $\operatorname{dom}\Phi_{e,s} \supsetneq \{0, \dots, \sup(\operatorname{dom}\Phi_{e,s-1})\}$, and $\operatorname{rng}\Phi_{e,s} \supsetneq \{0, \dots, \sup(\operatorname{rng}\Phi_{e,s-1})\}$. (Thus, if there are infinitely many e-expansionary stages, Φ_e is total, injective, and surjective.)

We begin at $s = 0$ with $\Gamma_0 = \varnothing$, $g_0 = \varnothing$ and $h_{e,0} = \varnothing$ for each $e \in \omega$. We assume by induction that $\mathcal{B} \models \Gamma_s[g_s]$ and for each $e \in \omega$, $\mathcal{B} \models \Gamma_s[h_{e,s}]$. At stage $s+1$ we find the least $e \leq s$, if any, such that R_e is not satisfied and one of the following conditions holds.

1. For some ε, $\mathcal{B} \models (\Gamma_s \wedge \theta_s^\varepsilon)[g_s \restriction r_{e,s}+1]$ and $\mathcal{A} \models \neg((\Gamma_s \wedge \theta_s^\varepsilon)[\Phi_{e,s}])$ or $\mathcal{B} \models \neg(\Gamma_s \wedge \theta_s^\varepsilon)[g_s \restriction r_{e,s}+1]$ and $\mathcal{A} \models ((\Gamma_s \wedge \theta_s^\varepsilon)[\Phi_{e,s}])$.
2. Not 1 and for some ε,
   (a) $\mathcal{B} \models (\Gamma_s \wedge \theta_s^\varepsilon)[g_s \restriction r_{e,s}+1]$,
   (b) $\mathcal{B} \models (\Gamma_s \wedge \theta_s^\varepsilon)[h_{e,s}]$, and
   (c) $s+1$ is an e-expansionary stage.
3. Not (1 or 2 a and b), and for some ε,
   (a) $\mathcal{B} \models (\Gamma_s \wedge \theta_s^\varepsilon)[g_s \restriction r_{e,s}+1]$,
   (b) $\mathcal{B} \models \neg((\Gamma_s \wedge \theta_s^{1-\varepsilon})[g_s \restriction r_{e,s}+1])$, and
   (c) $\mathcal{B} \models \neg((\Gamma_s \wedge \theta_s^\varepsilon)[h_{e,s}])$.

If such an e exists, we say that e is active at stage $s+1$. Let $r = r_{e,s}+1$. For each $i > e$, let $h_{i,s+1} = \varnothing$. For each $i < e$, let $h_{i,s+1} = h_{i,s}$. Declare all R_i, $i > e$, to be unsatisfied.

If 1 or 3 holds we must abandon the current attempt at the isomorphism h and so let $h_{e,s+1} = \varnothing$. If 1 holds, we have a witness to fact that Φ_e is not an isomorphism from $\mathcal{C}$ to $\mathcal{A}$ and we declare R_e to be satisfied.

If 2 holds, there are two cases. If $h_{e,s} = \varnothing$, we restart our definition of h_e using the assumed isomorphism between $\langle \mathcal{A}, a\rangle$ and $\langle \mathcal{B}, b\rangle$: Find the least tuple $\langle a_0, \ldots, a_{r-1}\rangle$ of distinct numbers such that $a_{z_{e,s}} = b$ and if we define $h_{e,s+1}$ to be the partial function mapping each $n < r$ to a_n, then

1. $\mathcal{B} \vDash \Gamma_{s+1}[h_{e,s+1}]$ and
2. for all $t \leq s$ and $\delta \in \{0, 1\}$, $\mathcal{B} \vDash (\Gamma_t \wedge \theta_t^\delta)[h_{e,s+1}] \Rightarrow \mathcal{A} \vDash (\Gamma_t \wedge \theta_t^\delta)[\Phi_{e,s} \restriction r_{e,s} + 1]$,

and define $h_{e,s+1}$ in this manner. (Such a tuple exists because, since R_e is not satisfied, $\mathcal{A} \vDash \Gamma_{s+1}[\Phi_{e,s}]$, so that $\mathcal{A} \vDash \Gamma_{s+1}[\{\langle z_{e,s}, a\rangle\}]$, and $\langle \mathcal{A}, a\rangle \cong \langle \mathcal{B}, b\rangle$.)

If $h_{e,s} \neq \varnothing$, we extend h_e so as to keep h_e and $h_e\Phi_e^{-1}$ looking like isomorphisms. If $|\operatorname{dom} h_{e,s}|$ is even, let k be the least number not in $\operatorname{rng} h_{e,s}$, let n be a number larger than any previously appearing in the construction, and define $h_{e,s+1} = h_{e,s} \cup \{\langle n, k\rangle\}$. If $|\operatorname{dom} h_{e,s}|$ is odd, let p be the least number not in $\operatorname{dom} h_{e,s}$, let m be such that $\mathcal{B} \vDash \Gamma_{s+1}[h_{e,s} \cup \{\langle p, m\rangle\}]$, and let $h_{e,s+1} = h_{e,s} \cup \{\langle p, m\rangle\}$.

If no such e exists, let ε be such that $\mathcal{B} \vDash (\Gamma_s \wedge \theta_s^\varepsilon)[g_s]$ and let $\max(\operatorname{dom} g_s) + 1 = r$. For each $i \in \omega$, let $h_{i,s+1} = h_{i,s}$.

In any case, we continue to extend the diagram $\Delta_\mathcal{C}$ and the isomorphism g. We add θ_s^ε to $\Delta_\mathcal{C}$ and let $\Gamma_{s+1} = \Gamma_s \wedge \theta_s^\varepsilon$. If $|\operatorname{dom}(g_s \restriction r)|$ is even, let k be the least number not in $\operatorname{rng}(g_s \restriction r)$, let n be a number larger than any previously appearing in the construction, and let $g_{s+1} = g_s \restriction r \cup \{\langle n, k\rangle\}$. If $|\operatorname{dom}(g_s \restriction r)|$ is odd, let p be the least number not in $\operatorname{dom}(g_s \restriction r)$, let m be such that $\mathcal{B} \vDash \Gamma_{s+1}[g_s \restriction r \cup \{\langle p, m\rangle\}]$, and set $g_{s+1} = g_s \restriction r \cup \{\langle p, m\rangle\}$.

Notice that, whichever case holds, $\mathcal{B} \vDash \Gamma_{s+1}[g_{s+1}]$ and for each $e \in \omega$, $\mathcal{B} \vDash \Gamma_{s+1}[h_{e,s+1}]$, which are the induction hypotheses needed for the next stage of the construction.

Verifications. Since at each stage $s + 1$ we added either θ_s or its negation to $\Delta_\mathcal{C}$, $\Delta_\mathcal{C}$ is the atomic diagram of a structure $\mathcal{C}$. Because $\mathcal{A}$ and $\mathcal{B}$ are 1-decidable, the construction is effective and so $\mathcal{C}$ is computable.

Suppose first that there is an e such that R_e is active infinitely often and let e be the least such number. We wish to show that $h_e\Phi_e^{-1}$ is the desired computable isomorphism from $\langle \mathcal{A}, a\rangle$ to $\langle \mathcal{B}, b\rangle$. Let s_0 be a stage such that no R_i is active for $i < e$ at any stage $t \geq s_0$. It follows from the definition of $r_{e,s}$ that there exists an $s_1 \geq s_0$ such that $r_{e,t} = r_{e,s_1}$ for all $t \geq s_1$. Let $r_e = r_{e,s_1}$. It follows from the definition of g_s that there exists $s_2 \geq s_1$ such that $g_t \restriction r_e + 1 = g_{s_2} \restriction r_e + 1$ for all $t \geq s_2$. As R_e is active infinitely often it is never satisfied after stage s_2. So condition 1 never holds after this stage. Thus $\mathcal{A} \vDash \Gamma_s[\Phi_e]$ for every $s \geq s_2$, and hence Φ_e is an isomorphism from $\mathcal{C}$ to $\mathcal{A}$.

We claim that it is not possible for condition 3 to hold infinitely often. Suppose otherwise. Let $s_3 \geq s_2$ be such that $\operatorname{dom} \Phi_{e,s_3} \supseteq \{0, \ldots, r_e\}$. Inspecting the way $h_{e,s+1}$ is defined when case 2 holds and $h_{e,s} = \varnothing$, we see that there is an $s \geq s_3$ such that $h_{e,s+1} = \{\langle n, a_n \rangle \mid n \leq r_e\}$ for a tuple $\langle a_0, \ldots, a_{r_e} \rangle$, $a_{z_{e,s}} = b$, such that for all $t > s$,

1. $\mathcal{B} \vDash \Gamma_{t+1}[\{\langle n, a_n \rangle \mid n \leq r_e\}]$ and
2. $\mathcal{B} \vDash (\Gamma_t \wedge \theta_t^{1-\varepsilon})[\{\langle n, a_n \rangle \mid n \leq r_e\}] \Rightarrow \mathcal{A} \vDash (\Gamma_t \wedge \theta_t^{1-\varepsilon})[\Phi_e \upharpoonright r_e + 1]$.

Such a tuple exists because $\langle \mathcal{A}, a \rangle \cong \langle \mathcal{B}, b \rangle$ and $\Phi_e(z_{e,s}) = a$.

Now suppose that $t+1$ is the first stage after $s+1$ at which condition 3 holds, and let ε be as in that condition. Then $\mathcal{B} \vDash \neg((\Gamma_t \wedge \theta_t^{\varepsilon})[h_{e,t}])$. On the other hand, $\mathcal{B} \vDash \Gamma_t[h_{e,t}]$. Thus $\mathcal{B} \vDash (\Gamma_t \wedge \theta_t^{1-\varepsilon})[h_{e,t}]$. Since $h_{e,t}$ is an extension of $h_{e,s+1}$, $\mathcal{B} \vDash (\Gamma_t \wedge \theta_t^{1-\varepsilon})[h_{e,s+1}]$. But then by 2 above, $\mathcal{A} \vDash (\Gamma_t \wedge \theta_t^{1-\varepsilon})[\Phi_e \upharpoonright r_e + 1]$. But by part b of condition 3, $\mathcal{B} \vDash \neg((\Gamma_t \wedge \theta_t^{1-\varepsilon})[g_t \upharpoonright r_e + 1])$. By Proposition 6.21(5), there exists a u and an ε such that $\mathcal{A} \vDash \neg(\theta_u^{\varepsilon}[\Phi_e])$ and $\mathcal{B} \vDash \neg(\theta_u^{1-\varepsilon}[g_t \upharpoonright r_e + 1])$. But then θ_u^{ε} must be in Γ_{u+1}, so that $\mathcal{A} \nvDash \Gamma_{u+1}$, contrary to our assumption.

So condition 3 holds only finitely often. Say it never holds after stage $s_4 \geq s_3$. Since condition 2 holds infinitely often, there are infinitely many e-expansionary stages. Thus, since R_e is never satisfied, Φ_e is a computable isomorphism from $\mathcal{C}$ to $\mathcal{A}$. Furthermore, $h_e = \lim_s h_{e,s}$ is well-defined, and in fact $h_e(x) = h_{e,s}(x)$ for the least $s > s_4$ for which $h_{e,s}(x)$ is defined. Since $\mathcal{B} \vDash \Gamma_s[h_{e,s}]$ for all $s > s_4$, h_e is a computable isomorphism from $\mathcal{C}$ to $\mathcal{B}$.

Thus $h_e \circ \Phi_e^{-1}$ is a computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$. But if we let $z = \lim_s z_{e,s}$, then $h_e \circ \Phi_e^{-1}(a) = h_e(z) = b$. Thus in fact $h_e \circ \Phi_e^{-1}$ is the desired computable isomorphism from $\langle \mathcal{A}, a \rangle$ to $\langle \mathcal{B}, b \rangle$.

Finally, suppose for the sake of a contradiction that every e is active only finitely often. It is not hard to see that at any e-expansionary stage, one of conditions 1, 2, or 3 must hold. Thus, if there are infinitely many e-expansionary stages then R_e is eventually permanently satisfied.

As we have mentioned, if s is a stage such that, for each $i < e$ and each $t \geq s$, R_i is not active at stage t and $r_{e,t} = r_{e,s}$, then for all $t \geq s$, $g_t \upharpoonright r_{e,s} + 1 = g_s \upharpoonright r_{e,s} + 1$ and $\mathcal{B} \vDash \Gamma_t[g_t \upharpoonright r_{e,s} + 1]$. So the fact that each e is active only finitely often implies that $g = \lim_t g_t$ exists and is an isomorphism from $\mathcal{C}$ to $\mathcal{B}$.

Thus $\mathcal{C}$ is isomorphic, but not computably isomorphic, to $\mathcal{A}$, contradicting the computable categoricity of $\mathcal{A}$. $\square$

Thus 1-decidability suffices to guarantee the persistence of computable categoricity. We will see in the next section that, without such an assumption, computable categoricity need not be persistent. Moreover, the equivalence of computable categoricity with having a Scott family established by Goncharov under the assumption of 2-decidability does not hold for all 1-decidable structures (Theorem 7.19).

6.2 Nonpersistence of Computable Categoricity

We now see that the addition of even a single constant for any element of a computably categorical structure can change its dimension.

Theorem 6.22 *(Cholak, Goncharov, Khoussainov and Shore [1999]) For each $k \in \omega$ there is a computably categorical $\mathcal{A}$ such that the expansion $\mathcal{A}'$ of $\mathcal{A}$ gotten by adding on a constant naming any element of $\mathcal{A}$ has dimension exactly k.*

Idea of Proof (for $k = 2$)**.** We first construct a (uniformly) computably enumerable family of distinct pairs of sets $S = \{f(i)|i \in \omega\} = \{(A_i, B_i)|i \in \omega\}$ which is *symmetric,* i.e. for every $i \in \omega$ there is a $j \in \omega$ such that $f(i) = (A_i, B_i) = (B_j, A_j)$. In addition to the computable enumeration f, there is one other natural computable enumeration of this family, $\tilde{f}$ defined by $\tilde{f}(i) = (B_i, A_i)$. This family S is constructed (by a $0''$ type priority argument) to have dimension 2 in the sense that there is no computable function g such that $f = \tilde{f}g$ but, for every one-one computable enumeration h of the family, there is a computable function g such that $f = hg$ or $\tilde{f} = hg$. The two enumerations of this family are then coded symmetrically into a graph so that the whole structure is computably categorical. If one adds on a constant, however, it distinguishes between the two coded enumerations and so one has a structure of dimension 2. □

For $\dot{k} > 2$, one can generalize the notion of symmetric family to one-one enumerations f of families S of k-tuples of sets. The combinatorial details become fairly complicated. A simpler approach to a proof of the general theorem is provided in the next section as a corollary to some results on degree spectra.

7. Degree Spectra of Relations

Another important topic in computable model theory that turns out to be closely connected to computable categoricity is that of the dependence of the computability properties of relations not included in the language of a given structure on its presentation. For example, in "standard" presentations of $\langle \mathbb{N}, \leq \rangle$ the successor function is computable but it is not computable in every presentation (Proposition 6.1). Similarly, standard presentations of the algebraically closed field $\widetilde{\mathbb{Q}}$ of characteristic 0 and infinite transcendence degree make the relation of algebraic dependence computable but not all presentations do. (Indeed, if algebraic dependence is computable in both of two isomorphic computable algebraically closed fields then they are computably isomorphic. However, Corollary 6.12 says that if they have infinite transcendence degree their dimension is infinite.) On the other hand, these particular relations are easily seen to always be co-computably enumerable and computably enumerable respectively. Others remain computable or computably enumerable in every presentation. Such relations were singled out and studied by Ash and Nerode [1981].

Definition 7.1 (Ash and Nerode) If $R \subseteq A^n$ is an n-ary relation on a structure $\mathcal{A}$, R is *intrinsically computable (computably enumerable)* if $f[R]$ is computable (computably enumerable) for every isomorphism $f : \mathcal{A} \to \mathcal{B}$.

Example 7.2 $\langle \mathbb{N}, \leq \rangle$: Successor is not intrinsically computable.

Example 7.3 $\langle \mathbb{N}, s \rangle$: Every computable relation is intrinsically computable.

We know that the two structures discussed above, $\langle \mathbb{N}, \leq \rangle$ and $\widetilde{\mathbb{Q}}$, are not computably categorical while similar ones (such as $\langle \mathbb{N}, s \rangle$ and algebraically closed extension of $\mathbb{Q}$ of finite transcendence degree) are computably categorical and in each of them this phenomena (of a relation being computable in one presentation and not in another) does not arise. One might naturally ask if computable categoricity guarantees that a relation computable in one presentation is computable in all. The answer is both yes and no. If we restrict our attention to relations that are definable or even *invariant* under all automorphisms the answer is yes.

Proposition 7.4 *If a structure $\mathcal{A}$ is computably categorical then every definable relation R (or one invariant under automorphisms) on $\mathcal{A}$ that is computable in any presentation of $\mathcal{A}$ is intrinsically computable, i.e. computable in every presentation of $\mathcal{A}$.*

Proof. Suppose $\mathcal{A}$ is computably categorical, $R^{\mathcal{A}}$ is computable, and g is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. We wish to show that $g[R^{\mathcal{A}}]$ is computable. As $\mathcal{A}$ is computably categorical, there is a computable isomorphism $f : \mathcal{A} \to \mathcal{B}$. $R^{\mathcal{A}}$ and $\overline{R}^{\mathcal{A}}$ are computable and so their images under f are computably enumerable and complementary and hence computable. As R is invariant under automorphisms, in particular under $g^{-1}f$, $f[R^{\mathcal{A}}] = g[R^{\mathcal{A}}]$ and so $g[R^{\mathcal{A}}]$ is also computable. $\Box$

So for computably categorical structures the effectiveness of definable properties is independent of the presentation. If we ask instead that every computable relation on $\mathcal{A}$ (definable or not) be intrinsically computable, the answer to our question is no. Computable categoricity does not suffice to guarantee that every computable relation is intrinsically computable. (See Example 7.7 below.) Instead we are led to a stronger notion.

Definition 7.5 $\mathcal{A}$ is *computably stable* if every isomorphism $f : \mathcal{A} \to \mathcal{B}$ is computable.

Example 7.6 $\langle \mathbb{N}, s \rangle$ is computably stable. Indeed, every isomorphism between two presentations is uniquely determined by the computable procedure of sending the least element in one presentation to the least one in the other and then proceeding by recursion as in Example 6.5.

Example 7.7 $\langle \mathbb{Q}, \leq \rangle$ is computably categorical but not computably stable. In fact, given any two presentations of $\langle \mathbb{Q}, \leq \rangle$, the usual back and forth argument shows that there are continuum many isomorphisms between them. Moreover, the usual back and forth argument can be run in each of countably many intervals to, for example, construct an automorphism taking a computable subset (such as $\mathbb{Z}$) to a noncomputable one (any set consisting of one element from each interval $(x, x+1)$ for $x \in \mathbb{Z}$).

Proposition 7.8 *(Ash and Nerode [1981]) $\mathcal{A}$ is computably stable if and only if every computable relation on $\mathcal{A}$ is intrinsically computable.*

Proof. As every isomorphism between presentations of $\mathcal{A}$ is computable, the argument of Proposition 7.4 shows that the image of any computable relation R on $\mathcal{A}$ under any isomorphism is computable. For the other (if) direction, consider $\mathcal{A}$ as a structure on the set $\mathbb{N}$ and the relation R giving, in $\mathcal{A}$, the successor function on $\mathbb{N}$. If $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism and $R^{\mathcal{B}} = f[R^{\mathcal{A}}]$ is computable then the construction of Example 6.5 computes f. $\square$

One can, in fact, give a more informative characterization of computable stability like that provided for computable categoricity in terms of Scott families by Theorem 6.16. In place of a sequence of formulas each of which determines a sequence of elements of the given structure $\mathcal{A}$ up to automorphisms, one needs a sequence of formulas that uniquely define the elements of $\mathcal{A}$. On the other hand, we now only need the 1-decidability of $\mathcal{A}$ for the characterization.

Theorem 7.9 *(Ash and Nerode [1981], Goncharov [1975]) If $\mathcal{A}$ is 1-decidable then $\mathcal{A}$ is computably stable if and only if there are constants $\overline{c} \in A$ and a computable sequence $\phi_i(\overline{c}, x)$ of existential formulas such that for each i there is a unique $a \in A$ satisfying ϕ_i and each $a \in A$ satisfies some ϕ_i.*

Proof sketch. It is easy to see that the existence of a family as described insures that every isomorphism $f : \mathcal{A} \to \mathcal{B}$ is computable as, once the image of the constants $\overline{c}$ are fixed, f must send the unique solution of each $\phi_i(\overline{c}, x)$ in $\mathcal{A}$ to the solution of the same formula in $\mathcal{B}$. The proof of the other direction (only if) of this theorem involves a finite injury priority argument. One attempts to build a $\mathcal{B}$ isomorphic to the given $\mathcal{A}$ by a Δ_2 isomorphism but not by any computable isomorphism. The least failure of this diagonalization requirement occurs only because the elements on which we might diagonalize are uniquely defined from those fixed by higher priority requirements. These already fixed elements are the constants $\overline{c}$ required. The portions of the diagram of $\mathcal{B}$ to which we have committed ourselves at various stages of the construction provide the desired formulas ϕ_i when various extra parameters are replaced by existentially quantified variables. $\square$

More generally, we would like to know when a specified computable (or computably enumerable) relation is intrinsically computable or computably enumerable. An examination of the two examples considered above, $\langle \mathbb{N}, \leq \rangle$ and $\widetilde{\mathbb{Q}}$, gives us a clue as to when a relation is intrinsically c.e. The relation $P(x, y)$ on $\mathbb{N}$ saying that y is not the immediate successor of x is definable in the structure $\langle \mathbb{N}, \leq \rangle$ by the existential formula $\exists z((x < z < y) \vee (y < z < x) \vee (x = y))$ and so is computably enumerable in any presentation of $\langle \mathbb{N}, \leq \rangle$. The binary relation $D(x, y)$ saying that x and y are algebraically dependent is equivalent to the disjunction of an infinite computable list of existential formulas ϕ_n each asserting (in the language of fields) that there is a nonzero polynomial of degree n in x and y which equals 0. Any such relation is again clearly computably enumerable in any presentation of a field. To enumerate the dependent pairs, one simply dovetails the searches for witnesses for each of the existential formulas ϕ_n. These phenomena suggest a definition.

Definition 7.10 A relation $R(x_1, \ldots, x_n)$ on a structure $\mathcal{A}$ is *formally computably enumerable* if it is equivalent to a disjunction $\bigvee \phi_i(x_1, \ldots, x_n)$ of a computable sequence of existential formulas ϕ_i with free variables $x_1, \ldots, x_n$. R is *formally computable* if both R and $\overline{R}$ are formally computably enumerable.

Clearly, any formally computable (computably enumerable) relation is intrinsically computable (computably enumerable). Ash and Nerode [1981] prove that, under mild decidability conditions, this condition is also necessary.

Theorem 7.11 *(Ash and Nerode [1981]) If $R \subseteq A^n$ and $\langle \mathcal{A}, R \rangle$ is 1-decidable, then R is intrinsically computably enumerable if and only if it is formally computably enumerable. R is intrinsically computable if and only if it is formally computable.*

Actually, the 1-decidability of $\langle \mathcal{A}, R \rangle$ is a bit stronger than what Ash and Nerode need. They only need to be able to decide for each $\overline{c}$ in A and each existential $\phi(\overline{c}, \overline{x})$ if there is an $\overline{a} \notin R$ such that $\mathcal{A} \models \phi(\overline{c}, \overline{a})$. However, some conditions are necessary as Goncharov [1980a] and Manasse [1982] have constructed examples of intrinsically c.e. relations which are not formally c.e. There has been a lot of work, primarily by Ash, Ash and Knight and their students generalizing these results (under stronger decidability conditions) to syntactic characterizations of relations being intrinsically Σ_α or Δ_α for all levels α of the hyperarithmetic hierarchy. They also provide similar generalizations of the notions and results on computable categoricity and stability to higher levels of the hierarchy of computable infinitary formulas. These papers include Ash [1986], [1986a], [1987]; Ash and Knight [1990], [1994], [1995], Barker [1988] and Chisholm [1990a]. Related results when the notions are relativized to the degree of noncomputable models can be found in Ash, Knight, Manasse and Slaman [1989], Ash, Knight and Slaman [1993] and Chisholm [1990]. Here the results are proven by forcing arguments and the extra decidability hypotheses are not needed.

Faced with a computable (or c.e.) relation R on $\mathcal{A}$ which is not intrinsically computable (or c.e.), what can we say about its image under isomorphisms? In particular, how complicated can $f[R]$ be for a (computable) relation R on $\mathcal{A}$ and an arbitrary isomorphism $f : \mathcal{A} \to \mathcal{B}$ (with $\mathcal{B}$ computable, of course). An approach to this question is suggested by the following definition.

Definition 7.12 If $R \subseteq A^n$ is an n-ary relation on $\mathcal{A}$, the *degree spectrum of* R, $DgSp(R)$, is $\{\deg_T(f[R]) \mid f : \mathcal{A} \to \mathcal{B} \text{ is an isomorphism}\}$.

There are a number of results giving conditions under which the degree spectrum of a computable relation consists of precisely some particular standard class of degrees such as all the degrees, the c.e. degrees, etc. We concentrate on the issue of finding instances where the spectrum is finite and the connections between this issue and the dimension of the given structure. The first results of this sort are due to Harizanov. Here is one example.

Theorem 7.13 *(Harizanov [1993]) There is an $\mathcal{A}$ and an R on $\mathcal{A}$ such $\mathcal{A}$ has exactly two computable presentations and $DgSp(R) = \{\mathbf{0}, \mathbf{c}\}$ with $\mathbf{c}$ noncomputable and Δ^0_2.*

The next problem (that remained open for some time) was whether Δ^0_2 could be replaced by c.e. in this result or, more generally, what is possible for intrinsically c.e. relations especially for structures of finite dimension. Goncharov has announced a solution, based on work with Khoussainov, constructing a structure $\mathcal{A}$ of dimension 2 with a relation R on $\mathcal{A}$ with degree spectrum consisting of $\mathbf{0}$ and a nonzero c.e. $\mathbf{c}$. He constructs families of c.e. sets and codes them into a structure. Khoussainov and Shore have independently directly constructed directed graphs of each finite dimension n with relations having various degree spectra. Moreover, these structures can be simply modified to provide examples of ones for each n which are computably categorical but when expanded by a constant have dimension n. We first state the main result for dimension 2.

Theorem 7.14 *(Khoussainov and Shore [1998]) There is a rigid directed graph $\mathcal{A}$ (i.e. one with no nontrivial automorphisms) of dimension 2 and a subset R of A such that $DgSp(R) = \{\mathbf{0}, \mathbf{c}\}$ with $\mathbf{c}$ noncomputable and c.e. Moreover, the relation $P = \{(x, y) | x \in R^{\mathcal{A}_0} \wedge y \in R^{\mathcal{A}_1} \wedge$ there is an isomorphism from $\mathcal{A}_0$ to $\mathcal{A}_1$ which extends the map $x \mapsto y\}$ is computable.*

We sketch the proof of this theorem in §9. For now we give some generalizations and corollaries.

Theorem 7.15 *(Khoussainov and Shore [1998]) For any computable partially ordered set $\mathcal{D}$ there is a rigid directed graph $\mathcal{A}$ of dimension the cardinality of $\mathcal{D}$ and a subset R of A such that $DgSp(R) \cong \mathcal{D}$. (The ordering on $DgSp(R)$ is given by Turing reducibility.) Indeed, we can also guarantee that $R^{\mathcal{B}}$ is c.e. for every computable presentation $\mathcal{B}$ of $\mathcal{A}$ and that, if $\mathcal{D}$ has a least element, then the least element in $DgSp(R)$ is 0. Moreover, there is a uniformly computable sequence $\mathcal{A}_i$ of representatives of the computable isomorphism types of $\mathcal{A}$ such that the relation $P = \{(x, y) | x \in R^{\mathcal{A}_i} \wedge y \in R^{\mathcal{A}_j} \wedge$ there is an isomorphism from $\mathcal{A}_i$ to $\mathcal{A}_j$ which extends the map $x \mapsto y\}$ is computable.*

Corollary 7.16 *For each natural number $k \geq 2$ there exists a computably categorical structure $\mathcal{B}$ whose expansion by finitely many constants has exactly k many computable isomorphism types.*

Proof. Take the structure $\mathcal{A}$ given by Theorem 7.15 for the partial order consisting of k many incomparable elements. Let $\mathcal{A}_i$, $1 \leq i \leq k$ be the computable representatives of the computable isomorphism types of $\mathcal{A}$. So, in particular the sets $R^{\mathcal{A}_i}$ are Turing incomparable. We use the computability of P to paste the $\mathcal{A}_i$ together to produce a $\mathcal{B}$ as required. More precisely, B consists of the disjoint union of the A_i and the edges of $\mathcal{B}$ are the ones in each $\mathcal{A}_i$. In addition, $\mathcal{B}$ has an extra binary predicate defined by the relation P in the theorem and an equivalence relation E whose equivalence classes are the A_i.

Clearly $\mathcal{B}$ is a computable structure. Now let $\mathcal{B}'$ be any computable presentation of $\mathcal{B}$. Let $\mathcal{A}_1'$ and $\mathcal{A}_2'$ be two equivalence classes in $\mathcal{B}'$. These two substructures of $\mathcal{B}'$ considered as graphs are isomorphic to $\mathcal{A}$. Hence $\mathcal{A}_1'$ is computably isomorphic to one of $\mathcal{A}_1$, …, $\mathcal{A}_k$. Without loss of generality suppose that $\mathcal{A}_1'$ is computably isomorphic to $\mathcal{A}_1$ via a computable function $f_1 : \mathcal{A}_1 \to \mathcal{A}_1'$. If $\mathcal{A}_2'$ were computably isomorphic to $\mathcal{A}_1$ via a computable function $f_2 : \mathcal{A}_1 \to \mathcal{A}_2'$, then we would be able to decide $R^{\mathcal{A}_1}$ in $\mathcal{A}_1$ as follows: x in A_1 belongs to $R^{\mathcal{A}_1}$ if and only if $(f_1(x), f_2(x)) \in P$. Hence all the structures $\mathcal{A}_1'$, …, $\mathcal{A}_k'$ are pairwise non-computably isomorphic and so represent all the computable isomorphism types of $\mathcal{A}$, i.e. are computably isomorphic to $\mathcal{A}_1, \ldots, \mathcal{A}_k$ (in some order). Hence $\mathcal{B}'$ is clearly computably isomorphic to $\mathcal{B}$ and so $\mathcal{B}$ is computably categorical.

Now let a be any element from $\mathcal{A}_1$. Consider the expanded structures $\mathcal{B}_i$ consisting of $\mathcal{B}$ with the new constant interpreted as a_i, the image of a in $\mathcal{A}_i$. It is clear that all the $\mathcal{B}_i$ are isomorphic but not computably so. Thus the dimension of $\mathcal{B}_i$ is at least k. On the other hand, as $\mathcal{A}$ is rigid there are no choices other than the a_i as the interpretation of a in $\mathcal{B}$. Thus, by the computable categoricity of $\mathcal{B}$, any structure isomorphic to say $\mathcal{B}_1$ must be computably isomorphic to one of the $\mathcal{B}_i$ and so the dimension of these structures is precisely k as required. □

Corollary 7.17 *(Khoussainov and Shore [1998]) There exists a computably categorical structure without a Scott family.*

Proof. If structure of previous corollary had a Scott family it would remain computably categorical when constants were added. □

A similar construction provides an example showing that even if the structure is persistently computably categorical it need not have a Scott family.

Theorem 7.18 *(Khoussainov and Shore [1998]) There exists a structure without a Scott family such that every expansion of the structure by a finite number of constants is computably categorical.*

Kudinov independently proved more.

Theorem 7.19 *(Kudinov [1996]) There is a computably categorical* 1*-decidable structure* $\mathcal{A}$ *with no Scott family.*

Proof sketch. Kudinov slightly modifies a family of computable enumerations constructed by Selivanov [1976] and then codes the family as a unary algebra in such a way as to produce a computably categorical structure with a decidable existential theory but no Scott family. □

Of course, this Theorem shows that the assumption of 2-decidability was necessary in Goncharov's characterization (Theorem 6.16) of computably categorical structures as ones with Scott families. By Millar's result on persistence (Theorem 6.20), Kudinov's structure is persistently computably categorical and so is also a witness to Theorem 7.18.

A very natural question is whether every c.e. degree can be realized (with **0**) as a degree spectrum. Hirschfeldt has recently answered this question by adapting and extending the methods presented here.

Theorem 7.20 *(Hirschfeldt [1999]) For every c.e. degree* $\mathbf{c}$ *there is an* $\mathcal{A}$ *and a relation* R *on* $\mathcal{A}$ *such that* $DgSp(R) = \{\mathbf{0}, \mathbf{c}\}$*. Indeed* $\mathbf{c}$ *can be replaced by any uniformly c.e. array of c.e. degrees.*

Hirschfeldt's construction precisely controls the degree spectrum of the relation R but does not control dimension of $\mathcal{A}$. Thus the following question is still open.

Question 7.21 (Goncharov and Khoussainov [1997]) Which n-tuples of c.e. degrees can be realized as the degree spectrum of a relation on a structure of dimension n?

If we move beyond the c.e. degrees there are a few results by Harizanov on possible degree spectra but not much is known. However, we should point out that several natural strengthenings of these results can ruled out by classical descriptive set theoretic results.

Remark 1 *For a given relation* R *on a computable structure* $\mathcal{A}$*, the set* $\{R^{\mathcal{B}} |\ \mathcal{B}$ *is a computable presentation of* $\mathcal{A}\}$ *is* Σ^1_1 *in* R*. Thus, there are countable partial orderings that cannot be realized in the c.e. degrees as the degree spectrum of any relation* R *on any computable structure* $\mathcal{A}$*. (Just consider one that is too complicated to be* Σ^1_1*.) Similarly, such a partial ordering with least element cannot be realized anywhere in the Turing degrees as the degree spectrum of a computable relation* R *on a computable structure* $\mathcal{A}$*. Nor can it be true that any finite set of degrees can be realized as the degree spectrum of any relation* R *on a computable structure* $\mathcal{A}$*. Indeed, any degree spectrum containing both a hyperarithmetic degree and a nonhyperarithmetic degree is uncountable as any* Σ^1_1 *set with a nonhyperarithmetic member is uncountable.*

8. Algebraic Examples

In §6 we saw several examples of theories whose models all have dimension 1 or ω and algebraic conditions characterizing the models in each class. The theories of this sort considered there were linear orderings, Boolean algebras and algebraically closed fields. We cite two more.

Theorem 8.1 *A real closed field of finite transcendence degree over $\mathbb{Q}$ is computably stable. One of infinite transcendence degree has dimension ω.*

Proof. If a real closed field $\mathcal{A}$ has finite transcendence degree over $\mathbb{Q}$ and $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism, let $a_1, \ldots, a_n$ be a transcendence basis for $\mathcal{A}$ over $\mathbb{Q}$ and $b_1, \ldots, b_n$ be their image in $\mathcal{B}$. Calculate $f(a)$ for any element a of A by first finding an equation over $\mathbb{Q}[a_1, \ldots, a_n]$ satisfied by a. Find all its solutions and the place of a among these solutions in the order of $\mathcal{A}$. Now, $f(a)$ must be the solution of the same equation over $\mathbb{Q}[b_1, \ldots, b_n]$ which lies in the same place among all the solutions in $\mathcal{B}$ listed in order. Thus f is computable. On the other hand, if $\mathcal{A}$ is of infinite transcendence degree then by Theorem 6.11, it has dimension ω. (Note that as the theory of real closed fields has effective quantifier elimination, every computable model is decidable. Moreover, the prime model of $Th(\mathcal{A}, \overline{c})$ for any finite list $\overline{c}$ of elements of $\mathcal{A}$ is of finite transcendence degree and so not $\mathcal{A}$ itself.) $\square$

Theorem 8.2 *(Goncharov [1981]) If $\mathcal{A}$ is an abelian group then it has dimension 1 or ω.*

The proof of this result is particularly interesting because it relies on important sufficient condition for a structure to have dimension ω.

Theorem 8.3 *(Goncharov [1982]) If there is a Δ^0_2 isomorphism between $\mathcal{A}$ and $\mathcal{B}$ but no computable one then $\mathcal{A}$ has dimension ω.*

On the other hand, the results described in §6.2 and §7, as well as many earlier papers, supply examples of structures of dimension n for each $n \in \omega$. Indeed, our results supply examples of structures of dimension n whose presentations are characterized by the Turing degree of a specific relation on the structure. Moreover, representatives of the n many computable isomorphism types of these structures can be pasted together to produce a single computably categorical structure such that an expansion by constants yields a structure of dimension n. Of course, when there are characterization theorems that show that the dimension must be 1 or ω such constructions are not possible. On the other hand, for many familiar algebraic theories for which we cannot provide such a dichotomy and characterization, it is possible to construct examples of models not only of each finite dimension but also ones exhibiting the additional properties enjoyed by the examples in §7.

Theorem 8.4 *(Goncharov [1980a], [1981]; Goncharov and Dobrotun [1989], Goncharov, Molokov and Romanovski [1989]; Kudinov (personal communication); Hirschfeldt, Khoussainov, Slinko and Shore [1999]) For each of the following theories and each $n \in \omega$, there is a model $\mathcal{A}$ with a subset R such that the dimension of $\mathcal{A}$ is n and the degree spectrum of R consists of n different c.e. degrees. Moreover, for each n there is a model $\mathcal{A}$ which is computably categorical but some expansion by constants has dimension n: graphs, lattices, partial orders, nilpotent groups, rings (with zero divisors) and integral domains. In each case the subset R can be taken to be a substructure of the appropriate type.*

The results on the existence of models of each of these theories of each finite dimension are due to various people (most to Goncharov and his coauthors, the one for integral domains is due to Kudinov). They were typically first proved by codings of families of c.e. sets. For graphs, the results involving degree spectra and extensions by constants are due to Khoussainov and Shore and are described in §7. (Actually, the original paper used directed graphs but an examination of the construction shows that it is possible to use undirected graphs instead.) All the other ones involving degree spectra and extensions by constants have been proven by Hirschfeldt, Khoussainov, Slinko and Shore [1999].

Although direct constructions are sometimes possible, these results can all be derived from the results on graphs by finding a sufficiently effective coding of graphs into models of each theory. The idea is that, if the coding is sufficiently effective, all the computability properties involved carry over. Thus all these theories are not only undecidable but the codings (of say graphs) needed to prove that they are universal (i.e. code all of predicate logic) are highly effective. (In addition to simple codings of the domain and edge relation on the initial graph, an important issue is the effective reversibility of the coding. That is, one wants the model coding a given graph to effectively determine the original graph.) On the other hand, the theories discussed in §6 whose models are all either computably categorical or of dimension ω are decidable and have strong structure theorems that are used in the proofs. We expect that there are natural theories that are neither "so decidable" as those of §6 nor "so undecidable" as the ones in Theorem 8.4. In particular, we suggest the theory of fields as a good test case as it is undecidable but the proofs of undecidability (that we know) interpret $\mathbb{N}$ in a rather specific way rather than arbitrary structures.

9. The Basic Theorem on Degree Spectra

In this section we sketch the proof of the Theorem 7.14, the case $n = 2$ of our main theorem on degree spectra.

Theorem 7.14 *(Khoussainov and Shore [1998]) There is a rigid directed graph $\mathcal{A}$ of dimension 2 with computable (but not computably isomorphic) presentations $\mathcal{A}_0$ and $\mathcal{A}_1$and a subset R of A such that $DgSp(R) = \{\mathbf{0}, \mathbf{c}\}$ with $\mathbf{c}$ noncomputable* *and c.e. Moreover, the relation $P = \{(x,y)|x \in R^{\mathcal{A}_0} \wedge y \in R^{\mathcal{A}_1} \wedge$ there is an isomorphism from $\mathcal{A}_0$ to $\mathcal{A}_1$ which extends the map $x \mapsto y\}$ is computable.*

Proof sketch. Our directed graph $\mathcal{A}$ will consist of disjoint components $[B_i]$ all of one special type. The graph we denote by $[B]$ is uniquely determined by the set $B \subseteq \{n|n \geq 5\}$. It consists of one 3-cycle and one n-cycle for each $n \in B$. In addition, there is one element of the 3-cycle, called the *top* of the graph, from which there is an edge to one element of each n-cycle for $n \in B$. This element of the n-cycle is called the *coding location for n*. For convenience, we also denote $[\{n\}]$ by $[n]$. We build up our graph using two operations, $+$ and $\cdot$. The sum $[A] + [B]$ of two graphs is simply their disjoint union. The product, $[A] \cdot [B]$, of two graphs of our special form is gotten by taking disjoint copies of $[A]$ and $[B - A]$ and identifying the top elements (and the associated 3-cycles) in each of the two graphs. For example $[5] \cdot [6] \cong [\{5, 6\}]$ and $[\{5, 6, 7\}] \cdot [\{6, 7, 8, 9\}] \cong [\{5, 6, 7, 8, 9\}]$. Note that $[A] \cdot [B] \cong [B] \cdot [A]$.

Our plan is to construct our graph $\mathcal{A} = [B_0] + [B_1] + \cdots + [B_n] + \cdots$ together with enumerations of the sets B_i so that $B_i - B_j \neq \emptyset$ for $i \neq j$ (and indeed we guarantee that $B_{i,s} - B_{j,s} \neq \emptyset$ for every s and $i \neq j$). So clearly

- $\mathcal{A}$ is rigid.

The required R will be a subset of the coding points in $\mathcal{A}$. We enumerate two presentations $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ as $\mathcal{A}_{0,s}$ and $\mathcal{A}_{1,s}$ each isomorphic to $[B_{0,s}] + [B_{1,s}] + \cdots$ and the interpretations R_i (as $R_{i,s}$) of R in $\mathcal{A}_i$ so that

- R_0 is computably enumerable but not computable: As the construction proceeds we enumerate the elements x of R_0 so as to make the set enumerated noncomputable by a standard diagonalization procedure.
- R_1 is computable: As we enumerate a number x into R_0, we make sure that the corresponding element y of $\mathcal{A}_1$ is a new large number. Thus R_1 is enumerated in increasing order.
- $P = \{\langle x, y\rangle\, |x \in R_0 \wedge y \in R_1 \wedge (\exists f : \mathcal{A}_0 \cong \mathcal{A}_1)(f(x) = y)\}$ is computable: By the procedure alluded to above for choosing the $y \in R_1$ corresponding to a given $x \in R_0$, the pairs $\langle x, y\rangle \in P$ are enumerated in increasing order.
- $\mathcal{A}_0 \not\cong_c \mathcal{A}_1$: This is guaranteed by the previous requirements that R_0 is computable but R_1 is not. By the rigidity of $\mathcal{A}$, there is only one isomorphism from $\mathcal{A}_0$ to $\mathcal{A}_1$ and it must take R_0 to R_1. If it were computable it would preserve the computability of the interpretation of R.
- Every computable presentation $\mathcal{G}_j$ of $\mathcal{A}$ is computably isomorphic to $\mathcal{A}_0$ or $\mathcal{A}_1$: Our plan here is to define maps $r_{j,s}$ so that at every stage s of the construction at which it still looks as if $\mathcal{G}_j$ might be isomorphic to $\mathcal{A}$, $r_{j,s}$ is a monomorphism from $\mathcal{G}_{j,s}$ into $\mathcal{A}_{i,s}$ (for $i = 0$ or 1) and that, at the end of stage s, if we cannot extend the current map $r_{j,s}$ then we switch so that $r_{j,s+1}$ is a monomorphism from $\mathcal{G}_{j,s+1}$ into $\mathcal{A}_{1-i,s+1}$. If, after some stage t, we never switch our potential isomorphism then $\cup\{r_{j,s}|t \leq s\}$ is, in fact, the desired computable isomorphism

from $\mathcal{G}_j$ to $\mathcal{A}_i$. On the other hand, if we switch infinitely often we guarantee that there is a *special component* S_j of $\mathcal{G}_j$ which is not a component of $\mathcal{A}$ and so $\mathcal{G}_j$ is not isomorphic to $\mathcal{A}$.

The crucial idea needed for the construction is how to diagonalize to make R_0 noncomputable while its isomorphic image R_1 remains computable and also maintain control over the potential isomorphisms between $\mathcal{G}_j$ and $\mathcal{A}_i$. The diagonalization procedure is based on two symmetric operations $\mathbf{L}$ (*left*) and $\mathbf{R}$ (*right*) on sequences of graphs $[B_i]$.

Definition 9.1 $\mathbf{L}([B_1], \ldots, [B_n])$ is the graph

$$[B_1] \cdot [B_2] + \cdots + [B_{n-1}] \cdot [B_n] + [B_n] \cdot [B_1].$$

$\mathbf{R}([B_1], \ldots, [B_n])$ is the graph

$$[B_1] \cdot [B_n] + [B_1] \cdot [B_2] + \ldots + +[B_{n-1}] \cdot [B_n].$$

We apply the $\mathbf{L}$ operation, for example, to a graph $\mathcal{G}$ whose components include the $[B_i]$ by removing all the $[B_i]$ and inserting $\mathbf{L}([B_1], \ldots, [B_n])$. We also adopt the convention that the elements of the component $[B_i]$ are the same ones in the corresponding subgraph in the component $[B_i] \cdot [B_{i+1}]$ of $\mathbf{L}([B_1], \ldots, [B_n])$ while those elements in the new graph corresponding to ones in $[B_{i+1}]$ of the original graph are new elements in $[B_i] \cdot [B_{i+1}]$ (with 1 for $n+1$ when $i = n$). This convention is important for establishing computability properties of the graphs being constructed.

We will apply an $\mathbf{R}$ operation in the construction (to $\mathcal{A}_1$) only when we also apply an $\mathbf{L}$ one (to $\mathcal{A}_0$). We also have the corresponding convention that the elements of the component $[B_{i-1}]$ are the same ones in the corresponding graph in the component $[B_{i-1}] \cdot [B_i]$ of $\mathbf{R}([B_1], \ldots, [B_n])$ while those elements in the new graph corresponding to ones in $[B_i]$ of the original graph are new elements in $[B_{i-1}] \cdot [B_i]$ (with 0 for n when $i = 1$).

The following lemma is immediate from the definitions.

Lemma 9.2 *For any sequence* $[B_1], \ldots, [B_n]$ *of graphs,* $\mathbf{L}([B_1], \ldots, [B_n])$ *and* $\mathbf{R}([B_1], \ldots, [B_n])$ *are isomorphic and extend* $[B_1] + \cdots + [B_n]$. *Moreover, if* $\mathcal{G}$ *has the* $[B_i]$ *as components then replacing their sum with* $\mathbf{L}([B_1], \ldots, [B_n])$ *or* $\mathbf{R}([B_1], \ldots, [B_n])$ *produces two isomorphic graphs each extending* $\mathcal{G}$. $\Box$

The plan for diagonalization is now easily described. To make sure that $R_0 \neq \phi_e$, we choose numbers a_e, b_e and c_e and insert copies of $[a_e], [b_e]$ and $[c_e]$ into $\mathcal{A}_0$ and $\mathcal{A}_1$. For definiteness, say that x_e is the (number which is) the coding location for a_e in these graphs. We now wait for $\phi_e(x_e)$ to converge to 0. If it never does we do nothing and so win as x_e is not in R_0. If $\phi_e(x_e)$ converges to 0 at stage s, we replace the components $[a_e], [b_e]$ and $[c_e]$ in $\mathcal{A}_0$ and $\mathcal{A}_1$ by $\mathbf{L}([b_e], [a_e], [c_e])$ and $\mathbf{R}([b_e], [a_e], [c_e])$, respectively; put x_e into R_0 and its image in $\mathcal{A}_1$ into R_1. The crucial point here is that, by our conventions, the image of x_e (as an element of

$\mathbf{L}([b_e],[a_e],[c_e])$ in $\mathcal{A}_0$) in $\mathbf{R}([b_e],[a_e],[c_e])$ and so in $\mathcal{A}_1$ is a new large number. Thus we diagonalize for R_0 but keep R_1 computable.

The remaining issue is how to simultaneously satisfy the requirements that, if isomorphic to $\mathcal{A}$, $\mathcal{G}_j$ is computably isomorphic to $\mathcal{A}_0$ or $\mathcal{A}_1$. Consider the requirement for a single $\mathcal{G}$. Following the idea described above, we choose a *special component* $[S]$ of $\mathcal{G}$ and make its image in the $\mathcal{A}_i$ participate in infinitely many of the left and right operations done for diagonalizations. We have some definition of expansionary stage that measures the extent of a possible isomorphism between $\mathcal{G}$ and $\mathcal{A}$. If there are only finitely many such expansionary stages then $\mathcal{G}$ is not isomorphic to $\mathcal{A}$ and no other actions are necessary. So suppose there are infinitely many expansionary stages.

At each expansionary stage s we have a monomorphism r_s from $\mathcal{G}$ into $\mathcal{A}_i$ and components $[S_{i,s}]$ of $\mathcal{A}_i$ (for $i = 0$ or 1) corresponding to the special component $[S]$ of $\mathcal{G}$. If we wish to diagonalize at a coding location x_e in the range of r, we wait for the next expansionary stage s and perform $\mathbf{L}$ and $\mathbf{R}$ operations in $\mathcal{A}_0$ and $\mathcal{A}_1$, respectively, on the sequence $[b_e],[a_e],[c_e],[P_e],[S_{i,s}],[Q_e]$. Here P_e and Q_e are either numbers chosen in advance for the requirement for $\mathcal{G}$ or sets that have participated in one of these two locations in a previous operation for $\mathcal{G}$. In any case, all of these components are in the range of r_s when we perform the operations. Suppose r_s mapped $\mathcal{G}$ into $\mathcal{A}_i$. The crucial point is that when we next get an expansionary stage at t and it is possible to extend r_s so as to keep $[S]$ mapped into $[S_{i,s}]$ then it is possible to extend r_s to be a map of $\mathcal{G}$ into $\mathcal{A}_i$ at t. The key idea here is that each component in the original sequence can "grow into" only one of two components in the final one, itself or the one immediately to its left (or right depending on whether $i = 0$ or 1). Thus if the image of one of the components remains fixed then we can see (in the reverse order of the operation performed) that each component in turn remains fixed as it has no other place to go. In this case, we extend r_s to r_t still mapping $\mathcal{G}$ into $\mathcal{A}_i$. If it is not possible to keep $[S]$ mapped into $[S_{i,s}]$ then we change r so as to define r_t as a map from $\mathcal{G}$ into $\mathcal{A}_{1-i}$. This also means that $[S_{i,t}]$ is not the same component in $\mathcal{A}_i$ as was $[S_{i,s}]$ (or we could have kept it fixed). (Actually it is the component that had been $[P_e]$ or $[Q_e]$ depending on the specifics of the situation.) We now guarantee never to use the old $[S_{i,s}]$ component in any future operation.

The ultimate consequence of such a procedure is that, if we change the range of r_t infinitely often, $[S]$ becomes infinite in $\mathcal{G}$ but each component $[S_{i,s}]$ that is a potential image of $[S]$ in $\mathcal{A}_i$ is involved in only finitely many operations and so is itself finite. Thus, in this case, $\mathcal{G}$ is not isomorphic to $\mathcal{A}$. On the other hand, if $\mathcal{G}$ actually is isomorphic to $\mathcal{A}$, we keep extending r_s from some stage on while never changing the $\mathcal{A}_i$ to which it maps $\mathcal{G}$. In this case, we arrange the definition of r_s so that if it eventually maps onto $\mathcal{A}_i$ and so determines the required computable isomorphism from $\mathcal{G}$ to $\mathcal{A}_i$.

We have, of course, omitted some of the combinatorics (particularly the way in which we extend the domain of r) even in this case of one $\mathcal{G}$ requirement. The full construction consists of using a module of this sort for each requirement on a typical $0''$ priority tree. Of course, the precise actions for a diagonalization requirement at a node α (e.g. which special components have to go into the sequence on which the operations are performed and in what order they go on this list) depend on the outcome of nodes β of higher priority contained in α which are devoted to various $\mathcal{G}_j$. (The choices here are whether there are infinitely many expansionary stages or not and if so whether the range of r_s is fixed as i from some point on or we change it infinitely often.) The details can be found in Khoussainov and Shore [1998].

We take this opportunity to point out two corrections that should be made to the details of the general construction found in Khoussainov and Shore [1998]. The first is that whenever one applies (or considers the application of) an operation for a node β to a sequence of the form

$$B^k_{n+1}, X^k, C^k_{n+1}, B^k_n, [S]^k_{\beta_n,t+1}, C^k_n, \ldots D^k_1, B^k_1, [S]^k_{\beta_1,t+1}, C^k_1$$

one should instead use its β-*transform* which is defined to be the sequence $B^k_{i_1}$, $[S]^k_{\beta_{i_1},t+1}, C^k_{i_1}, B^k_{i_2}, [S]^k_{\beta_{i_2},t+1}, C^k_{i_2}, \ldots, B^k_{i_m}, [S]^k_{\beta_{i_m},t+1}, C^k_{i_m}, B^k_{n+1}, X^k, C^k_{n+1}, B^k_{j_1}$, $[S]^k_{\beta_{j_1},t+1}, C^k_{j_1}, B^k_{j_2}, [S]^k_{\beta_{j_2},t+1}, C^k_{j_2}, \ldots, B^k_{j_r}, [S]^k_{\beta_{j_r},t+1}, C^k_{j_r}$ where $i_1, i_2, \ldots, i_m$ list, in order, the i such that the designated isomorphism for the $\beta_i = \beta \upharpoonright (3i+1)$ such that $\beta(3i+1) \neq w$ at $t+1$ is $r^0_{\beta_i,t+1}$ and $j_1, j_2, \ldots, j_r$ list, in order, the j such that the designated isomorphism for the $\beta_j = \beta \upharpoonright (3j+1)$ such that $\beta(3j+1) \neq w$ is $r^1_{\beta_{i,t+1}}$.

The second concerns the marking of numbers with the symbols $\square^\beta_w$. No numbers should be marked in Case 1 of the construction. As a result, condition 1 in Subcase 2.1 should be weakened by not requiring that the image of $r^k_{\beta_i,t+1}$ has nonempty intersection with B^k_i if the designated isomorphism for β_i is $r^0_{\beta_i,t+1}$ or with C^k_i if the designated isomorphism for β_i is $r^1_{\beta_i,t+1}$. Instead, the marking take place at the end of each stage of the construction as follows:

At the end of stage $t+1$ we do some additional cancelation and marking. Suppose $t+1$ is a γ recovery stage. If there are any uncancelled components isomorphic to $[b_{\beta,\gamma,u}]$ or $[c_{\beta,\gamma,u}]$ for $u \leq t$ (and $\beta \supseteq \gamma$) which, necessarily, have not participated in any operation, we cancel them and appoint new ones $[b_{\beta,\gamma,t+1}]$ or $[c_{\beta,\gamma,t+1}]$, respectively. We now mark all of the following with $\square^\gamma_w$ if they are not already so marked:

1. Any cancelled component.
2. Any component associated with a node β to the left of γ.
3. Any component of the form $[q_{\beta,t+1}]$ with $\beta \supseteq \gamma$.
4. Any components of the form $[b_{\beta,t+1}]$, $[c_{\beta,t+1}]$ or $[p_{\beta,t+1}]$ for $\beta \supseteq \gamma$.
5. Any components of the form $[b_{\beta,\beta_i,t+1}]$ or $[c_{\beta,\beta_i,t+1}]$ for $\beta \supseteq \beta_i \supset \gamma$.
6. Any components of the form $[b_{\beta,\gamma,t+1}]$ if $\beta \supseteq \gamma$ and the designated isomorphism for γ is $r^1_{\gamma,t+1}$or of the form $[c_{\beta,t+1}]$ if $\beta \supseteq \gamma$ and the designated isomorphism

for γ is $r^0_{\gamma,t+1}$.

7. If, at $t+1$, we performed an operation on the β-transform of a sequence $B^0_{n+1}, X^0, C^0_{n+1}, B^0_n, [S]^0_{\beta_n}, C^0_n, \ldots D^0_1, B^0_1, [S]^0_{\beta_1,t+1}, C^0_1$ (and so $\gamma \subseteq \beta$), then, for $k = 0, 1$, we mark B^k_i or C^k_i if $\beta_i \subseteq \gamma$ and it has previously participated in an operation; we mark B^k_i if $\beta_i \subseteq \gamma$ and the designated isomorphism for β_i is $r^1_{\beta_i,t+1}$; we mark C^k_i if $\beta_i \subseteq \gamma$ and the designated isomorphism for β_i is $r^0_{\beta_i,t+1}$; and we mark both B^k_i and C^k_i for $\beta_i = \gamma$.

The changes needed in the verifications to take advantage of these corrections are straightforward. A complete corrected version of the paper can be found at http://math.cornell.edu/~shore/.□

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