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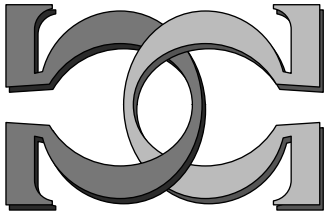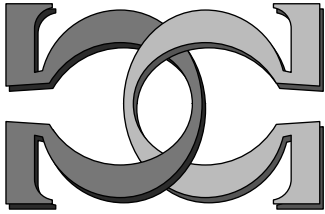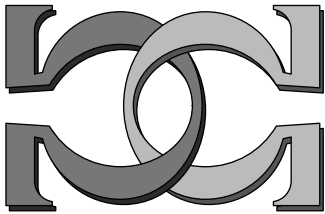**Report**
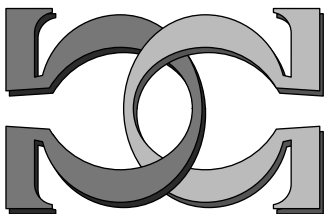**Series**
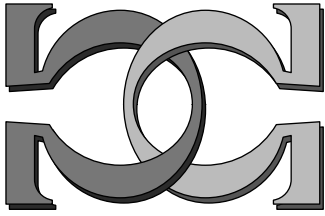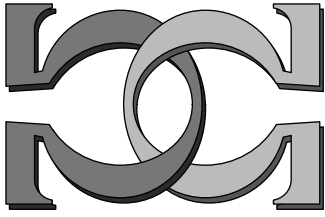
Finite Automata and
Isomorphism Types

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1   Introduction and Basic Results

In this paper we report on some results about finite automata presentable structures, an area of research first proposed in a Khoussainov-Nerode paper in 1995 [1]. The topic of automata presentable structures is new, under development and is on the edge of interactions between automata theory, (universal) algebra, model theory and complexity theory.

One motivation for the development of the theory of FA presentable structures comes from the theory of computable structures, the area devoted to understanding the effective content of results in model theory and algebra. We refer the interested reader to the Handbook of Recursive Mathematics [2], the Handbook of Computability Theory [3] and the survey paper [4]. In the theory of computable structures one naturally assumes that there are unbounded resources for performing computations and hence the basic model of computation is the one of Turing. Cenzer, Nerode and Remmel suggest the idea of considering feasible computations (e.g. polynomial time computations) and investigate the effect of such computations on results in algebra and model theory. Research in this area has grown into the theory of feasible structures (e.g. polynomial time structures), or more generally into feasible mathematics [5] [6] [7].

In [1] Khoussainov-Nerode suggest the development of a finer theory, the theory of structures presented by automata. Informally, a structure $\mathcal{A}$ of a finite signature is **automatic** if its domain and all the atomic relations are recognised by finite automata. Then any structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ is called **finite automaton (FA) presentable**. In this case $\mathcal{A}$ is a **FA presentation** of $\mathcal{B}$. In [1] it is shown that any automatic structure can be characterised in terms of congruences of finite index of a certain partial

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free monoid. This generalises the known Myhill-Nerode theorem about FA recognisable languages. Another somewhat unexpected result is the next theorem proved in [1] that shows how FA presentable structures differ from computable or even polynomial time computable ones.

Theorem 0 [1] *The first order theory of any FA presentable structure is decidable. Moreover, if $\mathcal{A}$ is automatic, then there exists an algorithm that applied to any first order definition of any relation produces a finite automaton that recognises the relation.*

Indeed there are computable (even polynomial) structures, such as the arithmetic $(\omega, \ 0, \ S, \ + \ \times, \ \leq)$ for instance, whose first order theories are not computable and may even decide the ω-jump of the computable degree.

Consideration of automatic structures poses several natural questions. We are concerned with one of them which is the following:

What structures have or do not have FA presentations?

As mentioned above, one possible answer to this question is given in [1] in terms of congruences of finite index over a certain partial free monoid. However, this answer is not satisfactory in the sense that it does not explain what the isomorphism types of FA presentable structures look like. In this paper we give some answers to this question by giving algebraic characterisations of certain FA presentable structures over unary alphabet. Here are some of these results which we wish to report at this stage.

The first theorem concerns structures of the type (A, f), where f is a bijection on A, called **permutation structures**.

Theorem 1 *A permutation structure (A, f) is FA presentable if and only if the number of infinite f-orbits is finite and there exits a finite bound on the sizes of all finite f-orbits.*

The next result concerns structures of the type (A, E), where E is an equivalence relation on A, called **equivalence structures**.

Theorem 2 *An equivalence structure (A, E) is FA presentable if and only if the number of infinite E-equivalence classes is finite and there exists a finite bound on the sizes of all finite E-equivalence classes.*

Finally, the third result concerns linearly ordered sets. We recall that ω is the type of the ordering of the natural numbers, $\omega^\star$ is the type of

the ordering of negative integers, and n is the type of a linear order on n elements.

Theorem 3 *A linearly ordered set* $(L, \leq)$ *is FA presentable if and only if it is isomorphic to a finite sum of linearly ordered sets of the type* ω *,* $\omega^\star$ *or* n.

We would like to make a few comments about the theorems above. On the one hand, we note that the notion of a FA presentable structure is a computability-theoretic (or more precisely automata-theoretic) notion. The notion of isomorphism type of a structure, on the other hand, is an algebraic notion (or if one wishes a set-theoretic notion). The theorems above show what effect FA presentability makes on isomorphism types and manifest the relationship between finite automata and isomorphism types of structures. We point out that the theorems do not refer to some computability–theoretic notions such as Kleene-Mostowski hierarchy, recursive ordinals or automata-theoretic terminology in describing the isomorphism types of FA presentable structures. This is, in fact, another essential difference between the theory of FA presentable structures and the theory of computable (polynomial time computable) structures. In the latter case, characterisations of computably presentable structures (if any) usually involve some computability-theoretic terminology by stating, for instance, that certain invariants of a structure belong to a some level of the Kleene–Mostwoski hierarchy or correspond to some recursive ordinal.

The paper is organised in four parts. In the next section we give a formal definition of FA presentable structures and provide several examples. In Section 3 we outline the proof of the theorems above. The last section reports on work in progress and discusses some open questions.

2 Definitions and Examples

We begin with giving definitions of automatic and FA presentable structures. We need some preliminary notions. Let Σ be a finite alphabet. Suppose that the symbol $\diamond$ does not belong to Σ. Consider words $u_i = \sigma_{i1} \ldots, \sigma_{in_i}$ from $\Sigma^\star$, where $i = 0, \ldots, n-1$. The **convolution** $u_0 \star \ldots \star u_{n-1}$ of these words is defined as follows. If for all $i, j < n$ $n_i = n_j$, then the convolution is

$$(\sigma_{01}, \ldots, \sigma_{(n-1)1}), \ldots, (\sigma_{0n_0}, \ldots, \sigma_{(n-1)n_0}).$$

Otherwise, let m be the maximal length of the words $u_0, \ldots, u_{n-1}$. Add to the right end of each u_i the necessary number of symbols $\diamond$ to get words of length m. Call these new words u_i', $i = 0, \ldots, n-1$. Then the **convolution** of the n-tuples $u_0, \ldots, u_{n-1}$ is $u_0' \star \ldots \star u_{n-1}'$. This convolution is a word of the new alphabet $(\Sigma \bigcup \{\diamond\})^n$. Thus, for any n–ary relation R on $\Sigma^\star$ we can consider the subset $R^\star \subset (\Sigma \bigcup \{\diamond\})^n$ obtained from R using convolution, that is,

$$R^\star = \{u_0 \star \ldots \star u_{n-1} | (u_0, \ldots, u_{n-1}) \in R\}.$$

Definition 1 1) **An n-variable automaton** *on Σ is a finite automaton over the alphabet $(\Sigma \bigcup \{\diamond\})^n$.* 2) *An n-ary relation R on $\Sigma^\star$ is* **FA recognisable***, if its convolution $R^\star$ is recognisable by an n–variable automaton.*

Let $\mathcal{A}$ be a structure of fixed signature $(P_1^{n_1}, \ldots, P_k^{n_k}, c_0, \ldots, c_t)$ that contains predicate and constant symbols only. If a structure contains operation symbols, then we identify this structure with the relational structure by replacing all the operations with their graphs.

Definition 2 *The structure $\mathcal{A}$ is* **automatic** *over Σ if its domain and the basic relations are FA recognisable. An isomorphism from $\mathcal{A}$ to a structure $\mathcal{B}$ is called a* **FA presentation** *of $\mathcal{B}$.*

Now we present several examples of FA presentable structures.

Example 1 *The structure $(Q, \leq)$, where Q is the set of rationals and $\leq$ is the usual ordering of the rationals, possesses a FA presentation.*

Indeed, we use the coding of Rabin [8]. Let $\Sigma = \{0, 1\}$ and let D be a set such that $\alpha 101 \in D$ if and only if $\alpha \in \Sigma^\star$ and α does not have the subword 101. It is clear that D is a recognisable subset of $\Sigma^\star$. Consider the lexicographic linear ordering $\preceq$ on the set $\Sigma^\star$. This ordering is recognisable by a 2–variable automaton. Thus the linear ordered set $(D, \preceq)$ is automatic. It is not hard to see that $(D, \preceq_l)$ is isomorphic to $(Q, \leq)$.

One can find automatic presentations of the structures in the examples below.

Example 2 *Any ordinal ω^n has a FA presentation.*

Example 3 *Any free unary structure $\mathcal{A}$ possesses a FA presentation.*

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Example 4 *Any finitely generated abelian group possesses a FA presentation.*

Example 5 *Any countable vector space over a finite field possesses a FA presentation.*

Example 6 *The structure $(\omega, +, \leq)$ possesses a FA presentation.*

At the end of this section we note that by Theorem 0 (see introduction) the first order theories of all structures provided in the examples above are decidable. In particular, the decidability of the theory of Presburger arithmetic $(\omega, +, \leq)$ follows. Along these lines, any structure whose first order theory is not decidable can not have FA presentation. In particular, the arithmetic $(\omega, 0, S, +, \times, \leq)$ and the ring $F[t]$ of polynomilas with one variable t over any field F can not have FA presentations since their theories are known to be undecidable. An example of a structure whose first order theory is decidable but which does not possess a FA presentation is the absolutely free algebra of signature (f, c), where c is a constant and f is a binary functional symbol. Some of these facts and above examples are known from [1].

3   Proofs of Theorems

In this section we basically outline (in more or less details) the proof of Theorem 1 and Theorem 2 stated in the introduction. The proof of Theorem 1 is exemplary since the proofs of Theorem 2 and Theorem 3 follow similar patterns and are even easier. We will not include the proof of Theorem 3 as it can be presented using methods developed in the proofs of Theorem 1 and Theorem 2.

Our assumption is that the alphabet Σ is unary and so $\Sigma = \{1\}$. We also restrict ourselves to considering those automatic structures over $\{1\}$ whose domains are $\{1\}^\star$. This will simplify our discussion and the technical side of the proofs.

Our proofs of the necessary parts of the theorems are somewhat technical and require consideration of certain types of finite automata and the introduction of some notions and notations. Since all the theorems concern structures containing binary predicate symbols (the graph of a unary function, the symbol for equivalence relation, the symbol for linear order), we will deal with 2-variable automata over the alphabet $\{1\}$ and convolutions

of binary relations on the set $\{1\}^\star$. So let $\Omega = \{1, \diamond\}^2$. Let $\mathcal{A} = (S, I, T, F)$ be a FA over Ω.

Definition 3 *The* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$*-chain is defined as the minimal set X of states such that:*

1. *$I \subset X$,*

2. *For all $s \in X$ if $\left(s, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, s'\right) \in T$, then $s' \in X$.*

Clearly there exists exactly one $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$-chain X. Elements of X are called X-states. Using states from X we give the next definition.

Definition 4 *For any $x \in X$, we define the $\left(x, \begin{pmatrix} 1 \\ \diamond \end{pmatrix}\right) - chain$ from x as the minimal set Y of states such that:*

1. *$x \in Y$,*

2. *For all $s \in Y$ if $\left(s, \begin{pmatrix} 1 \\ \diamond \end{pmatrix}, s'\right) \in T$, then $s' \in Y$.*

Clearly there exists exactly one $\begin{pmatrix} 1 \\ \diamond \end{pmatrix} - chain$ from x. Elements of the $\left(x, \begin{pmatrix} 1 \\ \diamond \end{pmatrix}\right) - chain$ other than x itself will be called $\left(x, \begin{pmatrix} 1 \\ \diamond \end{pmatrix}\right)$-states. Sometimes we call these chains $\begin{pmatrix} 1 \\ \diamond \end{pmatrix} - chains$ without specifying the X-states. The notion of $\left(x, \begin{pmatrix} \diamond \\ 1 \end{pmatrix}\right) - chain$ is defined similarly. Now we are ready to give the following definition.

Definition 5 *The automaton $\mathcal{A}$ is* **reduced** *if it satisfies the following conditions:*

1. *For all $s \in S, \sigma \in \Omega$, $T(s, \sigma)$ contains at most one element.*

2. *If s is a $\left(x, \begin{pmatrix} 1 \\ \diamond \end{pmatrix}\right)$-state ($\left(x, \begin{pmatrix} \diamond \\ 1 \end{pmatrix}\right)$-state, X-state), then $T(s, \sigma)$ is defined if and only if $\sigma = \begin{pmatrix} 1 \\ \diamond \end{pmatrix}$ ($\sigma = \begin{pmatrix} \diamond \\ 1 \end{pmatrix}$, $\sigma = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively).*

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3. For all distinct $x, x' \in X$, no pair of the sets $(x, \begin{pmatrix} 1 \\ \diamond \end{pmatrix})$ − chain, $(x, \begin{pmatrix} \diamond \\ 1 \end{pmatrix})$ − chain, $(x', \begin{pmatrix} 1 \\ \diamond \end{pmatrix})$ − chain, $(x', \begin{pmatrix} \diamond \\ 1 \end{pmatrix})$ − chain have elements in common.

4. For all $x \in X$, $T(x, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$ is defined.

5. For all $x \in S$, $T(x, \begin{pmatrix} \diamond \\ \diamond \end{pmatrix})$ is undefined.

Now we state the following lemma whose proof follows from the mentioned generalised Myhill-Nerode theorem and can be found in [1].

Lemma 1 *A binary relation R on $\Sigma^\star$ is FA recognisable if and only if the convolution of R can be recognised by a reduced automaton.* □

This lemma allows us to consider reduced automata only. We will be refering to this fact without specifically mentioning it. Now we begin our proofs.

Proof of Theorem 1. We first prove the sufficient condition. Assume that $\mathcal{U} = (U, f)$ is a permutation structure. For $a \in U$, the set $\{\ldots, f^{-1}(a), a, f(a), \ldots\}$ is called an **orbit** of the structure. The cardinality of the orbit is its **size**. We assume that $\mathcal{U}$ has finitely many orbits of infinite size and there exists a number n such that the sizes of all finite orbits are bounded by n. First, we note that the structure (Z, S), where $S(k) = k + 1$, is a permutation structure. It is easy to see that this structure has a FA presentation over the alphabet $\{1\}$. Let $\mathcal{U}_m$ be a permutation structure whose orbits are all of size m. Again it is easy to give a FA presentation of $\mathcal{U}_m$ over the alphabet $\{1\}$. If $\mathcal{U}_1 = (U_1, f_1)$ and $\mathcal{U}_2 = (U_2, f_2)$ are permutation structures with $U_1 \bigcap U_2 = \emptyset$, then the sum of these structures is the permutation structure $(U_1 \bigcup U_2, f_1 \bigcup f_2)$. We note that the sum of two FA presentable permutation structures is again FA presentable. It is clear that the permutation structure $\mathcal{U}$ is a finite sum of permutation structures isomorphic to (Z, S), $\mathcal{U}_m$ for some $m \leq n$, and a finite number of orbits of size less than n. Hence $\mathcal{U}$ has a FA presentation over the unary alphabet $\{1\}$.

We now prove the necessary part of the theorem. It is sufficient to show the result for automatic permutation structures since a FA presentable structure is isomorphic to some automatic structure.

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Let $\mathcal{U} = (U, f)$ be an automatic permutation structure, and $\mathcal{A}$ the 2-variable reduced finite automaton which recognises the graph of f. Consider the X-chain of $\mathcal{A}$. The chain can be thought of as consisting of its tail and its loop. Let t be the length of the tail and l be the length of the loop of X.

We write a for 1^a when unambiguous. Further we write $a \leq b$ to mean $|1^a| \leq |1^b|$. Now we define the following equivalence relation on the set of words $\{1\}^*$:

$$a \sim b \text{ if and only if } (a = b) \text{ or } (a, b \geq t \text{ and } a \equiv b(mod\ l))$$

We denote the $\sim$-equivalence class of a by $[a]$, or $[a]_\sim$ to avoid ambiguity. We set

$$T = \{[a] | a < t\} \text{ and } L = \{[a] | a \geq t\}.$$

Note that T and L depend on $\mathcal{A}$.

Remark 3.1 $|T| = t$ and $|L| = l$. Also, $[a] \in T$ if and only if $[a] = \{a\}$.

Now, with every orbit C, we associate a sequence $\{s_i(C)\}$ of $\sim$-equivalence classes

$$s_0(C), s_1(C), s_2(C), \ldots,$$

defined as follows. Let u be the shortest word in C. Then

$$s_i(C) = [f^i(u)],$$

where $f^0(w) = w$, $f^{n+1}(w) = f(f^n(w))$, $n \in \omega$.

Define $[a](C)$ to be the restriction of $[a]$ to the orbit C.

The following remark will be used without reference.

Remark 3.2 *An orbit C is infinite if and only if $f^i(u) \neq f^j(u)$ for all $i \neq j \in \omega$.*

Claim 3.1 *Suppose that $\{s_i(C)\} \subset L$. If $s_i(C) = s_{i+p}(C)$ for some i and $p \neq 0$, then $s_{i+1}(C) = s_{i+p+1}(C)$.*

Proof. Say $a = f^i(u)$ and $a' = f^{i+p}(u)$, where u is the shortest word in C. Further if $a \sim a'$ then $a' = a + k_0 l$ for some $k_0 \in Z$. Suppose that $f(a) = b$. Then we are required to show that $f(a') \equiv b(mod\ l)$.

We have the following cases:

1. $a > b$

   Then $\mathcal{A}$ accepts $\begin{pmatrix} 1 \\ 1 \end{pmatrix}^b \begin{pmatrix} 1 \\ \diamond \end{pmatrix}^j$, where $j = a - b$, since $f(a) = b$. Note that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{b+k_0 l} \begin{pmatrix} 1 \\ \diamond \end{pmatrix}^j$ is also accepted by $\mathcal{A}$ since $\{s_i(C)\} \subset L$. So $f(a') = f(a + k_0 l) = f(b + j + k_0 l) = b + k_0 l \equiv b (mod\ l)$, as required.

2. $a \leq b$

   Then $\mathcal{A}$ accepts $\begin{pmatrix} 1 \\ 1 \end{pmatrix}^a \begin{pmatrix} \diamond \\ 1 \end{pmatrix}^j$, where $j = b - a$, since $f(a) = b$.

   But then $\mathcal{A}$ also accepts $\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{a+k_0 l} \begin{pmatrix} \diamond \\ 1 \end{pmatrix}^j$ by the condition of the claim. So $f(a') = f(a + k_0 l) = a + j + k_0 l = b + k_0 l \equiv b (mod\ l)$, as required.

Claim 3.2 *Let C and D be any orbits with $\{s_i(C)\}, \{s_i(D)\} \subset L$ and $\{s_i(C)\} \cap \{s_i(D)\} \neq \emptyset$. Then $\{s_i(C)\}$ and $\{s_i(D)\}$ are eventually equal. Further, if $\{s_i(C)\}$ and $\{s_i(D)\}$ are both periodic, then $\{s_i(C)\} = \{s_i(D)\}$.*

Indeed, suppose that the sequences are not disjoint. Then there is some element common to both. So by repeated use of claim 3.1, we have that the sequences are eventually equal. The second part of the claim also follows from Claim 3.1.

Claim 3.3 *Let C be a finite orbit. Then $\{s_i(C)\}$ is periodic.*

Say $|C| = n$. Then for all $c \in C$, since f is a bijection, we have that $f^n(c) = c$. Hence $s_i(C) = s_{i+n}(C)$ for all $i \in \omega$. Thus $\{s_i(C)\}$ is periodic, with period n.

Note that the number of words in a finite orbit is equal to the period of its associated sequence.

Claim 3.4 *Let C be an infinite orbit. Then $\{s_i(C)\}$ is eventually periodic.*

If C were infinite, then $\{s_i(C)\}$ must contain a finite number of elements from T and some elements from L. If all elements were from T, then since T is finite, C would also be finite. Hence there exists a j such that $s_k(C) \in L$ for all $k \geq j$. Using the proof of 3.1, we see that $\{s_i(C)\}$ becomes eventually periodic.

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Claim 3.5 *There are finitely many sequences of the type $\{s_i(C)\}$ which are not contained in L.*

It follows from the fact that T is finite and any element from T is in exactly one orbit of $\mathcal{U}$.

Claim 3.6 *There are finitely many sequences $\{s_i(C)\}$ representing finite orbits which are completely contained in L.*

Consider two distinct finite orbits whose sequences are contained in L. Then since each is periodic, by claim 3.2 they are disjoint.

Thus the collection of such sequences forms a pairwise disjoint collection. Hence, since $|L|$ is finite, there are only finitely many such sequences.

Claim 3.7 *There is bound on the size of finite orbits.*

Combining, since there are finitely many associated sequences for finite orbits, and each sequence represents orbits of a fixed size (namely the periodicity of the sequence), there is a finite bound on the size of all finite orbits.

Claim 3.8 *There are finitely many sequences of the type $\{s_i(C)\}$ whose periodic parts are contained in L.*

Consider two sequences whose periodic parts are contained in L. Suppose further that the periodic parts are not equal. Then by claim 3.2 the periodic parts can not be eventually equal and hence must be disjoint. Thus, since $|L|$ is finite, the set of all distinct sequences with periodic parts contained in L is finite.

Claim 3.9 *Suppose that $\{s_i(C)\}$ represents some infinite orbit. Then it represents at most a finite number of infinite orbits.*

In order to prove the claim we need some simple subclaims. The subclaims below are not restricted to infinite orbits or their associated sequences.

Subclaim 3.1 *Let C be an orbit and $[a] \in \{s_i(C)\}$. Then there is some $j \in Z$ such that for all $a' \in [a](C)$ we have that $f(a') = a' + j$.*

Consider any $a' \in [a](C)$. Let $f(a') = b$.

We have two cases:

1. $a' \leq b$

    Then since $f(a') = b$, the automaton $\mathcal{A}$ accepts $\begin{pmatrix} 1 \\ 1 \end{pmatrix}^a \begin{pmatrix} \diamond \\ 1 \end{pmatrix}^j$, for some $j \in \omega$.

2. $a' > b$

    Then the computation of $f(a') = b$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}^b \begin{pmatrix} 1 \\ \diamond \end{pmatrix}^j$, for some $j \in \omega_+$.

Note that the required j is dependent only on the class $[a](C)$ and not on any of its elements.

Subclaim 3.2 *Let $\mathbf{s} = \{s_i\}$ be some sequence with period n. Then there is some $\Delta_{\mathbf{s}} \in Z$ such that for all C with $\{s_i(C)\} = \{s_i\}$, and for all $a \in C$ we have that $f^n(a) = a + \Delta_{\mathbf{s}}$.*

Given any $a \in C$, define $\Delta_a = |f^n(a)| - |a|$. Then

$$\Delta_{f(a)} = |f^{n+1}(a)| - |f(a)| = (|f^n(a)| + j) - (|a| + j) = |f^n(a)| - |a| = \Delta_a,$$

where j is from the previous subclaim. Hence, letting $\Delta_{\mathbf{s}} = \Delta_a$ for any $a \in C$ is well defined and has the required property.

Now, consider an infinite orbit C and the periodic part of the sequence $\{s_i(C)\}$ representing it. Then $\Delta_{\mathbf{s}} > 0$, for otherwise if $\Delta_{\mathbf{s}} = 0$ then we would have a finite orbit, and $\Delta_{\mathbf{s}} < 0$ is impossible since $\{s_i(C)\}$ is well defined for all $i \in \omega$. Letting a' be the smallest word in $[a]$, we have that if $[a']$ is in the periodic part of $\{s_i(C)\}$, then $[a'](C) = \{a' + k\Delta_{\mathbf{s}} | k \in \omega\}$ is infinite and contains all but finitely many elements from $[i]_{\Delta_{\mathbf{s}}} = \{i + k\Delta_s | k \in \omega\}$, for $i = a' \ (mod \ \Delta_{\mathbf{s}})$.

So let C and D be two distinct orbits with the same periodic part to their associated sequences. Then for all $a \in C$ and $b \in D$ with $a \sim b$ we have that $[a](C) \cap [b](D) = \emptyset$. For if the orbits shared at least one element in common, they would have to be identical.

Hence to every $[a](C)$ we can associate a unique $[i]_{\Delta_{\mathbf{s}}}$ in a 1-1 manner. Thus over the set of all such infinite orbits, we can have at most finitely many such $[a](C)$'s, each one contained in one of the finitely many $[i]_{\Delta_{\mathbf{s}}}$. And hence there are a finite number of infinite orbits with the same periodic parts to their sequences.

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Since orbits with identical sequences certainly have the periodic parts of their sequences the same, the claim is proved.

Claim 3.10 *There are finitely many infinite orbits.*

Combining, we have that the set of all sequences which represent infinite orbits is finite. Further, each sequence represents a finite number of orbits. Hence the claim follows.

Claims 3.7 and 3.10 prove the necessary part of Theorem 1. Hence we have proved Theorem 1.

Proof of Theorem 2. We recall that an **equivalence structure** $\mathcal{E}$ is a pair (E, ρ) where ρ is an equivalence relation on E. For $a \in E$, denote by $[a]_\rho$ the ρ-equivalence class containing a, and $|[a]_\rho|$ its cardinality. Define

$$Inv(\mathcal{E}) = \{(n, m) | \mathcal{E} \text{ has exactly } m \text{ } \rho\text{-equivalence classes of size n}\}$$

where m, n represent cardinals. It is not hard to see that two equivalence structures $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic if and only if $Inv(\mathcal{E}_1) = Inv(\mathcal{E}_2)$.

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalence structures with $E_1 \bigcap E_2 = \emptyset$, then the sum of these equivalence structures is $(E_1 \bigcup E_2, \rho_1 \bigcup \rho_2)$. The following is not hard to prove.

Lemma 2 *The sum of two FA presentable equivalence structures is FA presentable.* □

Let $\mathcal{E}_n$ be an infinite equivalence structure whose equivalence classes all have the same finite size n. Then one can see that $\mathcal{E}_n$ has a FA presentation over $\{1\}$. Clearly the equivalence structure whose elements are all equivalent is also FA presentable over $\{1\}$. Hence we have the next lemma which proves the sufficient part of Theorem 2.

Lemma 3 *If for the equivalence structure $\mathcal{E}$ the number of infinite equivalence classes is finite and the sizes of all finite equivalence classes are bounded by some natural number n, then $\mathcal{E}$ has a FA presentation over a unary alphabet.* □

Now we prove the necessary part of Theorem 2. Consider an automatic equivalence structure $\mathcal{E}$, with say $L(\mathcal{A}) = \rho$.

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Define I to consist of exactly one representative from every infinite ρ-class as follows. $a \in I$ if a is the smallest representative from the infinite class $[a]_\rho$ such that $[a]_\sim \in L$.

We shall show that I is finite.

Remark 3.3 *Distinct words from I are not ρ-related.*

Claim 3.11 *Consider some $a \in I$. Then there is some set of natural numbers, $\{n_1, \ldots, n_p\}$ and a number m such that $(a, a + n_j + mi) \in \rho$ for all $i \in \omega$ and $1 \le j \le p$.*

Indeed, let $s(a)$ be the last X-state in the computation of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{|} a|$. Since a is minimal, there is a $\begin{pmatrix} \diamond \\ 1 \end{pmatrix}$-chain from $s(a)$. This chain must contain a loop in order to recognise infinitely many elements. Let m be the length of the loop on the $\begin{pmatrix} \diamond \\ 1 \end{pmatrix}$-chain. Each accept state on this chain is at some distance from $s(a)$, namely n_j.

Claim 3.12 *Consider $a \in I$, and $b \in [a]_\sim$. Then if $a \equiv b(mod\ m)$ then $(a, b) \in \rho$.*

Indeed, suppose $b = a + mi_0$ for some $i_0 \in \omega$. Then $(a, b + n_j) \in \rho$, by claim 3.11. Similarly, $(b, b + n_j) \in \rho$. By the symmetry of ρ, we have that $(b + n_j, b) \in \rho$. By the transitivity of ρ we then have that $(a, b) \in \rho$, as required.

Now, let $a \ne b \in [a]_\sim \in L$ with $a, b \in I$. Then since $(a, b) \notin \rho$, we have that $a \not\equiv b(mod\ m)$. Hence we can associate a unique congruence class modulo m to each distinct element of $I \bigcap [a]_\sim$, in a 1-1 manner. Thus $|I \bigcap [a]_\sim| \le m$. But since L is finite, we have that I is finite.

Define $\mathcal{F}$ to consist of the smallest representative from each finite ρ-class. We shall show that there exists $B \in \omega$ such that for all $a \in \mathcal{F}$, $|[a]_\rho| \le B$.

Remark 3.4 *Consider $a \in \mathcal{F}$. Then $(a, a + n_j) \in \rho$ for finitely many n_j.*

Indeed, since a is the smallest word in its equivalence class, it is only related to words which are larger than it, other than itself. Hence $(a, a + n_j) \in \rho$ for finitely many n_j.

Then there is a $\begin{pmatrix} \diamond \\ 1 \end{pmatrix}$-chain from $s(a)$, with accept states at the n_j positions. Thus,

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Remark 3.5 $[a]_\rho$ *has the same cardinality as* $\{s \in (s(a), \binom{\diamond}{1})|s \subset F\}$.

But there are finitely many accept states, say B. Thus for all $a \in \mathcal{F}$, $|[a]_\rho| \leq B$, as required. Thus we have proved Theorem 2.

4   Some Questions and Plans For Future Work

We are in the process of characterising the isomorphism types of all FA presentable structures over a unary alphabet. Our goal is to give a general characterisation of the the isomorphism types of all FA presentable structures (over an arbitrary alphabet) in a suitable terminology.

We think that certain classes of FA presentable structures have decidable theories. For example, we intend to prove that the theory of the class of all FA presentable structures over a unary alphabet is decidable. So we pose a more general question.

Question 1 *Let* Σ *be an alphabet. Is it true that the first order theory of the class of all FA presentable structures (of fixed signature) over* Σ *is decidable?*

Theorem 0 in the introduction states that the first order theory of any automatic structure is decidable. On the other hand, it is well known that if a theory is decidable, then one can effectively carry out a Henkin type of construction to build a model of the theory whose satisfaction predicate is decidable (see for example [4] for a proof of this result). Of course the model constructed need not be an automatic one. A construction of an automatic model of the decidable theory would require a deeper understanding of the Henkin method and would be a bridging point between automata theory and model theory. Hence we naturally pose the next question.

Question 2 *Which decidable first order theories have automatic models?*

By Theorem 2 the least ordinal that does not have a FA presentation over a unary alphabet is ω^2. On the other hand, in [1] it is proved for all natural positive n, the ordinal ω^n has a FA presentation. So our next question concerns the problem as to what finite automata can or can not do in terms of presentation of ordinals.

Question 3 *What is the least ordinal without a FA presentation?*

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Finally, we note that one can develop the theory of structures presented by tree automata as well as Büchi and Rabin automata. However, we decided not to raise these topics in this paper since the study of FA presentable structures already presents its own interesting problems and promises further developments.

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