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# Interferometric Information Gain Versus Interaction-Free Measurement

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# Interferometric information gain versus interaction- free measurement

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**Abstract**

Interaction-free measurement schemes with ideal Mach-Zehnder interferometers promised to distinguish absorptive samples with lower average absorption than simple transmission schemes. We show that this is only true for an ensemble of two kinds of samples, where one kind is highly absorptive and the other is highly transmissive. As soon as a third kind of sample with intermediate transmission is introduced, but no phase shift is permitted, the cost of information gain in terms of absorbed particles in the samples is higher in the interferometric scheme. We also investigate the general case of samples with a continuous range of transmission *and* phase shift values, such that an interferometer's ability to measure both sample characteristics can be exploited. With an interferometer the number of principally distinguishable samples increases linearly with the number of probe particles, but with a simple transmission setup it increases as the square root. When wishing to distinguish twice as many samples from a continuous sample distribution with an interferometric scheme, the number of absorbed particles per sample only doubles, but it quadruples with a simple transmission scheme.

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# 1 Introduction

Interaction free measurement was put forward by Elitzur and Vaidman to probe for the presence of a perfectly absorbing sample in one path of a Mach-Zehnder type interferometer, without the probe particle being absorbed in the sample [1]. Their idea can be seen as an inversion of the common conclusion corresponding to complementarity [2]: Instead of considering the availability of path information as the cause for the absence of interference one may equally well consider the absence of interference as an indication for the availability of path information. In such a way the observation of non-interference can be used to detect the presence of a path measurement device, i.e. the absorber. Whereas without the absorber all particles will be observed in the same output, there will be counts in the other output as well if the detector is inserted. Therefore the observation of a particle in the latter output is a clear indication of the presence of the absorber. This leads to the conclusion that the particle which proves the presence of the detector never got into contact with it and therefore did not interact with it. The phenomenon may properly be called interaction-free measurement. Since then, several theoretical papers aimed at clarifying the paradoxical aspect of the proposal [3] [4], and at possible applications [5] [6] [7]. Several experiments demonstrated the feasibility of the scheme [8] [9].

The possibility of interaction free measurement leads to the question, whether this method could provide information about samples of arbitrary absorption with less interaction than conventional transmission techniques. The purpose of this paper is therefore the comparison of a conventional scheme, in which absorption is determined by a simple transmission measurement, with an interferometric scheme, which renders absorption and phase information. For the interferometric scheme we choose an ideal Mach-Zehnder interferometer. While non-symmetric interferometers or multi-loop interferometers may be superior in particular regimes of absorption and phase measurement, the Mach Zehnder interferometer exhibits technological simplicity and *both* output beams can be fully modulated. We would expect that this feature is statistically advantageous for obtaining information from general samples.

The sample will be treated as classical, because we are only interested in how many particles are absorbed in it, thereby depositing possibly harmful amounts of energy. Also, we will neglect scattering into momentum states other than the original momentum, because scattering could be treated as additional absorption. Therefore, the term *interaction* is here equivalent to *absorption*. Thus *interaction-free* shall mean that the particle was not absorbed, but a branch of its wavefunction may have passed through the sample and picked up a phase shift.

For comparing the performance of the two schemes we will employ Bayesian inference. We shall establish the conditional probability that the sample is identified correctly and then sum over a constant prior distribution of samples to obtain the average probability of correctly identifying a sample. We shall require

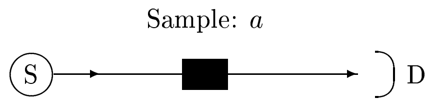


Figure 1: Particles emitted by the source S pass through a sample with transmission probability $\tau$ and are registered by detector D. For a given number of particles ($N$) emitted by the source, an experimenter may infer the transmission probability of the sample from the number of particles registered in D.

that this probability exceed a certain minimum for both methods. The method requiring fewer absorbed particles in the samples to obtain the information shall be judged superior.

# 2 Two kinds of samples

## 2.1 Black and white samples

First we assume that we have only fully absorbing (black) and fully transparent (white) samples, i.e., we have a transmission probability of $\tau_1 = 0$ or of $\tau_2 = 1$. The question we want to answer is, in which of the two schemes fewer particles are absorbed in the samples, on average, if we demand that both methods achieve a certain minimum average probability of correct identification of the samples. There is a very practical relevance to this question, as today's scanning methods with X-rays, be it in medicine or in materials testing, use the simple transmission method to obtain the desired information. If the interferometric methods turn out to require fewer absorbed particles in the sample - as is suggested by the interaction-free measurement scheme - this would lead to less radiation damage. The standard scheme of electron holography is one such interferometric method, to which our findings will be directly applicable [10], [11].

Let us assume further, that we know when a particle has been sent from the source. This is possible, in principle. In this manner we get rid of the source fluctuations, which usually reduce the amount of information obtainable about the sample from a give number of detected particles. As a further assumption throughout this paper, we will neglect background noise in the detectors, and we will assume detectors of an efficiency of 100%.

For the simple transmission case (Fig.1), we need only one particle. If the particle arrives at the detector, we conclude that the sample is white, and if it does not, we conclude that the sample is black. The probability of correct interpretation of each kind of sample is 1, such that this is also true for the average probability of correct interpretation, i.e.,

$$C_T = 1. \tag{1}$$

When testing a large number of samples, where black and white samples occur equally often, the average number of particles absorbed per sample is

$$A_T = \frac{1}{2}. \tag{2}$$

Now, consider the ideal Mach-Zehnder interferometer in Fig.2. The transmission and reflection amplitudes at the beam splitters are $1/\sqrt{2}$ and $i/\sqrt{2}$, respectively. The sample in path $I$ has trsnamission probability $\tau$ and induces a phase shift $\varphi$. The probabilities for detection of the particles in $D_1$ or in $D_2$ are

$$p_1 = \frac{1+\tau}{4} + \frac{\sqrt{\tau}}{2}\cos\varphi, \tag{3}$$

$$p_2 = \frac{1+\tau}{4} - \frac{\sqrt{\tau}}{2}\cos\varphi. \tag{4}$$

The probability that the particle is absorbed in the sample is

$$p_3 = 1 - p_1 - p_2 = \frac{1-\tau}{2}. \tag{5}$$

A white sample is characterized by $\tau = 1$ and $\varphi = 0$. It always results in the particle hitting $D_1$. A black sample has $\tau = 0$ and blocks path I in the interferometer. There is a probability of $\frac{1}{2}$ that the particle is absorbed in the sample. The probability that it hits detector $D_1$ is $\frac{1}{4}$ and the same is true for $D_2$. Therefore, if we send one particle, and it is detected in $D_1$, we cannot decide whether the sample is black or white. We must send several particles to obtain sufficient confidence about the transmission property of the sample. Then we can devise the following measurement procedure. Send particles until a particle is detected either in $D_2$ or does not arrive at a detector, which means it is absorbed in the sample. But send at most $N$ particles per sample. The interpretation of the result is as follows. If all particles are detected at $D_1$ we conclude the sample is white, otherwise we conclude it is black. The probability of correct interpretation of a white sample is 1, because with a white sample the particle will always go to $D_1$. The probability of misinterpreting the sample as white, while it is in fact black, is equal to the probability, that with a black sample we get all particles into $D_1$. This is $\left(\frac{1}{4}\right)^N$, such that the probability of correct interpretation of a black sample is $1 - \left(\frac{1}{4}\right)^N$. Therefore the average probability of correct interpretation of the samples with the interferometer is

$$C_I = 1 - \frac{1}{2}\left(\frac{1}{4}\right)^N. \tag{6}$$

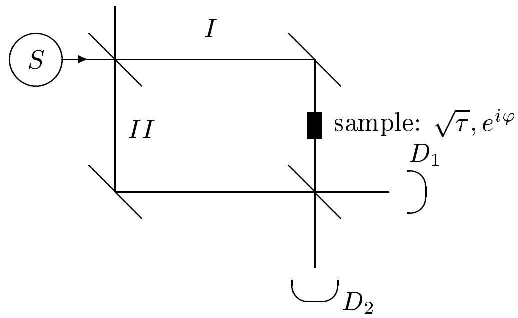


Figure 2: A particle emitted from source $S$ impinges on the Mach-Zehnder interferometer where it can follow path I or path II. The particle can be detected in $D_1$, in $D_2$, or be absorbed in the sample.

If we wish to have $C_I \geq 0.99$ we can confine ourselves to sending at most $N = 3$ particles per sample.

We must now establish how many particles will get absorbed in the samples, on average. With a black sample, the probability that the first particle is absorbed in the sample is $\frac{1}{2}$. The probability that the second particle is absorbed is equal to the probability that the first particle goes to $D_1$, such that the test will not be stopped, times the probability that the second particle is absorbed in the sample. This is $\frac{1}{4} \times \frac{1}{2}$. If also the second particle goes to $D_1$ a third particle has to be sent, etc. These considerations yield for the total probability that a particle is absorbed in the black sample, when sending at most $N$ particles per sample,

$$\frac{1}{2} \sum_{j=0}^{N-1} \left(\frac{1}{4}\right)^j . \tag{7}$$

In a test of a large number of samples, where black and white samples occur equally often, the average number of absorbed particles per sample will therefore be

$$A_I = \frac{1}{2} \left[ 0 + \frac{1}{2} \sum_{j=0}^{N-1} \left(\frac{1}{4}\right)^j \right] = \sum_{j=1}^{N} \left(\frac{1}{4}\right)^j . \tag{8}$$

In our experiment we need to send at most $N = 3$ particles, so that we get $A_I = 0.328$. Nevertheless, with the interferometric setup the number of correctly identified samples is *smaller* than with the transmission setup, because we have $C_I < C_T$. In order to get the same performance with the interferometer, $C_I \to 1$, we have to increase the number of particles sent through it to $N \to \infty$. Of course this will also affect the number of particles absorbed on average, which will increase to $A_I = \frac{1}{3}$. With the interferometric scheme the number of absorbed particles per tested sample is thus smaller than in the simple transmission scheme. So the interferometric method is superior here. Its advantage can be even increased. With the improved version of interaction-free measurement as proposed by Kwiat et al. [8], one can ultimately test for black and white samples without ever absorbing a particle in a sample. This would require an infinite number of interferometer loops. But in this study we will limit ourselves to one-loop interferometers.

## 2.2 Grey and white samples

Now we replace the black sample by a dark-grey one to investigate the transition from the ideal interaction-free case to a more realistic situation. Our grey samples shall be very dark, i.e. their transmission shall be very low: $0 < \tau_1 \ll 1$. The white samples shall have perfect transmission, $\tau_2 = 1$, and no phase shift. It is also necessary to fix the phase shift of the grey samples. We will set it to $\varphi_1 = 0$, because this leads to a smaller difference in the probabilities of the outcomes between grey and white samples in the interferometer, than if we had any other value of $\varphi_1$. Therefore, setting $\varphi_1 = 0$ constitutes the most stringent test of the performance of the interferometer.

Samples will be measured in the same way as before. For a given sample in the interferometer we send at most $N$ particles. As soon as one particle is absorbed in the sample or detected in $D_2$ we stop the test and say we have a grey sample. If all $N$ particles are detected in $D_1$, we interpret this to be due to the white sample. Obviously, the probability of correctly recognizing a white sample is 1, as before. The probability of correctly recognizing the grey sample is the complement of the probability of not recognizing it correctly. The latter is given by the probability that despite a grey sample in the interferometer all $N$ particles are detected in $D_1$, which is $[p_1(\tau_1)]^N$. Hence, the average probability of correct interpretation of samples with the interferometer is

$$C_I = \frac{1}{2}(1 + 1 - [p_1(\tau_1)]^N) = 1 - \frac{1}{2}\left(\frac{1 + \tau_1 + 2\sqrt{\tau_1}}{4}\right)^N, \tag{9}$$

where we have made use of eq.(3). For the case of the simple transmission setup we will again do the tests with only one particle per sample. Since we are only interested in $\tau_1 \approx 0$ and $\tau_2 = 1$, a sufficient probability of correct interpretation is achieved already with one particle: If the particle is absorbed in the sample, we infer the sample is grey, otherwise we infer it is white. We will therefore always recognize a white sample correctly, but will sometimes misinterpret a grey sample as white. The average probability of correct interpretation of the samples with the simple transmission setup is thus

$$C_T = \frac{1}{2}(1 - \tau_1 + 1) = 1 - \frac{\tau_1}{2}. \tag{10}$$

If we wish to have $C_T \geq 0.99$, we can only permit $0 \leq \tau_1 \leq 0.02$. Let us set $\tau_1 = 0.02$. How many particles must we send into the interferometer, such that we get at least the same probability of correct interpretation as with the transmission setup, i.e., $C_I \geq 0.99$? We find $N \geq 4$.

It is interesting to note that now it is very well possible that the average probability of correct interpretation of the sample can be *larger* with the interferometer than with the transmission setup, i.e. $C_I > C_T$, as is indeed the case for the present example, whereas with black and white samples we found that we always have $C_I \leq C_T$ [eqs.(1) and (6)].

What will be the average number of particles absorbed in the samples with these testing procedures? For the transmission setup we have

$$A_T = \frac{1}{2}\left[(1-\tau_1) + (1-\tau_2)\right] = \frac{1-\tau_1}{2}. \tag{11}$$

With our values of $\tau_1 = 0.02$ and $\tau_2 = 1$ we obtain $A_T = 0.49$. For the interferometer we must add up the probabilities of those cases where, with the grey sample in place, the final test particle is absorbed in the sample. When sending up to $N$ particles, the total probability of this to happen is obtained in a straightforward manner as

$$\frac{1}{2}(1-\tau_1)\sum_{j=0}^{N-1}[p_1(\tau_1)]^j. \tag{12}$$

Since no particle will be absorbed in the white sample, the average number of particles absorbed in the samples with the interferometric setup is

$$A_I = \frac{1}{4}(1-\tau_1)\sum_{j=0}^{N-1}[p_1(\tau_1)]^j. \tag{13}$$

With the chosen values of $\tau_1 = 0.02$ and $N = 4$ we obtain $A_I = 0.359$. Therefore, the interferometer leads to a lower average absorption than the transmission setup when we want to distinguish dark grey and white samples, just as with the black and white samples we investigated in the previous section. When we take a closer look at the numbers we see that, we have a smooth transition of the average number of absorbed particles from the case of black and white samples to the case of grey and white samples. But we also note that, as soon as the black sample becomes a bit transparent, the difference in the average number of absorbed particles per sample between the interferometric and the transmission setup becomes smaller. This suggests that the advantage of the interferometric method in terms of lower average absorption will be lost when lighter shades of grey are used. Indeed, we will see that the cases of black and white samples and of grey and white samples are narrow domains in which the interferometer performs better than the transmission setup, and that in the general case of several different transmission values of samples the reverse is true.

### 2.3 Black and grey samples

Now we permit black ($\tau_1 = 0$) and light grey ($0 \ll \tau_2 < 1$) samples. With a grey sample in path I of the interferometer (Fig. 2) we can no longer expect that all particles sent through the interferometer will be arriving at $D_1$. Some particles will be absorbed by the sample and some will arrive at $D_2$. Therefore the observation of a particle in $D_2$ is no longer a unique indication for a black sample in path I of the interferometer, as it was with black and white samples. Nevertheless, one expects a continuous transition from the case of black and white samples. In particular, the replacement of the white sample ($\tau_2 = 1$) by a nearly white one ($\tau_2 \approx 1$) should conserve the advantage of interaction free measurement.

As before, the phase shift induced by the grey sample will be assumed as $\varphi_2 = 0$. Other phase shifts would, in fact, reduce the statistical difference between the outcomes with black and grey samples. Thus we are giving the interferometer a little advantage here. The method of testing a sample with the interferometer will also be the same as before: We send $N$ particles per sample. As soon as a particle is absorbed in the sample or detected in $D_2$ we say we have a black sample and stop the test. If all $N$ particles go to $D_1$ we say we have a grey sample. However, what is new here, in contrast to the cases of black and white and of grey and white samples, is that we will now make mistakes of interpretation with *both* kinds of samples.

The probability of correct interpretation of a black sample is, as before, $\frac{1}{2}\left[1-(\frac{1}{4})^N\right]$. The probability of correct interpretation of the grey sample is equal to the probability that with a grey sample all particles go to detector $D_2$, which is

$$\left(\frac{1+\tau_2+2\sqrt{\tau_2}}{4}\right)^N . \tag{14}$$

The average probability of correct interpretation of a sample therefore results in

$$C_I = \frac{4^N - 1 + (1+\sqrt{\tau_2})^{2N}}{2^{2N+1}}. \tag{15}$$

The average number of absorbed particles per tested black sample is given by eq.(7). The average number of absorbed particles per tested grey sample is the sum of the probabilities that the first $j-1$ particles went into detector $D_1$ and the $j^{th}$ particle was absorbed in the grey sample, such that the test was stopped and the sample was mistakenly called a black sample. With the use of eqs.(3)-(5) this sum is

$$p_3(\tau_2) \sum_{j=0}^{N-1} \left[p_1(\tau_2)\right]^j . \tag{16}$$

The average number of absorbed particles for all samples then becomes

$$A_I = \frac{1}{4}\left[\sum_{j=0}^{N-1}\left(\frac{1}{4}\right)^j + (1-\tau_2)\sum_{j=0}^{N-1}\left(\frac{1+\tau_2+2\sqrt{\tau_2}}{4}\right)^j\right]. \tag{17}$$

The corresponding expressions for the transmission setup are

$$C_T = \frac{1+\tau_2}{2}, \tag{18}$$

and

$$A_T = 1 - \frac{\tau_2}{2}. \tag{19}$$

Looking at numerical examples we notice that there is only a very narrow domain for the grey sample, for a given minimum of the average probability of correct interpretation of the samples with the interferometric method. For instance, let us again demand $C_I \geq 0.99$. We must then send at most $N = 4$ particles, but can lower the transmission of the grey sample only to $\tau_2 = 0.992$. The average number of particles absorbed in the sample is then $A_I = 0.340$. With the transmission setup it would be $A_T = 0.504$. The average absorption is therefore still lower with the interferometer, and we benefit from the "interaction-free effect". However, we also find, that the average probability of correct interpretation of the samples is *higher* with the transmission setup, because we have $C_T = 0.996$, whereas we only have $C_I = 0.990$. Let us see, whether we can have equal probability of correct interpretation of the samples of the two methods. We have

$$C_I - C_T = \frac{(1+\sqrt{\tau_2})^{2N} - 4^N\tau_2 - 1}{2^{2N+1}}. \tag{20}$$

This expression is negative for $0 < \tau_2 < 1$. It is zero for $\tau_2 = 0$ for all values of $N > 0$ and $N \to \infty$. If the grey sample becomes white, i.e. $\tau_2 = 1$, it evaluates to 0 in the limit of $N \to \infty$, which corresponds to what we found for the case of black and white samples. This means that for grey samples condition (20) is never fulfilled and with the interpretation rules we adopted the average probability of correctly identifying a sample is *always* greater with the transmission setup.

It is worth while to try to reverse this situation by changing the experimental procedure and the interpretation rules. We make use of the fact that, beginning from a certain value of $N$ the outcome of getting $N-1$ particles at $D_1$ and one particle absorbed in the sample is more likely for the grey sample than for the black sample. We could therefore establish the following new rules: Send at most $N$ particles per sample. If all $N$ particles are detected at $D_1$, or if $N-1$ are detected in $D_1$ and one is absorbed in the sample, then interpret this as a grey sample. As soon as one particle is detected in $D_2$, or as soon as a second particle is absorbed in the sample, stop the test of the sample and interpret it as a black sample.

If condition (20) is reformulated using these new interpretation rules one will indeed find values of $N$ such that the average probability of correct identification of a sample with the interferometer is equal to, or larger, than with the transmission setup. However, such an improvement has its price in terms of increased absorption, because after absorption of one particle we can now not terminate the test and conclude that a black sample is in path $I$ of the interferometer. Rather, we have to send further particles. Calculating all possibilities of outcomes and their respective number of absorbed particles, it can be shown that, the average number of absorbed particles per sample increases to 10/9, when black samples are tested. When testing black *and* grey samples ($\tau_2 \approx 1$), 5/9 particles are absorbed per sample, on average. This is significantly more than what we had found for all cases of two different kinds of samples using our original experimental procedure and interpretation rules. And it is also more than what we had obtained for the simple transmission setup, eq.(19). We therefore come to the surprising conclusion that, *if* we require the probability of correct interpretation of the samples with the interferometer to be *at least equal* to that with the transmission setup, the transmission setup is *less* absorption consuming than the interferometer in distinguishing black from nearly white samples.

# 3 Black, white and grey samples

In this section we permit three different kinds of samples, which have transmission probabilities for the particle of $\tau_1 = 0$, $0 < \tau_2 < 1$, and $\tau_3 = 1$. The exact transmission of sample 2 (grey sample) will be chosen such that the average number of particles absorbed in the interferometric testing scheme will become minimal. As before, the phase shift induced by the samples will be assumed to be 0. Whether this choice ensures the most stringent test of the interferometer's capability to distinguish the samples depends on the exact value of $\tau_2$. At worst, it gives the interferometer an advantage relative to the transmission setup.

First we look at the interferometric scheme. Using eqs.(3)-(5) we note that detector $D_1$ can fire with any of the three samples, and that detector $D_2$ can fire with samples 1 and 2. Clearly, each sample must be tested with several particles to obtain a statistically significant result. It is now very cumbersome to check through all the possibilities of what one can conclude after each additionally detected particle, as we did in the previous sections. Therefore, we will analyse the more practical method of sending a definite number of particles, $N$, into the interferometer, and to draw a conclusion then. We will try to keep $N$ as low as possible. And we will interpret an observed result to be due to that sample, for which one expects the highest probability for the particular result. (This is equivalent to a ranking according to likelihood as used in the next section.) For sample $i$ the probability of getting $N_1$ particles in detector $D_1$, $N_2$ particles in detector $D_2$ and $N_3 = N - N_1 - N_2$ particles absorbed in the sample is given by the trinomial expression

$$Prob(N_1, N_2|N, \tau_i) = \frac{N!}{N_1!N_2!N_3!} [p_1(\tau_i)]^{N_1} [p_2(\tau_i)]^{N_2} [p_3(\tau_i)]^{N_3} . \qquad (21)$$

We must also fix the minimum probability of correct identification of a sample. We will require that, for each kind of sample, this probability shall exceed a certain value $C_{min}$. This is a small change to the cases with just two kinds of samples, where we had required the *average* probability of correct identification to exceed a certain minimum. However, in the general case to be discussed in the next section, equal statistical distinguishability of samples will be the important criterion. This amounts to requiring equal probability of correct identification for all samples, such that it is useful to introduce this criterion already now. Let us demand $C_{min} = 0.99$. Then, using eq.(24), a little numerical analysis shows that we must send up to $N = 19$ particles per sample, and that we must have a transmission of the grey sample of $\tau_2 = 0.555$. (With other values of $\tau_2$ even more particles may be necessary.) If each kind of sample occurs equally often, the average number of particles absorbed per sample is

$$A_I = \frac{N}{3} \sum_{i=1}^{3} \frac{1 - \tau_i}{2} = 4.576. \qquad (22)$$

This represents a significant jump compared to the findings in the previous section, where we had just two kinds of samples, which were essentially black and white.

The situation is similarly worsened, when we go to the simple transmission setup. In order to be able to conclude that the sample is neither black nor white, we must send particles until both kinds of outcomes have happened. With the grey sample in the beam, the probability that all particles are absorbed in the sample or that all particles are transmitted, is given by

$$W = \tau_2^N + (1 - \tau_2)^N. \qquad (23)$$

Since we want $1 - W > C_{min}$, and we set $C_{min} = 0.99$, we must have $N \geq 9$ for our value of $\tau_2$. We take $N = 9$. The average number of particles absorbed per sample is here given by

$$A_T = \frac{N}{3} \sum_{i=1}^{3} (1 - \tau_i) = 4.335. \qquad (24)$$

This is *less* than with the interferometric setup. In fact, the simple transmission setup could perform even better, if, instead of always sending 9 particles, we stop as soon as both outcomes have happened, because we can then be confident that we are faced with the grey sample. Hence, the advantage of the interferometric setup, which is due to its *interaction-free measurement capability*, is definitely lost as soon as we permit grey samples in addition to (almost) black and (almost) white ones. It should be mentioned that, with *multi-loop* interferometers such as in [8], the interferometric method is still superior to the simple transmission method, even with the three kinds of samples discussed here: For a white sample the particle would end up in one detector, for a black sample in the other, and for a grey sample it would be absorbed in the sample. (This suggests that a multi-loop arrangement with *many* output beams might permit distinguishing various shades of grey with just a single test particle. We will look at this in a future paper.) The results of this section do, however, not imply that the ideal Mach-Zehnder interferometer is always worse than the simple transmission setup, as soon as more than two kinds of samples are to be distinguished. We shall see this in the next section, where we include the phase shift a sample imprints on the particle's wavefunction and permit continuous values of phase shift and transmission [12].

# 4 Continuous range of samples

First we look at the simple transmission setup of Fig.1. If we send $N$ particles, and the sample has a transmission probability of $\tau$, the probability of getting $N_1$ particles into the detector is given by the binomial expression

$$Prob(N_1|N,\tau) = \frac{N!}{N_1!(N-N_1)!}\tau^{N_1}(1-\tau)^{N-N_1}. \tag{25}$$

However, we are interested in the reverse question: Given that we sent $N$ particles and received $N_1$ in the detector, what is the *likelihood* that the sample has transmission $\tau$? The likelihood function is by definition proportional to (25), the proportionality factor being arbitrary [14]. Because one is most often interested in the likelihood of one value of $\tau$ relative to the most likely value of $\tau$, one normalizes the likelihood function such that its maximum is 1. Thus we have

$$L(\tau|N,N_1) = \left(\frac{\tau}{\tau_{max}}\right)^{N_1}\left(\frac{(1-\tau)}{(1-\tau_{max})}\right)^{N-N_1}, \tag{26}$$

where $\tau_{max} = N_1/N$, which is where the likelihood function reaches its maximum. As $N$ becomes large, the likelihood function approaches the Gaussian

$$L(\tau|N,N_1) \approx \exp\left[-\frac{N(\tau_{max}-\tau)^2}{2\tau_{max}(1-\tau_{max})}\right]. \tag{27}$$

Clearly, the true value of $\tau$ need not be $\tau_{max}$. As in any probabilistic process, for a specific experimental result $N_1$ the true value of $\tau$ can only be determined to within a confidence (or uncertainty) interval. For this we must decide on a confidence level. We could, for instance, accept all those values of $\tau$ as quite likely, whose likelihood is above .01. With (27) this gives a confidence interval whose full width $w$ is given by

$$w = 2\sqrt{\frac{2\tau_{max}(1-\tau_{max})}{N}\ln(100)}, \qquad (28)$$

except for $N_1$ very close to 0 or very close to $N$, where the width has to be determined from the exact likelihood function (26). The center of the confidence interval is at $\tau = \tau_{max}$. By means of eq.(25) it can then be shown that an experimenter's conclusion *'The true value of $\tau$ is within $\tau_{max} \pm w/2$.'* has a probability of being correct in excess of .99 for any possible $\tau$.

In fact, eq.(28) can immediately be used to count how many different samples we can distinguish when we send $N$ particles per sample. We plot the likelihood function for $\tau_{max} = .5$, then we find those neighboring ones, which intersect it where it drops to .01. Then we find the outer neighbors of the neighbors by the same criterion, etc. This has been done in Fig.3 for $N = 100$, $N = 200$, and $N = 300$. It can be seen that the number of distinguishable samples, $Z_T(N)$, turns out to be: $Z_T(100) \approx 5$, $Z_T(200) \approx 7$, $Z_T(300) \approx 9$. This suggests that $Z_T(N)$ increases with $\sqrt{N}$.

FIGURE 3

$Z_T(N)$ can also be calculated analytically as pointed out by Wootters [13]. The calculation is a continuous formulation of the considerations just presented. The number of confidence intervals passed when going with $\tau_{max}$ from 0 to 1 is given by the integral

$$Z_T(N) = \int_0^1 \frac{d\tau_{max}}{w(\tau_{max})} = \frac{\pi}{\sqrt{8\ln(100)}}\sqrt{N}, \qquad (29)$$

proving that $Z_T(N)$ does indeed increase with the square root of $N$ and showing good agreement with Fig.3.

Now we will apply the same considerations to the interferometric setup. The unknown sample is characterized by transmission probability $\tau$ and phase shift $\varphi$. In analogy to eqs.(25) and (26) we obtain the likelihood for $\tau$ and $\varphi$, given that $N$ particles were sent into the interferometer, of which $N_1$ were detected in $D_1$ and $N_2$ in $D_2$, respectively:

$$L(\tau,\varphi|N_1,N_2,N_3) = \left[\frac{p_1(\tau,\varphi)}{s_1}\right]^{N_1}\left[\frac{p_2(\tau,\varphi)}{s_2}\right]^{N_2}\left[\frac{p_3(\tau)}{s_3}\right]^{N_3}. \qquad (30)$$

As in eq.(21) we have again defined $N_3 = N - N_1 - N_2$. The probabilities $p_1$, $p_2$ and $p_3$ are as in eqs.(3) to (5). The normalization parameters $s_i$, $(i = 1,2,3)$, are given by

$$s_i = \frac{N_i}{N}. \qquad (31)$$

Noting that the likelihood attains its maximum of 1 when $p_i = s_i$ for all $i$, the most likely values of $\tau$ and $\varphi$ can be derived as

$$\tau_{max} = 1 - \frac{2N_3}{N} \tag{32}$$

and

$$\varphi_{max} = \arccos\left(\frac{N_1 - N_2}{\sqrt{2N_3N - N^2}}\right). \tag{33}$$

To eliminate the ambiguity of $\varphi_{max}$, we shall only be interested in the interval $[0, \pi]$.

The likelihood function (30) can again be used to count how many different samples can be distinguished if $N$ particles are sent into the interferometer per sample. This has been done graphically in Fig.4 in the following way. We assumed a certain $N$ and started with the likelihood function for $\tau_{max} = .5$ and $\varphi_{max} = \pi/2$. Then we kept $\varphi_{max}$ constant and determined those two neighboring likelihood functions whose $\tau_{max}$ was such that they intersected the original likelihood function where it had a value of .01. Then further neighbors along the $\tau$-axis were determined in the same fashion, until the limits were reached. After this, the same procedure was applied to each of the likelihood functions found so far, but keeping $\tau_{max}$ constant and varying $\varphi_{max}$. In this manner the polar plane of $\tau$ and $\varphi$ was filled with regions, each representing a confidence area. Although this is a crude way of counting how many kinds of different samples are distinguishable by the interferometric method, it still gives a good idea of the general dependence on the number of particles sent into the interferometer per sample. From Fig.4 we deduce $Z_I(100) \approx 10$, $Z_I(150) \approx 17.5$, and $Z_I(150) \approx 23.5$. (Regions cut at $\varphi = 0$ or at $\varphi = \pi$ were counted as 1/2.) This suggests a linear increase with $N$.

FIGURE 4

We can verify this by performing an analytic count. Let us first look at how many different phase shifts we can distinguish for samples of the same transmission probability $\tau$. For the interferometric setup shown in Fig.2 the probability, that a particle is detected either at $D_1$ or at $D_2$ is given by

$$p_{12} = p_1 + p_2 = \frac{1+\tau}{2}. \tag{34}$$

The total number of particles in these detectors will therefore be around

$$M = Np_{12} = \frac{N}{2}(1+\tau). \tag{35}$$

The number of statistically distinguishable results for a given $\tau$ and $N$ is therefore obtained by evaluating how many outcomes at detectors $D_1$ and $D_2$ we can consider as different:

$$U(\tau) = \int_{p_{1,min}}^{p_{1,max}} \frac{dp_1}{2\sqrt{2\ln(100)}\Delta p_1}. \tag{36}$$

Here, we have again assumed that a result is distinguishable from a neighboring one, if the two respective likelihood functions do overlap only up to those points, where both have dropped to .01. In this manner the analytic result will be directly comparable to what we found graphically in Fig.4. The standard deviation of $p_1$ is $\Delta p_1$, and it is obtainable from the binomial distribution, which governs the statistics of the counts in $D_1$ versus those in $D_2$. It is given by

$$\Delta p_1 = \sqrt{\frac{p_1(1-p_1)}{M}}. \tag{37}$$

Evaluating the integral (36) yields

$$U(\tau) = \frac{\sqrt{N(1+\tau)}}{4\sqrt{\ln(100)}} \left[\arcsin\left(\frac{1-\tau}{2} + \sqrt{\tau}\right) - \arcsin\left(\frac{1-\tau}{2} - \sqrt{\tau}\right)\right]. \tag{38}$$

Now we have to consider how many different values of $\tau$ are statistically distinguishable. Hence, we must weight each identifiable interval on $\tau$ with its respective number of distinguishable phase shifts, $U(\tau)$, and sum over them. Then we obtain the total number of statistically distinguishable samples as

$$Z_I(N) = \int_0^1 d\tau \frac{U(\tau)}{2\sqrt{2\ln(100)}\Delta\tau}, \tag{39}$$

where $\Delta\tau$ is the standard deviation of the inferred value of $\tau$ from the binomial probability distribution of the particles absorbed versus the particles detected in either $D_1$ or $D_2$. We have

$$\Delta\tau = \left|\frac{d\tau}{dp_{12}}\right| \Delta p_{12}, \tag{40}$$

where $\Delta p_{12}$ is the standard deviation of the probability $p_{12}$ (34), given by

$$\Delta p_{12} = \sqrt{\frac{p_{12}(1-p_{12})}{N}}, \tag{41}$$

such that we obtain

$$\Delta\tau = \sqrt{\frac{(1+\tau)(1-\tau)}{N}}. \tag{42}$$

Inserting this into (39) and substituting $x = \sqrt{1-\tau}$ yields

$$Z_I(N) = \frac{N}{4\sqrt{2}\ln(100)} \int_0^1 \left[\arcsin\frac{x^2}{2} + \sqrt{1-x^2} - \arcsin\frac{x^2}{2} - \sqrt{1-x^2}\right] dx \approx \frac{.42}{\ln(100)} N. \tag{43}$$

A comparison of the values of $Z_I$ for the values of $N$ as used in Fig.4 shows reasonably good agreement. But, what is important about this result is that the number of statistically distinguishable samples does indeed increase *linearly* with the number of particles sent into the interferometer.

It is now also useful to obtain the average number of absorbed particles per sample. For this, we must fix a distribution of sample characteristics $(\tau, \varphi)$ of the ensemble to be tested. Let us assume that, when blindly picking a sample from our ensemble, all values of $\tau$ and $\varphi$ shall be equally likely, where we restrict $\varphi$ to the interval $[0, \pi]$. Thus, we have a constant a priori probability density of sample characteristics,

$$f(\tau, \varphi) = \frac{2}{\pi} \tag{44}$$

since we must have

$$\int_0^\pi \int_0^1 f(\tau, \varphi) \tau d\tau d\varphi = 1. \tag{45}$$

In the simple transmission setup a test with $N$ particles of a sample with transmission $\tau$ will lead to a mean number of absorbed particles of $N(1-\tau)$, independent of the sample's phase shift $\varphi$. The average number of particles absorbed per sample when testing the whole ensemble is thus

$$A_T(N) = \int_0^\pi \int_0^1 f(\tau, \varphi) N(1-\tau) \tau d\tau d\varphi = \frac{N}{3}. \tag{46}$$

Testing the whole ensemble of samples also permits to class them into $Z_T(N)$ distinguishable groups. A useful number of merit is then the average number of particles absorbed per sample, per distinguishable group of samples. This is

$$S_T(N) = \frac{A_T(N)}{Z_T(N)} = \frac{\sqrt{8 \ln(100)}}{3\pi} \sqrt{N}. \tag{47}$$

The quantity $S_T(N)$ can be understood as the absorption cost per sample which we must pay for a desired amount of information about the ensemble. It increases with the square root of the number of probe particles sent per sample, which means that additional information about the samples becomes ever more costly, the more information we already have about the samples. It is worth noting, that this conclusion is independent of the particular form of $f(\tau, \varphi)$, as long as it is smooth, because the change of $f(\tau, \varphi)$ would only change the numerical constant in $S_T(N)$, but not its functional dependence on $N$.

For the interferometric setup we can form the analogous quantities. When sending $N$ particles, the mean number of absorbed particles in the sample is now only half as large as in the simple transmission setup, $\frac{N}{2}(1-\tau)$, and is again independent of $\varphi$. The average number of particles absorbed per sample when testing the whole ensemble is thus also just half,

$$A_I(N) = \frac{N}{6}. \tag{48}$$

Our number of merit, the number of particles absorbed per sample, per distinguishable group of samples is thus

$$S_I(N) = \frac{A_I(N)}{Z_I(N)} \approx .40 \ln(100). \tag{49}$$

This is a constant! It means, that additional information about the samples does *not* become more expensive the more information we already have about our ensemble. If we wish to double the number of experimentally resolved sample groups, we just have to pay twice the "absorption prize" per sample, and not the fourfold price, as would be the case with the simple transmission setup. Again, the fact that $S_I$ is a constant is independent of the particular form of the ensemble's sample distribution $f(\tau, \varphi)$, as long as it is smooth, but the particular value of $S_I$ does, of course, depend on $f(\tau, \varphi)$.

# 5 Discussion

Interaction free measurement as a method to obtain information about samples not otherwise accessible is certainly an intriguing possibility [15]. Applications could range from learning about fragile atomic or molecular states to materials testing and X-ray interferometry in medicine. For this purpose we compared the performance of a Mach-Zehnder interferometer (Fig.2) and of a simple beam transmission setup as devices for identifying samples with varying absorptivity. Of course, the restriction to a single loop interferometer excludes the advantages of many loop interferometers as proposed by Kwiat et al. [8] but, nevertheless, it gives an idea whether the performance of an interferometer can be expected to be superior.

Interaction-free measurement in it's original form [1] can be considered as a method of distinguishing black and white samples. In a real experimental situation we will have to send a certain number of particles through the interferometer in order to identify the sample with a certain confidence. Repeating the experiment many times we get an average number of particles absorbed per identified sample ($A_I$). This number is then compared to the corresponding number in the transmission setup ($A_T$). For black and white samples we have seen that $A_I$ is always smaller than $A_T$. Of course, this is not surprising, since we know that with the interferometer the black sample is identified without any absorption in 25% of the cases. Much more interesting is the result for cases in which either the black sample is no longer perfectly black but dark-grey or the white sample is no longer perfectly white but light-grey. These cases represent transitions from the ideal interaction-free measurement to general situations.

With white and grey samples we have found that $A_I$ may be smaller than $A_T$ if the grey sample is dark enough. It seems plausible that in this case many-loop interferometers could even perform better. Similarly, if we want to distinguish a black sample from a nearly white one, we also find that $A_I$ is smaller than

$A_T$. However, there is only a narrow range by which the nearly white sample may deviate from a perfectly white sample in order to ensure less absorption in the interferometric separation of black and nearly white samples than in a test with a simple transmission setup. And the confidence of correct identification with the interferometer is in this case *always* smaller than in the transmission setup, such that the interferometer's superiority rests on being content with a certain minimum probability of correct identification of the samples. It is not clear yet if many-loop interferometers may lead to an improvement here, but we will focus on that in a future publication.

The interferometer's stand becomes worse as soon as we wish to distinguish samples from an ensemble of black, grey and white samples, where the grey and the white sample both produce no phase shift (or one of multiples of $2\pi$), but where we choose the transmission of the grey sample such that it leads to the least absorption over the whole ensemble in the interferometric test, rather than in the test with the simple transmission setup. For a given confidence probability of correct identification of the samples, the average number of particles absorbed in a sample turns out to be *higher* in the interferometer than in the simple transmission setup. In fact, as long as only absorptivity is used to characterize samples, the interferometer tends to perform worse, the more samples we wish to be able to distinguish. *We are therefore lead to conclude that interaction-free identification of samples is a peculiar property of an interferometer, which comes to the fore only in the limiting situation where just two different kinds of samples with very different absorption are to be distinguished.*

However, when samples are characterized by the two continuous parameters which they can influence in a test particle's forward going wave function, namely amplitude and phase shift, the interferometer is the proper tool. Since a particle may end up in one of the two detectors or in the sample, one measures a trinomial probability distribution. Such a distribution is fully described by the number of trials and two parameters. Because of this, the number of principally distinguishable samples increases *linearly* with the number of test particles per sample. The number of particles absorbed in a sample also increases in direct proportion with the number of test particles. As a consequence, if we have an ensemble of samples whose absorption and phase shift values are homogeneously distributed, the average number of particles that must necessarily be absorbed per distinguishable sample, turns out to be a constant. Doubling the number of test particles per sample permits grouping the samples into twice as many distinct categories, but the average number of absorbed particles per category is always the same. We have a situation, where the "absorption prize" for additional information is a fixed value, independent of how much we already know.

In contrast, the simple transmission setup measures a binomial distribution, for which the number of distinguishable samples increases only with the square root of the number of test particles. But the number of absorbed particles in a sample is still directly proportional to the number of test particles. Raising the desired number of distinguishable samples therefore increases the number of absorbed particles per distinguished sample category proportional to the square root of the number of test particles per sample. Here we have a situation, where the "absorption prize" for additional information becomes increasingly higher, the more we already know about the samples.

## 6 Acknowledgment

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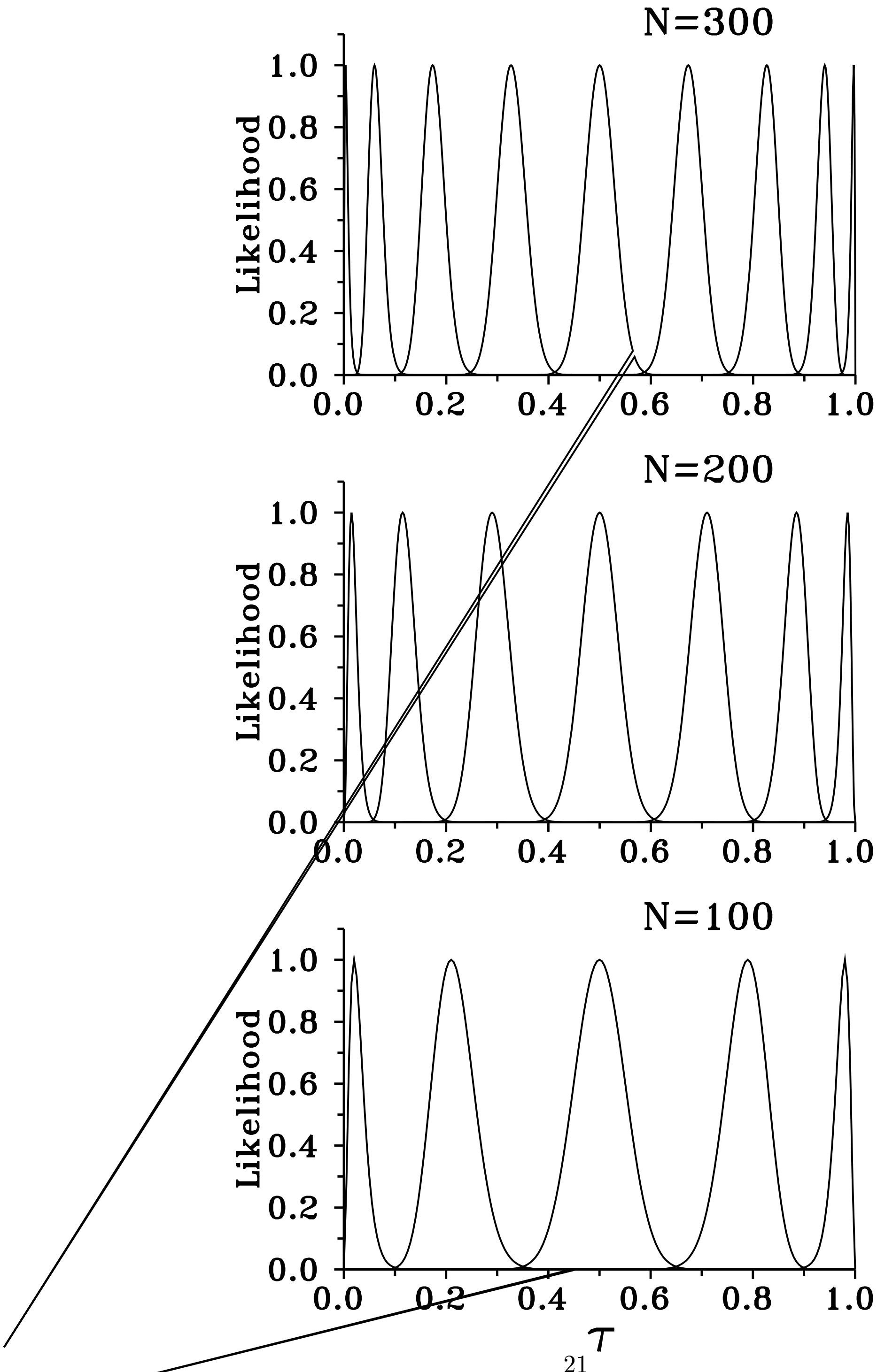




Figure 3: Likelihood functions for the distinguishable results of a simple transmission experiment, where a sample is tested with either $N = 100$ or $N = 200$ or $N = 300$ particles.

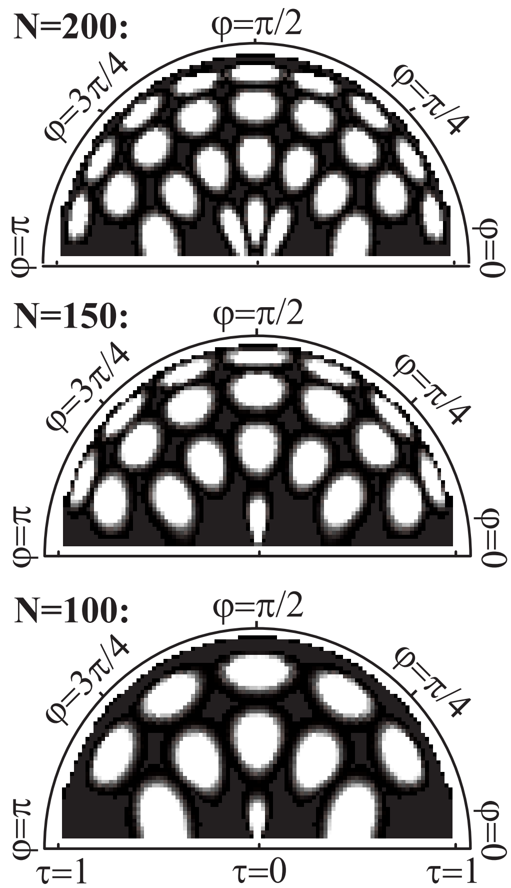


Figure 4: Likelihood in steps of grey (0=black, 1=white) as a function of $\tau$ and $\varphi$ when testing with the Mach-Zehnder interferometer of Fig.2. Each region demarcates the confidence area deduced from an experimental result. The number of distinguishable samples increases linearly with the number $N$ of test particles per sample. Plots are for $N = 100$, $N = 150$ and $N = 200$.