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Abstract

The minimization of nondeterministic automata without initial states (developed within a game-theoretic framework in Calude, Calude, Khoussainov [3]) is presented in terms of bisimulations; the minimal automaton is unique up to an isomorphism in case of reversible automata. We also prove that there exists an infinite class of (strongly connected) nondeterministic automata each of which is not bisimilar with any deterministic automaton. This shows that in the sense of bisimilarity nondeterministic automata are more powerful than deterministic ones. It is an open question whether the method of bisimulations can produced, in general, the unique minimal nondeterministic automaton.

1 Introduction

Minimal *deterministic* automata (with initial states) accepting the same language are isomorphic; in contrast, minimal nondeterministic automata (with initial states) accepting the same language may be *non*-isomorphic (for the classical theory of automata see [2, 15, 22, 25, 3]). Automata without initial states have been studied as toy models for quantum uncertainty (see [19, 23, 5]). This motivated the study of automata without initial states, in particular, the minimization problem for these automata (see [4, 7, 8, 3]). In [3] a game-theoretic solution was presented for the minimization problem for nondeterministic automata without initial states: it leads to a solution that is unique up to an isomorphism. The equivalence relation used to collapse states satisfies a condition (called in [3] "well-behaveness") which is very similar to Park's bisimulation notion for concurrent branching processes.[1] Inspired by Goguen [11, 12], Kozen [16] and Rutten

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[1]Milner [17] defined the notion of "observation equivalence" and later changed to "bisimulation" as introduced by Park [20]. Our notion of bisimulation comes close to the notion of "stronger bisimulation" from concurrency theory, see [14, 17, 18].

[21], we present the theory in [3] in terms of bisimulations. The minimal automaton exists and is unique for reversible automata. We prove that there exists an infinite class of nondeterministic (strongly connected) automata each of which is not bisimilar with any deterministic automaton. This shows that in the sense of bisimilarity nondeterministic automata are more powerful than deterministic ones.

The paper is structured as follows. Section 2 is devoted to notation and main definitions. In Section 3 we study the relation of bisimulation for nondeterministic automata. Section 4 discusses a solution to the minimization problem for nondeterministic reversible automata without initial states in terms of bisimulations. Section 5 contrasts deterministic and nondeterministic automata; in particular one proves that in the sense of bisimulation nondeterministic automata are more powerful than deterministic ones.

2 Notation

If S is a finite set, then $\#(S)$ denotes the cardinality of S. If $f : S \to T$ is a function and $X \subset S$, then $f(X) = \{f(x) \mid x \in X\}$. Any relation $\asymp \subset S \times T$ extends naturally to sets $X \subset S, Y \subset T : X \asymp Y$ if for every $x \in X$ there exists $y \in Y$ such that $x \asymp y$ and for every $y \in Y$ there exists $x \in X$ such that $x \asymp y$.

Let Σ be a finite set (sometimes called alphabet); the set $\Sigma^\star$ stands for the set of all finite words over Σ with the empty word denoted by λ. The length of a string x is denoted by $|x|$. If $X, Y \subset \Sigma^*$, then their concatenation is $X \cdot Y = \{xy \mid x \in X, y \in Y\}$.

We fix two finite alphabets Σ and O: Σ contains **input** symbols, and O contains **output** symbols. A **nondeterministic finite automaton** (without initial states) over the alphabet Σ and O is a triple $A = (S_A, \nabla_A, F_A)$, where

- S_A is a finite nonempty set of **states**,
- ∇_A is a function from $S_A \times \Sigma$ to the set 2^{S_A} of all subsets of S_A, called the **transition table**,
- F_A is a mapping from the set of states S_A into the output alphabet O, called the **output function**.

The output $F_A(s)$ is "emitted" by A on state s. In what follows we will take $O = \{0, 1\}$, so states will split into two categories, final states s in case $F_A(s) = 1$ and nonfinal states s in case $F_A(s) = 0$.

The transition function can be naturally extended to $\widetilde{\nabla}_A : 2^{S_A} \times \Sigma^* \to 2^{S_A}$ by the equations

$$\widetilde{\nabla}_A(X, \lambda) = X,$$
$$\widetilde{\nabla}_A(X, w\sigma) = \bigcup_{q \in \widetilde{\nabla}_A(X,w)} \nabla_A(q, \sigma),$$

for all $X \subset S_A$, $w \in \Sigma^*, \sigma \in \Sigma$. It is seen that $\widetilde{\nabla}_A(\widetilde{\nabla}_A(X, u), v) = \widetilde{\nabla}_A(X, uv)$, for all $X \subset S_A$, and $u, v \in \Sigma^*$.

If $\#(\widetilde{\nabla}_A(p, \sigma)) = 1$, for every $p \in S_A, \sigma \in \Sigma$, then the automaton is deterministic; in this case the transition function will be denoted by Δ_A.

All automata will act on a fixed input alphabet Σ. Unless specified in the context, automata will be nondeterministic, so, the adjective "nondeterministic" will be frequently omitted.

A **morphism** from A to B is a function $h : S_A \to S_B$ compatible with the transition and output functions of A, that is, a function h satisfying the following two conditions: a) $h(\nabla_A(p,\sigma)) = \nabla_B(h(p),\sigma)$, for all $p \in S_A, \sigma \in \Sigma$, b) $F_A(p) = F_B(h(p))$, $p \in S_A$. Note that condition a) is an equality between sets of states, so $\#(\nabla_A(p,\sigma)) \geq \#(\nabla_B(h(p),\sigma))$. It is easy to prove that every morphism h satisfies the equality $h(\nabla_A(p,w)) = \nabla_B(h(p),w)$, for all $p \in S_A, w \in \Sigma^*$. An **isomorphism** is a bijective morphism.

Next we follow Moore's Gedanken-experiments [19] to define the "response" of an automaton $A = (S_A, \nabla_A, F_A)$ to an input sequence of signals: the only thing one can observe about the state of an automaton is whether it is final or not, i.e., the value of the output function F_A. Take $w = \sigma_1 \ldots \sigma_n \in \Sigma^\star$ and $s_0 \in S_A$. A **trajectory** of A on s_0 and w is a sequence

$$s_0, s_1, \ldots, s_n$$

of states such that $s_{i+1} \in \nabla_A(s_i, \sigma_{i+1})$, for all $0 \leq i \leq n-1$.[2] A trajectory $s_0, s_1, \ldots, s_n$ **emits** the output $F_A(s_0)F_A(s_1)\cdots F_A(s_n)$.

The **response** of the automaton A is the function $R_A : S_A \times \Sigma^\star \to 2^{\{0,1\}^*}$ which to any (s,w) assigns the set of all outputs emitted by all trajectories of A on s and w.

The function R_A extends naturally to $\widetilde{R}_A : 2^{S_A} \times \Sigma^* \to 2^{\{0,1\}^*}$ by the equations:

$$\widetilde{R}_A(X, \lambda) = F_A(X),$$

$$\widetilde{R}_A(X, w\sigma) = \widetilde{R}_A(X, w) \cdot F_A(\widetilde{\nabla}_A(X, w\sigma)).$$

An automaton A is **connected** if for every pair of states $p, q \in S_A$ there is a word w such that $q \in \widetilde{\nabla}_A(p,w)$ or $p \in \widetilde{\nabla}_A(q,w)$. An automaton A is **strongly connected** if for every pair of states $p, q \in S_A$ there is a word w such that $p \in \widetilde{\nabla}_A(q,w)$. An automaton is A **reversible** in case for every pair of states $p, q \in S_A$, if there is a word w such that $q \in \widetilde{\nabla}_A(p,w)$, then $p \in \widetilde{\nabla}_A(q,u)$, for some word u. Reversible automata play a special role in the theory of reversible computations, computations which can be undone (see, for instance, [6, 24]). It is easy to see that *an automaton is strongly connected if and only if it is connected and reversible.*

3 Bisimulations

The aim of this section is to model "observationally undistinguishable" states, i.e. states that appear identical for every Moore Gedanken-experiment: the states emit the same output and performing on both states the same experiment will lead to new states which are still undistinguishable (see Calude, Calude, Svozil, Yu [5] for a detailed discussion).

Let $A = (S_A, \nabla_A, F_A)$ and $B = (S_B, \nabla_B, F_B)$ be two automata. A non-empty relation $\asymp \subset S_A \times S_B$ is a **bisimulation** if the following two conditions hold true for all $p \asymp q$ ($p \in S_A, q \in S_B$):

1. $\nabla_A(p,\sigma) \asymp \nabla_B(q,\sigma)$, for all $\sigma \in \Sigma$,

[2] If $s_0, s_1, \ldots, s_n$ is a trajectory of A on s_0 and $w = \sigma_1 \ldots \sigma_n$, then $\nabla_A(s_i, \sigma_{i+1}) \neq \emptyset$, for all $0 \leq i \leq n-1$.

2. $F_A(p) = F_B(q)$.

So, two states related by a bisimulation are observationally undistinguishable.

Proposition 1 *Let A and B be two automata and $h : S_A \to S_B$ a function. Then, the following two conditions are equivalent:*

1. *The function h is a morphism.*
2. *The graph of h is a bisimulation.*

Proof. Assume that $h : S_A \to S_B$ is a morphism. For $p \in S_A, q \in S_B$ define $p \asymp q$ if $q = h(p)$. We will prove that $\asymp$ is a bisimulation. Clearly, from the compatibility between h and the output functions we get $F_A(p) = F_B(q)$ provided $q = h(p)$. Now assume again that $p \asymp q$ and prove that $\nabla_A(p,\sigma) \asymp \nabla_B(q,\sigma)$, for every $\sigma \in \Sigma$. If $s \in \nabla_A(p,\sigma)$, then $h(s) \in h(\nabla_A(p,\sigma)) = \nabla_B(h(p),\sigma) = \nabla_B(q,\sigma)$. Finally, if $t \in \nabla_B(q,\sigma)$, then $t \in h(\nabla_A(p,\sigma))$ (h is morphism), so $t = h(s)$ for some state $s \in \nabla_A(p,\sigma)$.

Conversely, assume that the graph of the function $h : S_A \to S_B$ is a bisimulation, i.e., the relation $p \asymp q$ if $q = h(p)$ is a bisimulation. We shall prove that h is a morphism. Again, $F_A(s) = F_B(h(s))$ is immediate. Next we prove that $h(\nabla_A(s,\sigma)) = \nabla_B(h(s),\sigma)$, for all $s \in S_A$. If $q \in h(\nabla_A(s,\sigma))$, then there is a state $p \in \nabla_A(s,\sigma)$ such that $q = h(p)$ as $\nabla_A(s,\sigma) \asymp \nabla_B(h(s),\sigma)$. So, $p \asymp t$ for some $t \in \nabla_B(h(s),\sigma)$, that is $t = h(p) = q \in \nabla_B(h(s),\sigma)$. Conversely, if $q \in \nabla_B(h(s),\sigma)$, then $q \in \nabla_A(s,\sigma)$ (because $s \asymp h(s)$ and $\nabla_A(s,\sigma) \asymp \nabla_B(h(s),\sigma)$). Consequently, there is a state $p \in \nabla_A(s,\sigma)$ such that $q = h(p)$, so $q \in h(\nabla_A(s,\sigma))$. □

Two automata A, B are called **bisimilar** (we write $A \equiv B$) in case there is a bisimulation $\asymp \subset S_A \times S_B$. It is easy to see that bisimilarity is an equivalence relation (if $A \equiv B$ via $\asymp_1$ and $B \equiv C$ via $\asymp_2$, then $A \equiv C$ via the composition $\asymp_1 \circ \asymp_2$).

Remark 2 Bisimilarity can be very "superficial": two automata A and B may be very different, but still bisimilar, if, for example, there are two states $p \in S_A, q \in S_B$ such that $F_A(p) = F_B(q)$ and $\nabla_A(p,\sigma) = \nabla_B(q,\sigma) = \emptyset$, for all $\sigma \in \Sigma$. The notion of bisimulation defined here is quite different from that defined in usual process algebras. In contrast with processes in process algebras, which have an *implicit initial state*, here we are dealing with automata *without initial states.* As it is clear from Kozen [16], if we assume an initial state, the notions become more similar.

Proposition 3 *Let A and B be two bisimilar automata via $\asymp$. Let $X \subset S_A, Y \subset S_B$ be such that $X \asymp Y$. Then $\widetilde{\nabla}_A(X,w) \asymp \widetilde{\nabla}_B(Y,w)$, for all $w \in \Sigma^*$.*

Proof. The proof is done by induction on the length of w. □

Proposition 4 *Let A and B be two bisimilar automata via $\asymp$. Let $X \subset S_A, Y \subset S_B$ be such that $X \asymp Y$. Then $\widetilde{R}_A(X,w) = \widetilde{R}_B(Y,w)$, for all $w \in \Sigma^*$.*

Proof. As $X \asymp Y$, $F_A(X) = F_B(Y)$, so for every $w \in \Sigma^*$, $F_A(\widetilde{\nabla}_A(X,w)) = F_B(\widetilde{\nabla}_B(Y,w))$. □

Recall that the language recognized by the automaton A with initial states $I \subset S_A$ is the set of words leading to a terminal computation, that is, $L(A,I) = \{w \in \Sigma^* \mid 1 \in F_A(\widetilde{\nabla}_A(I,w))\}$.

Corollary 5 *Let A and B be two bisimilar automata via $\asymp$, and let $I_A \subset S_A$, $I_B \subset S_B$ be sets of initial states. If $I_A \asymp I_B$, then $L(A,I_A) = L(B,I_B)$.*

Proof. Note that $F_A(\widetilde{\nabla}_A(I_A,w))$ is the last letter of the word $\widetilde{R}_A(I_A,w)$. □

A simple verification proves the following:

Proposition 6 *Let A and B be two bisimilar automata. If $(\asymp_i)_{i\in I}$ is a family of bisimulations from $S_A \times S_B$, then their union, $\asymp = \bigcup_{i\in I} \asymp_i$ is also a bisimulation.*

Let A and B be two bisimilar automata. Denote by $\Xi(A,B)$ the set of all bisimulations from $S_A \times S_B$. In view of Proposition 6, $\Xi(A,B)$ *contains a coarsest bisimulation.* This result will be crucial for constructing the minimal automaton bisimilar to a given automaton.

4 Minimization

This section contains the main result of the paper: the existence and unicity of the minimal nondeterministic reversible automaton.

An automaton B is **minimal for** A if $B \equiv A$ and for every automaton $C \equiv A$, we have $\#(S_B) \leq \#(S_C)$.

Lemma 7 *Every minimal automaton is connected.*

Proof. Deleting a state which cannot be accessed we get a bisimilar automaton with less states. □

Example 8 *There exists a minimal automaton which which is not strongly connected.*

Proof. Let A have three states $\{p,q,r\}$, the one-letter alphabet $\Sigma = \{a\}$, $\nabla_A(p,a) = \{q,r\}, \nabla_A(r,a) = \{p\}, \nabla_A(q,a) = \emptyset$, and $F_A(p) = F_A(q) = 1, F_A(r) = 0$. □

Example 9 *There exist two bisimilar nondeterministic automata A and B, a word w and a state $p \in S_A$ such that for every state $q \in S_B$ we have $R_A(p,w) \neq R_B(q,w)$.*

Proof. Let $\Sigma = \{a,b\}$ and A be given by $S_A = \{p_1,p_2,p_3\}$, $\nabla_A(p_1,a) = \{p_1,p_2\}, \nabla_A(p_1,b) = \{p_2\}, \nabla_A(p_2,a) = \nabla_A(p_2,b) = \nabla_A(p_3,a) = \{p_2\}, \nabla_A(p_3,b) = \{p_1,p_3\}$, $F_A(p_1) = F_A(p_3) = 0, F_A(p_2) = 1$, and B be given by $S_B = \{q_1,q_2,q_3\}$, $\nabla_B(q_1,a) = \{q_1,q_2\}, \nabla_B(q_1,b) = \nabla_B(q_3,b) = \{q_3\}, \nabla_B(q_2,a) = \{q_2\}, \nabla_A(q_3,a) = \nabla_B(q_2,b) = \{q_2,q_3\}$, $F_B(q_1) = 0, F_B(q_2) = F_B(q_3) = 1$.

The automata A and B are bisimilar via $\{(p_1,q_1),(p_2,q_2),(p_2,q_3)\}$, but they produce different responses on $w = ba$: $R_A(p_3,w) \neq R_B(q,w)$, for all $q \in S_B$. □

We use bisimulations to prove Theorem 7.4 in [3] for reversible automata. *It is an open question whether the method of bisimulations can produced, in general, the unique minimal nondeterministic automaton.*

Theorem 10 *Let A be a reversible automaton. There is a reversible automaton $M(A)$ satisfying the following properties:*

1. $A \equiv M(A)$.
2. *$M(A)$ is minimal for A.*
3. *If B is reversible and minimal for A, then $M(A)$ is isomorphic to B.*

Proof. Notice first that $\Xi(A,A)$ is non-empty because the equality is an autobisimulation for A. Denote by $\asymp_A$ the greatest bisimulation of $\Xi(A,A)$ (see Proposition 6) and note that $\asymp_A$ is actually an equivalence relation on S_A. Construct the automaton $M(A) = (S_{M(A)}, \nabla_{M(A)}, F_{M(A)})$ by factoring the elements of A to $\asymp_A$: note that the construction is well-defined because $\asymp_A$ is a bisimulation. Denote by $[p]$ the $\asymp_A$—class of the state $p \in S_A$.

The automata A and $M(A)$ are bisimilar via the bisimulation (denoted by $\bowtie$) induced by the projection function $p \in S_A \mapsto [p] \in S_{M(A)}$, which is a morphism (see Proposition 1). The automaton $M(A)$ is reversible because A is reversible.

Intermediate Step. *If $\asymp \in \Xi(M(A),M(A))$ and $[p] \asymp [q]$, then $p \asymp_A q$.*[3]

Indeed, assume by absurdity that there exists a bisimulation $\asymp \subset S_{M(A)} \times S_{M(A)}$ such that there exist two different classes $[p_0],[q_0] \in S_{M(A)}$ such that $[p_0] \asymp [q_0]$. Consider the bisimulation on S_A defined by

$$\cong = \bowtie \circ \asymp \circ \bowtie^r,$$

where $[p] \bowtie^r q$ if $q \bowtie [p]$ (the composition of two bisimulations is again a bisimulation). As $\asymp_A$ is the greatest bisimulation on S_A, it follows that $\cong$ is a subset of $\asymp_A$. However, $p_0 \not\asymp_A q_0$ (as $[p_0]$ and $[q_0]$ are distinct), but $[p_0] \asymp [q_0]$, so $p_0 \cong q_0$ ($\asymp_A$ is an equivalence relation hence $p_0 \asymp_A p_0$, $q_0 \asymp_A q_0$, and $[p_0] \asymp [q_0]$), a contradiction.

We can now argue that $M(A)$ is minimal and unique for A. Let B be a reversible, minimal automaton bisimilar to A, so $B \equiv M(A)$ via a bisimulation $\asymp \subset S_{M(A)} \times S_B$.

[3] In general, $\asymp$ may not be the diagonal of $M(A)$: it's just a subset of the diagonal. For example, consider the automaton A having three states p,q,r, the one-letter alphabet $\Sigma = \{a\}$, the transition $\nabla_A(p,a) = \{q,r\}, \nabla_A(q,a) = r, \nabla_A(r,a) = \emptyset$ and output function $F_A(p) = F_A(r) = 1, F_A(q) = 0$. It is seen that A is minimal and the relation $\{(q,q),(r,r)\}$ is an autosimulation of A. The diagonal $\{(p,p),(q,q),(r,r)\}$ is also an autosimulation of A.

In view of the Intermediate Step, the composition of $\cong = \asymp \circ \asymp^r$ is a subset of the identity. This means that if $[p] \asymp u$ and $[q] \asymp u$, then $[p] = [q]$, i.e., $\asymp$ is one-to-one. Let $[p_0] \asymp u_0$ ($\asymp$ is non-empty) and $v \in S_B$. In view of reversibility, the automaton B is strongly connected (see Lemma 7), that is, there is a word $w \in \Sigma^*$ such that $v \in \widetilde{\nabla}_B(u_0, w) \asymp \widetilde{\nabla}_{M(A)}([p_0], w)$, so there is a state $[p] \in S_{M(A)}$ such that $[p] \asymp v$. This state $[p]$ is unique by virtue of the Intermediate Step. Consequently, we have obtained a bijection from S_B to $S_{M(A)}$. Its graph is a bisimulation, so by Proposition 1, it is a morphism, in fact an isomorphism. □

5 Contrasting Deterministic and Nondeterministic Automata

The subset construction shows that from the point of view of recognized languages, deterministic automata are as powerful as the nondeterministic ones (see [15, 22, 16]). It is not difficult to see that if A and B are bisimilar automata, then the deterministic automata obtained from A and B by the subset construction are also bisimilar (see, for example Proposition 17). Does the subset construction provide a way to pass from a nondeterministic automaton to a bisimilar deterministic automaton? The answer is **negative**.

Theorem 11 *There exist infinitely many nondeterministic (strongly connected) automata each of which is not bisimilar with any deterministic automaton.*

Proof. Let A be a (strongly connected) nondeterministic automaton such that for every state $s \in S_A$ there are $\sigma \in \Sigma$ and $p_{s,\sigma}, q_{s,\sigma} \in \nabla_A(s, \sigma)$ such that $F_A(p_{s,\sigma}) \neq F_A(q_{s,\sigma})$. Then no deterministic automaton $B = (S_B, \Delta_B, F_B)$ is bisimilar with A. Indeed, assume by absurdity, that there is a deterministic automaton B and a bisimulation $\asymp \subset S_A \times S_B$. Consider two states s, t, $s \asymp t$ ($\asymp$ is non-empty) and consider the states $p_{s,\sigma}, q_{s,\sigma} \in \nabla_A(s, \sigma)$ such that $F_A(p_{s,\sigma}) \neq F_B(q_{s,\sigma})$. As $\asymp$ is a bisimulation, $\nabla_A(s, \sigma) \asymp \Delta_B(t, \sigma)$, so there is a state $r_1 \in \Delta_B(t, \sigma)$ such that $p_{s,\sigma} \asymp r_1$. Similarly, there is a state $r_2 \in \Delta_B(t, \sigma)$ such that $q_{s,\sigma} \asymp r_2$. Because B is deterministic, $r_1 = r_2$ and $F_A(p_{s,\sigma}) = F_B(r_1) = F_B(r_2) = F_A(q_{s,\sigma})$, a contradiction.

□

Proposition 12 *Let A and B be two deterministic automata. The relation*

$$\rho = \{(p, q) \in S_A \times S_B \mid R_A(p, w) = R_B(q, w), \text{ for all } w \in \Sigma^*\} \tag{1}$$

is a bisimulation provided it is not empty.

Proof. Clearly, $F_A(p) = R_A(p, \lambda) = R_B(q, \lambda) = F_B(q)$. If $R_A(p, w) = R_B(q, w)$, for all $w \in \Sigma^*$, then $R_A(\Delta_A(p, \sigma), u) = R_B(\Delta_B(q, \sigma), u)$, for all $u \in \Sigma^*$. This can be proved by induction on u using the relation $F_A(\Delta_A(p, u)) = F_B(\Delta_B(q, u))$. □

Proposition 12 is not true for all nondeterministic automata.

Example 13 *There exist two strongly connected nondeterministic automata A and B such that the relation ρ given by (1) is not empty but not a bisimulation.*

Proof. Take $\Sigma = \{a\}$, and consider the automata A and B defined by $S_A = \{p_1, p_2, p_3, p_4\}$, $\nabla_A(p_1, a) = \{p_2\}, \nabla_A(p_2, a) = \{p_3, p_4\}, \nabla_A(p_3, a) = \nabla_A(p_4, a) = \{p_1\}$, $F_A(p_1) = F_A(p_2) = F_A(p_4) = 0, F_A(p_3) = 1$ and $S_B = \{q_1, q_2, q_3, q_4, q_5\}$, $\nabla_B(q_1, a) = \{q_2, q_4\}, \nabla_B(q_2, a) = \{q_3\}, \nabla_B(q_3, a) = \nabla_B(q_5, a) = \{q_1\}, \nabla_B(q_4, a) = \{q_5\}$, $F_B(q_1) = F_B(q_2) = F_B(q_4) = 0, F_B(q_5) = F_B(q_3) = 1$. It is routine to check that $R_A(p_1, w) = R_B(q_1, w)$, for all $w \in \Sigma^*$, that is $(p_1, q_1) \in \rho$, but $(\nabla_A(p_1, a), \nabla_B(q_1, a)) \notin \rho$ as $R_A(p_2, a) = 01 \neq 00 = R_B(q_4, a)$. □

Remark 14 Example 13 is similar to a well-known phenomenon in process algebras: trace equivalence (the process-algebra analogue of the equivalence of automata, see Hoare [13]) is weaker (i.e., relates more pairs of processes) than bisimilarity; see Bloom, Istrail, Meyer [1], van Glabbeek [9, 10].

Example 15 *There exist two strongly connected nondeterministic automata A and B for which the relation (1) is a bisimulation.*

Proof. Let $\Sigma = \{a, b\}$ and A be given by $S_A = \{p_1, p_2, p_3, p_4\}$, $\nabla_A(p_1, a) = \{p_2\}, \nabla_A(p_1, b) = \{p_3\}, \nabla_A(p_2, a) = \{p_2, p_4\}, \nabla_A(p_2, b) = \{p_4\}, \nabla_A(p_4, a) = \{p_1, p_4\}, \nabla_A(p_4, b) = \nabla_A(p_3, b) = \{p_4\}, \nabla_A(p_3, a) = \{p_3, p_4\}$, $F_A(p_1) = F_A(p_4) = 0, F_A(p_2) = F_A(p_3) = 1$, and B be given by $S_B = \{q_1, q_2, q_3\}$, $\nabla_B(q_1, a) = \nabla_B(q_1, b) = \{q_2\}, \nabla_B(q_2, a) = \{q_2, q_3\}, \nabla_B(q_3, b) = \nabla_B(q_2, b) = \{q_3\}, \nabla_B(q_3, a) = \{q_1, q_3\}$, $F_B(q_1) = F_B(q_3) = 0, F_B(q_2) = 1$.

The relation (1), $\rho = \{(p_1, q_1), (p_2, q_2), (p_3, q_2), (p_4, q_3)\}$, is a bisimulation.

Note that B is minimal: it follows from collapsing states p_2 and p_3 in A. □

We show now the compatibility between the bisimulation approach and the simulation approach for deterministic automata (see Calude, Calude, Khoussainov [4]). Recall that the deterministic automaton A is **simulated** by the deterministic automaton B if there is a function $h : S_A \to S_B$ preserving responses, that is, $R_A(p, w) = R_B(h(p), w)$, for all $w \in \Sigma^*$. The automaton A is **strongly simulated** by the deterministic automaton B if there is a function $h : S_A \to S_B$ which preserves responses and internal transitions (that is $h(\Delta_A(p, \sigma)) = \Delta_B(h(p), \sigma)$, for all $p \in S_A, \sigma \in \Sigma$).

Proposition 16 *Every morphism of deterministic automata preserves responses.*

Proof. We prove by induction on w the formula: $R_A(p, w) = R_B(h(p), w)$. For $w = \lambda$ we have: $R_A(p, \lambda) = F_A(p) = F_B(h(p)) = R_B(h(p), \lambda)$. If $R_A(p, w) = R_B(h(p), w)$ and $\sigma \in \Sigma$, then

$$\begin{aligned} R_B(h(p), w\sigma) &= R_B(h(p), w) \cdot F_B(\tilde{\nabla}_B(h(p), w\sigma)) \\ &= R_A(p, w) \cdot F_B(\tilde{\nabla}_B(h(p), w\sigma)) \\ &= R_A(p, w) \cdot F_A(\tilde{\nabla}_A(p, w\sigma)) \\ &= R_A(p, w\sigma). \end{aligned}$$

□

Theorem 17 *Let A and B be deterministic automata and $h : S_A \to S_B$ a function. Then, the following statements are equivalent:*

1. *The function h is a morphism.*

2. *The graph of h is a bisimulation.*

3. *The automaton A is strongly simulated by the automaton B via h.*

Proof. In view of Propositions 1 and 16 we need to prove only the implication 3. $\Rightarrow$ 1., that is, $F_A(p) = R_A(p, \lambda) = R_B(h(p), \lambda) = F_B(h(p))$. □

Remark 18 In Theorem 17 we cannot replace the condition "the automaton A is strongly simulated by the automaton B via h" by "the automaton A is simulated by the automaton B via h". Indeed, the last condition is weaker. For example, consider the automata $A = (\{p, q\}, \{a\}, \Delta_A, F_A)$, where $p \neq q$, $\Delta_A(p, a) = \Delta_A(q, a) = q$, $F_A(p) = F_A(q) = 0$, and $B = (\{p', q'\}, \{a\}, \Delta_B, F_B)$, where $p' \neq q'$, $\Delta_B(p', a) = q', \Delta_B(q', a) = p'$, $F_B(p') = F_B(q') = 0$. Every function h from $\{p, q\}$ to $\{p', q'\}$ respects outputs, but it's not a morphism.

Combining Theorem 17 and Corollary 2.2 in [4] we get:

Corollary 19 *Let A and B be deterministic minimal automata and $h : S_A \to S_B$ a function. Then, the following statements are equivalent:*

1. *The function h is a morphism.*

2. *The graph of h is a bisimulation.*

3. *The automaton A is strongly simulated by the automaton B via h.*

4. *The automaton A is simulated by the automaton B via h.*

5. *The automata A and B are isomorphic via h.*

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