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Abstract

Including the range of a rational function over an interval is an important problem in numerical computation. A direct interval arithmetic evaluation of a formula for the function yields in general a superset with an error linear in the width of the interval. Special formulas like the centered forms yield a better approximation with a quadratic error. Alefeld posed the question whether in general there exists a formula whose interval arithmetic evaluation gives an approximation of better than quadratic order. In this paper we show that the answer to this question is negative if in the interval arithmetic evaluation of a formula only the basic four interval operations $+, -, \cdot, /$ are used.

Keywords : Interval arithmetic, range enclosure, approximation of quadratic order, centered form.

1 Introduction

In numerical computations one often wishes to compute the interval $f(I)$ for a given continuous or rational function f and a given closed interval I such that f is defined at all points in I. Since in general the exact computation can be difficult or very costly one is often content with "including the range" of f over I by computing an interval J which contains $f(I)$ and such that the difference set $J \setminus f(I)$ is small. This is a central theme in interval arithmetic, which started with the book [9] by Moore. For interval arithmetic in general see also Moore [10], Alefeld and Herzberger [4], and for range enclosure see also Hansen [7], Krawczyk and Nickel [8], Cornelius and Lohner [6], Ratschek and Rokne [12], Neumaier [11], Alefeld [3, 2] and many more. Interval arithmetic provides the following approach to this problem (precise definitions of all notions used in the introduction will be given in Section 2). If $F(x_1, \ldots, x_m)$ is an arithmetical

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expression in the variables $x_1, \ldots, x_m$, furthermore using some or all of the symbols $+, -, \cdot, /$, the brackets (,), and symbols for real constants, then F defines a rational function $F_{\mathbb{R}}$ on the reals in the obvious way. It defines also a function $F_{\mathbb{IR}}$ mapping intervals to intervals when $+, -, \cdot, /$ are interpreted as functions on intervals. It is important that the interval $F_{\mathbb{IR}}(I)$ contains the interval $F_{\mathbb{R}}(I)$ if $F_{\mathbb{IR}}(I)$ is defined. But for most expressions F the interval $F_{\mathbb{IR}}(I)$ is much larger than $F_{\mathbb{R}}(I)$: usually the error is linear in the width of the interval I. However, locally one can do better: given a rational function f and a point z at which f is defined, one can always find an arithmetical expression F such that for any small enough interval I containing z the interval $F_{\mathbb{IR}}(I)$ contains $f(I)$ and such that the error is at most quadratic in the width of I. These are the so–called "centered forms" and similar forms like the "mean value form", compare Moore [9], Alefeld and Herzberger [4]. Is it possible to achieve in this way an approximation with an error of smaller than quadratic order? That means, is it possible to find in general an arithmetical expression F such that its interval evaluation $F_{\mathbb{IR}}$ with input I yields an interval $F_{\mathbb{IR}}(I)$ which approximates $f(I)$ better than quadratically in the width of I? This question was posed explicitly by Alefeld [2, page 63]. It is the purpose of this paper to give a negative answer to this question. Of course, it is well known that in a restricted sense one can give a positive answer to this question in special cases, using so–called "higher order centered forms" (Cornelius, Lohner [6], Alefeld, Lohner [5], Alefeld [3]). We shall discuss this in Section 5.

We give a short overview over the paper. In the following section we introduce some notation and provide precise definitions from interval arithmetic as well as fundamental results as far as they are important for this paper. In Section 3 we formulate the main result and simpler versions of it. Section 4 consists of the proof of the main result. We conclude the paper with remarks about higher order forms and about further, related problems.

2 Prerequisites from Interval Arithmetic

In this section we introduce some notation and fundamental notions and results from interval arithmetic. General references are Moore [9, 10] and Alefeld and Herzberger [4].

By $\mathbb{R}$ we denote the set of real numbers. For two real numbers a, b with $a \leq b$ we denote by $(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$ the open interval with bounds a and b and by

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

the closed interval with bounds a and b. When we write simply "interval" we always mean a closed interval. The set of all closed intervals is denoted by $\mathbb{IR}$. We embed the set $\mathbb{R}$ of real numbers into the set $\mathbb{IR}$ by identifying the real number x with the interval $[x, x]$. For a positive integer m the set of m–vectors of real numbers (of m–vectors of closed intervals) is denoted by $\mathbb{R}^m$ (by $\mathbb{IR}^m$). In the following we shall always identify a vector $I = (I_1, \ldots, I_m) \in \mathbb{IR}^m$ of intervals with the direct product $I_1 \times \ldots \times I_m \subseteq \mathbb{R}^m$. In this sense, for $D \subseteq \mathbb{R}^m$ we define

$$\mathbb{IR}^m(D) := \{I \in \mathbb{IR}^m \mid I \subseteq D\}\,.$$

If $D \subseteq \mathbb{R}^m$ and $f : D \to \mathbb{R}$ is a continuous function defined on D (where D is endowed with the subspace topology) with real values, then for an arbitrary interval vector $I \in \mathbb{IR}^m(D)$ the set $f(I) := \{f(x) \mid x \in I\}$ is a closed interval, because the continuous image of a nonempty, connected, compact set is again a nonempty, connected, compact set, and the nonempty, connected, compact subsets of $\mathbb{R}$ are exactly the closed intervals. Therefore the *induced function* $f : \mathbb{IR}^m(D) \to \mathbb{IR}$ is well defined (called "united extension" by Moore [9]). We write it also as f. In this way for example the basic arithmetic operations addition $+$, subtraction $-$, multiplication $\cdot$ (all defined on $\mathbb{R}^2$) and division $/$ (defined on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$) induce corresponding operations on closed intervals. Notice that the quotient $[a,b]/[c,d]$ of two intervals $[a,b]$ and $[c,d]$ is defined if and only if $0 \notin [c,d]$.

Often for a given continuous or rational function f and an interval I (or an interval vector I) such that f is defined at all points in I one wishes to compute the interval $f(I)$. Since in general the exact computation can be difficult or expensive one is often content with computing an interval J which contains $f(I)$ and such that the difference set $J \setminus f(I)$ is small. This "smallness" is measured by the Hausdorff distance between J and $f(I)$. For intervals the Hausdorff distance $d_H : \mathbb{IR}^2 \to \mathbb{R}$ is given by

$$d_H([a,b],[c,d]) = \max\{|a-c|, |b-d|\}\,.$$

The Hausdorff distance (with respect to the maximum distance on $\mathbb{R}^m$) between interval vectors (remember that we identify $(I_1,\ldots,I_m)$ with $I_1 \times \ldots \times I_m$) is given by

$$d_H(([a_1,b_1],\ldots,[a_m,b_m]),([c_1,d_1],\ldots,[c_m,d_m])) = \max_{1\le i\le m} d_H([a_i,b_i],[c_i,d_i])\,.$$

Notice that these equations define indeed a metric on the sets $\mathbb{IR}$ and $\mathbb{IR}^m$. Therefore we also have the notion of continuity for functions on the spaces of intervals or interval vectors. If a function $f : D \subseteq \mathbb{R}^m \to \mathbb{R}$ is continuous, then also the induced function $f : \mathbb{IR}^m(D) \to \mathbb{IR}$ is continuous (compare Moore [9, Theorem 4.1]).

Rational functions are defined via arithmetical expressions. These are defined recursively. We assume that we have an infinite set $V = \{x_1, x_2, x_3, \ldots\}$ of symbols for variables. Furthermore we use the real numbers $\mathbb{R}$, the symbols $\{+,-,\cdot,/\}$ and the brackets $\{(,)\}$. Arithmetical expressions are defined recursively as words over the union of these alphabets by the following conditions:

1. Each real number and each symbol for a variable is an arithmetical expression.

2. It t_1 and t_2 are arithmetical expressions, then also $(t_1 + t_2)$ and $(t_1 - t_2)$ and $(t_1 \cdot t_2)$ and (t_1/t_2) are arithmetical expressions.

3. No other words are arithmetical expressions.

If F is an arithmetical expression containing exactly the variables $x_1, \ldots, x_m$ (in the future we shall indicate this by saying "let $F(x_1, \ldots, x_m)$ be an arithmetical expression"), then F defines in an obvious way a rational function $F_{\mathbb{R}} : D \subseteq \mathbb{R}^m \to \mathbb{R}$ where also the domain D of $F_{\mathbb{R}}$ is determined by F. This function $F_{\mathbb{R}}$ is obtained by associating with each of the symbols $+, -, \cdot, /$ the corresponding real number function and by evaluating F recursively. In the same way F defines also a function $F_{\mathbb{IR}} :\subseteq \mathbb{IR}^m \to \mathbb{IR}$ mapping interval vectors to intervals: one associates with each real number c the interval $[c, c]$ and with each of the symbols $+, -, \cdot, /$ the corresponding function on intervals. One checks easily that $F_{\mathbb{R}}$ is the restriction of $F_{\mathbb{IR}}$ to $\mathbb{R}^m$. But notice that it may happen that $F_{\mathbb{R}}$ is defined at each point in an interval I while $F_{\mathbb{IR}}(I)$ is not defined.

Example 2.1 Let the arithmetical expression $F(x)$ be defined by

$$F(x) := (1/((x+2) + (1/x)))\,.$$

Then the rational function $F_{\mathbb{R}}$ is defined on $\mathbb{R} \setminus \{-1, 0\}$ and can also be written as $F_{\mathbb{R}}(x) = x/(x+1)^2$ for $x \notin \{-1, 0\}$. It is defined at each point in the interval $[-3/4, -1/2]$, but $F_{\mathbb{IR}}([-3/4, -1/2]$ is not defined:

$$\begin{aligned} F_{\mathbb{IR}}([-3/4,-1/2]) &= (1/(([-3/4,-1/2]+2) + (1/[-3/4,-1/2]))) \\ &= (1/([5/4,3/2] + [-2,-4/3])) \\ &= (1/[-3/4,1/6]) \\ &= \text{undefined}\,. \end{aligned}$$

However, if $F_{\mathbb{IR}}(I)$ is defined, then $F_{\mathbb{R}}$ is defined at all points in I. The following fundamental facts are already contained in Moore [9].

Proposition 2.2 *Let $F(x_1, \ldots, x_m)$ be an arithmetical expression and $I \in \mathbb{IR}^m$ such that $F_{\mathbb{IR}}(I)$ is defined.*

1. *For $J \subseteq I$ also $F_{\mathbb{IR}}(J)$ is defined and $F_{\mathbb{IR}}(J) \subseteq F_{\mathbb{IR}}(I)$.*
2. *$F_{\mathbb{R}}(I) \subseteq F_{\mathbb{IR}}(I)$.*

The interval $F_{\mathbb{IR}}(I)$ is an approximation to the interval $F_{\mathbb{R}}(I)$. How good is this approximation? This is usually measured by the Hausdorff distance between the two intervals and by comparing this distance with the width of the interval I. The *width* $w([a, b])$ of an interval vector $[a, b] = ([a_1, b_1], \ldots, [a_m, b_m]) \in \mathbb{IR}^m$ is defined by

$$w([a, b]) := \max\{b_i - a_i \mid i \in \{1, \ldots, m\}\}\,.$$

For the width of the interval $F_{\mathbb{IR}}(I)$ compared with I one obtains the following result, see Moore [10, Lemma 4.1].

Proposition 2.3 *Let $F(x_1, \ldots, x_m)$ be an arithmetical expression and $J \in \mathbb{IR}^m$ such that $F_{\mathbb{IR}}(J)$ is defined. Then there exists a constant $c > 0$ such that for all $I \in \mathbb{IR}^m(J)$*

$$w(F_{\mathbb{IR}}(I)) \leq c \cdot w(I)\,.$$

The proof is based on the fact that the real operations addition, subtraction, multiplication, and division are Lipschitz continuous on every compact subset of their domain. For a related, stronger result on Lipschitz continuity of $F_{\mathbb{IIR}}$ see Neumaier [11, Theorem 2.1.1].

Corollary 2.4 *Let $F(x_1, \ldots, x_m)$ be an arithmetical expression and $J \in \mathbb{IIR}^m$ such that $F_{\mathbb{IIR}}(J)$ is defined. Then there exists a constant $c > 0$ such that for all $I \in \mathbb{IIR}^m(J)$*

$$d_H(F_{\mathbb{IIR}}(I), F_{\mathbb{R}}(I)) \leq c \cdot w(I) \, .$$

Proof. Let $I \subseteq J$ be an interval vector. Using $F_{\mathbb{R}}(I) \subseteq F_{\mathbb{IIR}}(I)$ (Proposition 2.2.2) and a constant c as in Proposition 2.3 one obtains $d_H(F_{\mathbb{IIR}}(I), F_{\mathbb{R}}(I)) \leq w(F_{\mathbb{IIR}}(I)) \leq c \cdot w(I)$. □

In general, this trivial linear error bound for the inclusion $F_{\mathbb{R}}(I) \subseteq F_{\mathbb{IIR}}(I)$ (trivial in view of Proposition 2.3: the width itself of the computed interval $F_{\mathbb{IIR}}(I)$ is linear in the width of I) is realistic for a direct interval arithmetic evaluation of an arithmetical expression. Is it possible to find better approximations? Locally this is possible via the so–called centered forms, introduced by Moore [9], and since then extensively treated; for a presentation of the development up to 1984 see Ratschek and Rokne [12]. We formulate a version for dimension one. The "quadratic order property" in the following proposition was conjectured by Moore [9] and first proved by Hansen [7]. A proof of the following result can be obtained by using Proposition 2.3 and following Alefeld, Herzberger [4, pp. 36, 37].

Proposition 2.5 *Let $D \subseteq \mathbb{R}$ open, $f : D \to \mathbb{R}$ be a rational function, $z \in D$ a point, and $H(x)$ an arithmetical expression such that the arithmetical expression*

$$F(x) := (f(z) + ((x - z) \cdot H(x)))$$

satisfies $F_{\mathbb{R}}(y) = f(y)$ for all real numbers $y \in D$. For any interval $J \subseteq D$ containing z such that $F_{\mathbb{IIR}}(J)$ is defined (such an interval J exists!) there exists a constant $c > 0$ such that for all $I \in \mathbb{IIR}(J)$ with $z \in I$ we have

$$d_H(F_{\mathbb{IIR}}(I), f(I)) \leq c \cdot w(I)^2 \, .$$

Informally speaking, at least locally, at an arbitrary point, there always exists an arithmetical expression which leads to a range enclosure with a quadratic error for intervals containing this point. In fact, it is not necessary to take an arithmetical formula H such that the combined formula F (as in the proposition) has the property $F_{\mathbb{R}}(y) = f(y)$ for all $y \in D$, i.e. such that the combined formula F gives exactly the rational function. Instead, one can also take simpler expressions H, for example an expression given by the mean value theorem (Moore [9], Alefeld, Herzberger [4]).

Above we have formulated a version of the result about the centered form where we have fixed a point z. One can also choose the "center" of the centered form in dependence of the given interval I (it does not have to be the midpoint of the interval). This leads to a class of arithmetical expressions with an additional parameter or to a more general notion of an arithmetical expression. Here we are more interested in another question: is it possible to approximate in this way the range of a function with an error of smaller than quadratic order? This is indeed possible in special cases. In fact, sometimes the interval arithmetic evaluation of an arithmetical expression leads even to the exact range of the function over an interval; compare Alefeld [1] and Stahl [13]. Other special cases are given by the higher order centered forms by Cornelius and Lohner [6], Alefeld, Lohner [5], Alefeld [3], and others, which, however, require more general kinds of arithmetical expressions. We shall come back to this in Section 5. In the following two sections we shall formulate and prove a result which gives a negative answer to the above question when we interpret it strictly by using arithmetical expressions and their interval arithmetic evaluations as they have been introduced so far.

3 The Main Result

The following theorem is the technical main result of the paper.

Theorem 3.1 *Let $n > 0$, $D \subseteq \mathbb{R}$ open, $f : D \to \mathbb{R}$ a $2n$ times continuously differentiable function, and $z \in D$ a point with $f^{(i)}(z) = 0$ for $1 \leq i \leq 2n-1$ and $f^{(2n)}(z) \neq 0$. If $F(x)$ is an arithmetical expression in one variable x and $\tilde{c} > 0$ a constant such that the interval $F_{\mathbb{IR}}([z-\tilde{c}, z+\tilde{c}])$ is defined and for all $\delta \in [0, \tilde{c})$*

$$f([z-\delta, z+\delta]) \subseteq F_{\mathbb{IR}}([z-\delta, z+\delta]),$$

then for every $\varepsilon \in (0,1)$ there exists a positive number $c_\varepsilon < \tilde{c}$ such that for all $\delta \in [0, c_\varepsilon)$

$$d_H(F_{\mathbb{IR}}([z-\delta, z+\delta]), f([z-\delta, z+\delta])) \geq \frac{|f^{(2n)}(z)|}{(2n)!} \cdot (1-\varepsilon) \cdot \delta^{2n}.$$

The proof will be given in the following section. The most important case is the case $n = 1$. By taking $n = 1$ and $\varepsilon = 1/2$ one obtains the following simplified version.

Corollary 3.2 *Let $D \subseteq \mathbb{R}$ be open, $f : D \to \mathbb{R}$ be twice continuously differentiable, and $z \in D$ a point with $f'(z) = 0$ and $f''(z) \neq 0$. If $F(x)$ is an arithmetical expression and $\tilde{c} > 0$ a constant such that for all $\delta \in (0, \tilde{c})$ the interval $F_{\mathbb{IR}}([z-\delta, z+\delta])$ is defined and contains $f([z-\delta, z+\delta])$, then there exists a $c > 0$ such that for all $\delta \in [0, c)$*

$$d_H(F_{\mathbb{IR}}([z-\delta, z+\delta]), f([z-\delta, z+\delta])) \geq \frac{|f''(z)|}{4} \cdot \delta^2.$$

This shows that one cannot in general achieve an approximation with an error of smaller than quadratic order if one approximates the range of a function f

by interval arithmetic evaluation of an arithmetical expression and if there are points z with $f'(z) = 0$ and $f''(z) \neq 0$.

One can apply this even to the square function, which maps each real number x to its square x^2: there does not exist an arithmetical expression F whose interval arithmetic evaluation $F_{\mathbb{IIR}}(I)$ (in the sense of Section 2) gives a better than quadratic approximation to $I^2 = \{x^2 \mid x \in I\}$ for all small intervals I with center 0.

By using Proposition 2.2.2 we obtain the following corollary.

Corollary 3.3 *Let $F(x)$ be an arithmetical expression and $z \in \mathbb{R}$ be a point such that $F_{\mathbb{R}}(z)$ is defined, $F'_{\mathbb{R}}(z) = 0$, and $F''_{\mathbb{R}}(z) \neq 0$. Then there is a $c > 0$ such that for all $\delta \in (0, c)$ we have*

$$d_H(F_{\mathbb{IIR}}([z-\delta, z+\delta]), F_{\mathbb{R}}([z-\delta, z+\delta])) \geq \frac{|F''_{\mathbb{R}}(z)|}{4} \cdot \delta^2 .$$

Notice that if $F_{\mathbb{R}}(z)$ is defined, then for small enough $\delta > 0$ also $F_{\mathbb{IIR}}([z - \delta, z + \delta])$ is defined. This corollary gives explicitly a negative answer to the question posed by Alefeld [2, page 63]. In Section 5 we will shortly explain how by using an extension of the notion of an arithmetical expression one can give a positive answer in special cases.

We have formulated these negative results only for dimension one. It is clear that one can apply them also to multivariate functions by considering the partial derivatives in one direction.

4 Proof of the Main Result

This section contains the proof of Theorem 3.1. We shall use the following notation: for two functions $f :\subseteq \mathbb{R} \to \mathbb{R}$ and $g :\subseteq \mathbb{R} \to \mathbb{R}$ we write

$f(\delta) \in o(g(\delta)) :\iff$ there exists a $c > 0$ such that f and g are defined at each point in $(0, c)$, and $g(\delta) \neq 0$ for all $\delta \in (0, c)$, and $\lim_{x \searrow 0} f(x)/g(x) = 0$.

Before we give the detailed proof of Theorem 3.1 we sketch the idea for the case $n = 1$. The basic observation is that interval addition is a "symmetric" operation in the following sense: If I_1 and I_2 are intervals with centers z_1 and z_2, then $I_1 + I_2$ is an interval with center $z_1 + z_2$. The corresponding statement is true for subtraction. For multiplication (and division) it is not exactly true but at least in a restricted sense: if for $i \in \{1, 2\}$ the interval I_i either has center $z_i = 0$ or is a "small" interval "far away" from 0 with center z_i, then $I_1 \cdot I_2$ is an interval whose center is "close" to $z_1 \cdot z_2$. Altogether one might say that all four operations $+, -, \cdot, /$ on intervals are approximately symmetric. By induction this is true also for the interval arithmetic evaluation $F_{\mathbb{IIR}}$ of an arithmetical expression F. On the other hand, if the function $F_{\mathbb{R}}$ satisfies $F'_{\mathbb{R}}(z) = 0$ and $F''_{\mathbb{R}}(z) > 0$ at a point z, then close to z its graph looks like a parabola, and hence with $c := \frac{1}{2} F''_{\mathbb{R}}(z)$

$$F_{\mathbb{R}}([z-\delta, z+\delta]) = [F_{\mathbb{R}}(z), F_{\mathbb{R}}(z) + c \cdot \delta^2 + o(\delta^2)]$$

for $\delta > 0$ small enough. Now $F_{\mathbb{R}}([z-\delta, z+\delta]) \subseteq F_{\mathbb{IR}}([z-\delta, z+\delta])$ and the property of $F_{\mathbb{IR}}$ of being "approximately symmetric" as explained above imply

$$F_{\mathbb{IR}}([z-\delta, z+\delta]) \supseteq [F_{\mathbb{R}}(z) - c \cdot \delta^2 + o(\delta^2), F_{\mathbb{R}}(z) + c \cdot \delta^2 + o(\delta^2)] \,.$$

We conclude

$$d_H(F_{\mathbb{IR}}([z-\delta, z+\delta]), F_{\mathbb{R}}([z-\delta, z+\delta])) \geq c \cdot \delta^2 + o(\delta^2) \,.$$

This ends the sketch of the proof for Theorem 3.1 for the case $n = 1$.

We come to the detailed proof.

Definition 4.1 Let $x_0 \in \mathbb{R}$ and $(x_1, \ldots, x_m) \in \mathbb{R}^m$.

1. A *strict almost symmetrical class of intervals with center* x_0 is a function $S : [0,1) \to \mathbb{IR}$ with the following properties: there exist positive numbers c, e and functions $s^{(l)}, s^{(r)} : [0,1) \to \mathbb{R}$ with $s^{(l)}(\delta) \in o(\delta^e)$ and $s^{(r)}(\delta) \in o(\delta^e)$ such that

$$S(\delta) = [x_0 - c \cdot \delta^e + s^{(l)}(\delta), x_0 + c \cdot \delta^e + s^{(r)}(\delta)]$$

   for all $\delta \in [0,1)$.

2. An *almost symmetrical class of intervals with center* x_0 is a function $S : [0,1) \to \mathbb{IR}$ which is either a strict almost symmetrical class of intervals with center x_0 or has constant value $S(\delta) = [x_0, x_0]$ for all $\delta \in [0,1)$.

3. A *(strict) almost symmetrical class of interval vectors with center* $x = (x_1, \ldots, x_m)$ is a function $S = (S_1, \ldots, S_m) : [0,1) \to \mathbb{IR}^m$ such that each component $S_i : [0,1) \to \mathbb{IR}$ for $1 \leq i \leq m$ is a (strict) almost symmetrical class of intervals with center x_i.

Lemma 4.2 *Let $D \subseteq \mathbb{R}^m$ open and $f : D \to \mathbb{R}$ be a continuously differentiable function. Let $z = (z_1, \ldots, z_m) \in D$ be a point with $(\frac{\partial f}{\partial x_1}(z), \ldots, \frac{\partial f}{\partial x_m}(z)) \neq (0, \ldots, 0)$. If $S : [0,1) \to \mathbb{IR}^m(D)$ is a strict almost symmetrical class of interval vectors with center z, then the function $T : [0,1) \to \mathbb{IR}$ with*

$$T(\delta) := f(S(\delta))$$

is a strict almost symmetrical class of intervals with center $f(z)$.

Proof. Let $S : [0,1) \to \mathbb{IR}^m$ be a strict almost symmetrical class of interval vectors with center z, and let c_i, e_i be positive numbers and $s_i^{(l)}, s_i^{(r)}$ be functions such that they describe the components S_i of S as in Definition 4.1. We define a subset M of the index set $\{1, \ldots, m\}$ by

$$M := \{i \in \{1, \ldots, m\} \mid \ \tfrac{\partial f}{\partial x_i}(z) \neq 0 \ \text{ and } \ e_i = \min\{e_j \mid j \in \{1, \ldots, m\} \ \text{ and } \ \tfrac{\partial f}{\partial x_j}(z) \neq 0\}\} \,.$$

We set $e := e_i$ for an arbitrary $i \in M$ (all numbers e_i for $i \in M$ are identical). Since we have for $y = (y_1, \ldots, y_m) \in D$

$$f(y) = f(z) + \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(z) \cdot (y_i - z_i) + o(\max_i |y_i - z_i|)\,,$$

we conclude that there are functions $t^{(l)}, t^{(r)} : [0,1) \to \mathbb{R}$ with $t^{(l)}(\delta) \in o(\delta^e)$ and $t^{(r)}(\delta) \in o(\delta^e)$ and

$$\begin{aligned} f(S(\delta)) &= f([z_1 - c_1 \cdot \delta^{e_1} + s_1^{(l)}(\delta), z_1 + c_1 \cdot \delta^{e_1} + s_1^{(r)}(\delta)], \ldots, \\ &\qquad [z_m - c_m \cdot \delta^{e_m} + s_m^{(l)}(\delta), z_m + c_m \cdot \delta^{e_m} + s_m^{(r)}(\delta)]) \\ &= [f(z) - \sum_{i \in M} |\frac{\partial f}{\partial x_i}(z)| \cdot c_i \cdot \delta^e + t^{(l)}(\delta), \\ &\qquad f(z) + \sum_{i \in M} |\frac{\partial f}{\partial x_i}(z)| \cdot c_i \cdot \delta^e + t^{(r)}(\delta)] \end{aligned}$$

for all $\delta \in [0,1)$. That proves the assertion. □

The key for the proof of Theorem 3.1 is the following proposition.

Proposition 4.3 *Let $op \in \{+, -, \cdot, /\}$ be one of the basic four real arithmetic operations. If $S = (S_1, S_2) : [0,1) \to \mathbb{IR}^2$ is an almost symmetrical class of interval vectors with some center $z = (z_1, z_2) \in \mathbb{R}^2$ (in the case $op = /$ we assume additionally that $0 \notin S_2(\delta)$ for all $\delta \in [0,1)$), then the function $T : [0,1) \to \mathbb{IR}$ with $T(\delta) := op(S(\delta))$ is an almost symmetrical class of intervals with center $op(z)$.*

Proof. We treat the four operations separately.

"*op* = +": If both components of $S = (S_1, S_2)$ are constant functions, i.e. $S_i(\delta) = [y_i, y_i]$, then also T is a constant function with $T(\delta) = [y_1 + y_2, y_1 + y_2]$ for all $\delta \in [0,1)$.

If only the first component S_1 is constant with value $y \in \mathbb{R}$, then we can apply Lemma 4.2 to the function $+_y : \mathbb{R} \to \mathbb{R}$ with $+_y(x) := y + x$ which has nonzero derivative: $(+_y)'(z_2) = 1$ for all $z_2 \in \mathbb{R}$.

The same can be done if only the second component S_2 is constant.

If both components of S are nonconstant, then we can apply Lemma 4.2 directly to the addition function $+ : \mathbb{R}^2 \to \mathbb{R}$, which has nonzero derivative: $(\frac{\partial +}{\partial x_1}, \frac{\partial +}{\partial x_2})(z) = (1,1)$ for arbitrary $z \in \mathbb{R}^2$.

"*op* = −": Subtraction can be treated in exactly the same way as addition.

"*op* = ·": If both components of $S = (S_1, S_2)$ are constant functions, i.e. $S_i(\delta) = [y_i, y_i]$, then also T is a constant function with $T(\delta) = [y_1 \cdot y_2, y_1 \cdot y_2]$ for all $\delta \in [0,1)$.

If only the first component S_1 is constant with value $y \in \mathbb{R} \setminus \{0\}$, then we can apply Lemma 4.2 to the function $\cdot_y : \mathbb{R} \to \mathbb{R}$ with $\cdot_y(x) := y \cdot x$ which has nonzero derivative y everywhere.

If the first component S_1 is constant with value 0, then T is the constant function with $T(\delta) = [0, 0]$ for all $\delta \in [0, 1)$.
Analogously the case that only the second component S_2 is constant is treated. Let us assume that both components of S are nonconstant. If the center $z = (z_1, z_2)$ of S is nonzero, then the multiplication function $\cdot : \mathbb{R}^2 \to \mathbb{R}$ has nonzero derivative in the point z: $(\frac{\partial \cdot}{\partial x_1}, \frac{\partial \cdot}{\partial x_2})(z) = (z_2, z_1)$, and we can apply Lemma 4.2. Finally, if $z = (0, 0)$, then we obtain

$$\begin{aligned} T(\delta) &= S_1(\delta) \cdot S_2(\delta) \\ &= [-c_1 \cdot \delta^{e_1} + o(\delta^{e_1}), c_1 \cdot \delta^{e_1} + o(\delta^{e_1})] \cdot [-c_2 \cdot \delta^{e_2} + o(\delta^{e_2}), c_2 \cdot \delta^{e_2} + o(\delta^{e_2})] \\ &= [-c_1 c_2 \delta^{e_1+e_2} + o(\delta^{e_1+e_2}), c_1 c_2 \delta^{e_1+e_2} + o(\delta^{e_1+e_2})] \end{aligned}$$

where c_1, c_2, e_1, e_2 are positive numbers determined by S. This proves the assertion in the case of the multiplication $\cdot$.

"$op = /$": The division can be treated in a way similar to the multiplication. In fact, not all cases which had to be treated in the multiplication can occur because division by zero is not allowed: the center z_2 of the second component S_2 of the almost symmetrical class S of interval vectors can never be 0.
This ends the proof of Proposition 4.3. □

Corollary 4.4 *Let $F(x_1, \ldots, x_m)$ be an arithmetical expression and $S : [0, 1) \to \mathbb{IR}^m$ be an almost symmetrical class of interval vectors with some center $z \in \mathbb{R}^m$ such that $F_{\mathbb{IR}}(S(\delta))$ is defined for all $\delta \in [0, 1)$. Then the function $T : [0, 1) \to \mathbb{IR}$ defined by*

$$T(\delta) := F_{\mathbb{IR}}(S(\delta))$$

is an almost symmetrical class of intervals with center $F_{\mathbb{R}}(z)$.

Proof. This follows from Proposition 4.3 and by induction over the structure of the arithmetical expression F. □

Corollary 4.4 is the last step towards the proof of Theorem 3.1.

Proof of Theorem 3.1. Let n, D, f, z, F, and $\tilde{c}$ be as in Theorem 3.1. We assume without loss of generality that $f^{(2n)}(z) > 0$. Our assumptions imply that for numbers x close to z

$$f(x) = f(z) + \frac{f^{(2n)}(z)}{(2n)!} \cdot (x - z)^{2n} + o(|x - z|^{2n}) .$$

Hence, we can fix a (sufficiently small) positive constant $c' < \tilde{c}$ such that there is a function $s : [0, c') \to \mathbb{R}$ with $s(\delta) \in o(\delta^{2n})$ and such that for $\delta \in [0, c')$

$$f([z - \delta, z + \delta]) = [f(z), f(z) + \frac{f^{(2n)}(z)}{(2n)!} \cdot \delta^{2n} + s(\delta)] . \qquad (1)$$

From (1) and the assumptions in Theorem 3.1 we conclude

$$F_{\mathbb{IR}}([z - \delta, z + \delta]) \supseteq [f(z), f(z) + \frac{f^{(2n)}(z)}{(2n)!} \cdot \delta^{2n} + s(\delta)] \qquad (2)$$

for all $\delta \in [0, c')$. Applying Corollary 4.4 to the almost symmetrical class of intervals $S : [0,1) \to \mathbb{IIR}$ with $S(\delta) := [z - c'\delta, z + c'\delta]$ tells us that the function $T : [0,1) \to \mathbb{IIR}$ with $T(\delta) := F_{\mathbb{IIR}}([z - c'\delta, z + c'\delta])$ is an almost symmetrical class of intervals with center $F_{\mathbb{R}}(z) = f(z)$. From (2) we conclude that there is a function $t : [0, c') \to \mathbb{R}$ with $t(\delta) \in o(\delta^{2n})$ such that for all $\delta \in [0, c')$

$$F_{\mathbb{IIR}}([z-\delta, z+\delta]) \supseteq \\ [f(z) - \frac{f^{(2n)}(z)}{(2n)!} \cdot \delta^{2n} - s(\delta) + t(\delta), f(z) + \frac{f^{(2n)}(z)}{(2n)!} \cdot \delta^{2n} + s(\delta)] \,.$$

Hence, if for $\varepsilon \in (0,1)$ we choose a positive $c_\varepsilon < c'$ small enough, then for all $\delta \in [0, c_\varepsilon)$

$$F_{\mathbb{IIR}}([z-\delta, z+\delta]) \supseteq [f(z) - \frac{f^{(2n)}(z)}{(2n)!} \cdot (1-\varepsilon) \cdot \delta^{2n}, f(z) + \frac{f^{(2n)}(z)}{(2n)!} \cdot (1-\varepsilon) \cdot \delta^{2n}] \,.$$

Using (1) we conclude that for all $\delta \in [0, c_\varepsilon)$

$$d_H(F_{\mathbb{IIR}}([z-\delta, z+\delta]), f([z-\delta, z+\delta])) \geq \frac{f^{(2n)}(z)}{(2n)!} \cdot (1-\varepsilon) \cdot \delta^{2n} \,.$$

That was to be shown. □

5 Final Remarks

If a function f on the reals is monotonic, e.g. nondecreasing, then the computation of the range of f over an interval is trivial: $f([a,b]) = [f(a), f(b)]$ in case f is nondecreasing. If f is sufficiently often differentiable, then the only points at which f is locally not monotonic are the points z such that the minimal positive integer i with $f^{(i)}(z) \neq 0$ is even. It is striking that these are just the points which cause the problems in the interval arithmetic evaluation of an arithmetical expression for f as explained in Theorem 3.1.

Cornelius and Lohner [6], Alefeld and Lohner [5, 3], and others have shown that in special cases there exist so–called "higher order centered forms" which give better than quadratic approximation. Why does this not contradict our main result? The answer is that for these higher order centered forms one does not consider only arithmetical expressions F and the interval arithmetic evaluation $F_{\mathbb{IIR}}$ as defined in Section 2. Instead of allowing only the basic four interval operations induced by the real operations $+, -, \cdot, /$, one uses also interval functions induced by more complicated functions like x^k for some $k \geq 2$ (compare Alefeld and Lohner [5, 3]) or of even more complicated rational functions (compare Cornelius and Lohner [6]). For example the real function $x \mapsto x^2$ induces the interval function

$$\begin{aligned} [a,b] \mapsto \text{square}([a,b]) &:= \{x^2 \mid x \in [a,b]\} \\ &= \begin{cases} [\min\{a^2, b^2\}, \max\{a^2, b^2\}] & \text{if } 0 \notin [a,b] \\ [0, \max\{a^2, b^2\}] & \text{if } 0 \in [a,b] \,. \end{cases} \end{aligned}$$

We might extend the definition of an arithmetical expression by saying that also t^2 is an arithmetical expression if t is an arithmetical expression. If $H(x)$ is an arithmetical expression as defined in Section 2 and z is a point at which $H_{\mathrm{I\!R}}$ is defined, then for an arbitrary $c \in \mathrm{I\!R}$ the extended arithmetical expression

$$F(x) = (c + ((x - z)^2 \cdot H(x)))$$

defines a rational function $F_{\mathrm{I\!R}}$ with $F'_{\mathrm{I\!R}}(z) = 0$ and $F''_{\mathrm{I\!R}}(z) = 2 \cdot H(z)$. In that case, there exists a constant $c > 0$ such that the extended interval arithmetic evaluation yields a superset

$$\widehat{F_{\mathrm{II\!R}}}(I) = (c + (\mathrm{square}(I - z) \cdot H(I)))$$

of $F_{\mathrm{I\!R}}(I)$ with

$$d_H(\widehat{F_{\mathrm{II\!R}}}(I), F_{\mathrm{I\!R}}(I)) \leq c \cdot w(I)^3$$

for all sufficiently small intervals I containing z, compare Alefeld and Lohner [5, 3].

The last remark leads to another problem: to characterize which "basic" interval operations are necessary and sufficient such that the interval arithmetic evaluation of extended arithmetical expressions containing these operations yields the exact range for certain rational functions over intervals, — at least locally. Certainly, our main result can be generalized to certain larger classes of interval operations. More generally, one can analyze the following question not only for the class of interval operations $\{+, -, \cdot, /\}$ but also for larger classes: how well can one approximate the range of a given rational function over an interval by interval arithmetic evaluation of appropriate arithmetical expressions by using basic interval arithmetic operations from a given class? Which classes of basic interval operations are useful?

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