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Computable Approximations of Reals: An Information-Theoretic Analysis

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Abstract

How fast can one approximate a real by a computable sequence of rationals? We show that the answer to this question depends very much on the information content in the finite prefixes of the binary expansion of the real. Computable reals, whose binary expansions have a very low information content, can be approximated (very fast) with a computable convergence rate. Random reals, whose binary expansions contain very much information in their prefixes, can be approximated only very slowly by computable sequences of rationals (this is the case, for example, for Chaitin's Ω numbers) if they can be computably approximated at all.

We show that one can computably approximate any computable real also very slowly, with a convergence rate slower than any computable function. However, there is still a large gap between computable reals and random reals: any computable sequence of rationals which converges (monotonically) to a random real converges slower than any computable sequence of rationals which converges (monotonically) to a computable real.

Keywords: Computable reals, random reals, approximations, program–size complexity, information content.

1 Introduction

In this paper we analyze the possible rates of convergence of computable sequences of rationals by using information–theoretic arguments. We show several results stating that the maximal and the minimal convergence rate of converging, computable sequences of rationals is closely related to the information content of the finite prefixes of the binary expansions of their limits.

In practice, if one wishes to compute a real one computes a sequence of rationals which converges to the real. To do this efficiently is a problem of fundamental importance in many branches of mathematics and computer science, ranging from constructive mathematics (see Bishop and Bridges [1]), computable analysis (see Weihrauch [16] and

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Ko [9]), information based complexity (see Traub, Wasilkowski, and Woźniakowski [15]) to numerical analysis in general. The most important class of reals in this context is certainly the set of computable reals. In order to define them we introduce the notions of a computable sequence of rationals and of a computable convergence rate. We call a sequence $(a_i)_{i\geq 0}$ of rationals a_i *computable* if there is a Turing machine which, given a binary name for a nonnegative integer n, computes a name for the rational a_n, with respect to a standard notation of rationals. A sequence $(\alpha_i)_{i\geq 0}$ of reals α_i is said to *converge computably* if it converges and there is a computable function $g : \mathbf{N} \to \mathbf{N}$ such that $|\alpha_i - \lim_{k\to\infty} \alpha_k| \leq 2^{-j}$ for all i, j with $i \geq g(j)$. A real α is called *computable* if there exists a computable sequence of rationals which converges computably to α. For example, all algebraic numbers, π, the Euler number e, and all numbers commonly used in numerical analysis are computable reals. Given a computable sequence $(a_i)_i$ of rationals which converges computably to a computable real α, and given a computable function $g : \mathbf{N} \to \mathbf{N}$ as in the definition above, by computing $a_{g(n)}$ one obtains a rational approximation of α with precision 2^{-n}. By considering an appropriately chosen computable subsequence of the sequence $(a_i)_i$ one can speed up the convergence to a great extent. The motivating question for our analysis is the following: can the convergence be also very slow? To be more precise, is it possible that for a computable real α there exists a computable sequence $(a_i)_i$ of rationals which converges to α, but which does not converge computably? In Section 3 we shall answer this question affirmatively: for every computable real α there exists a computable sequence of rationals which converges non-computably to α. This answer poses the next question: is it possible to slow down the convergence arbitrarily much? To this question we shall give several negative answers. Here the program–size complexity, that is, the information content of the prefixes of the binary expansion of the limit will play an essential role.

The information content of a finite binary string is measured by its program–size complexity; this is a notion from algorithmic information theory, developed by Chaitin [6, 7], Kolmogorov [10], Solomonoff [12], Martin-Löf [11], and others (see Calude [2]). Roughly speaking, the program–size complexity of a finite string is the minimal length of a program for a universal self–delimiting Turing machine such that it produces the finite string. An infinite binary sequence is called random if the program–size complexity of its prefixes grows at least linearly with their length. Precise definitions will be given in Section 2. A real number is called random, if its fractional part possesses a random binary expansion.

In order to compare the information contents of reals and the convergence rates of nondecreasing sequences of rationals Solovay [13] (see also Chaitin [7]) introduced a relation between nondecreasing, converging sequences called domination. This relation can be extended to arbitrary converging sequences as follows. Let $(a_i)_i$ and $(b_i)_i$ be converging sequences of rationals, and let $\alpha = \lim_{i\to\infty} a_i$, $\beta = \lim_{i\to\infty} b_i$. We say that $(a_i)_i$ *dominates* $(b_i)_i$ if there exists a constant $c > 0$ such that for all i

$$c \cdot |\alpha - a_i| \geq |\beta - b_i| .$$

Informally, if $(a_i)_i$ dominates $(b_i)_i$, then $(b_i)_i$ converges modulo a multiplicative constant at least as fast as $(a_i)_i$. For further investigations about the domination relation between nondecreasing computable sequences of rationals and especially about the role Chaitin's Ω numbers play in this context, the reader is referred to Calude, Hertling, Khoussainov, Wang [3, 4]. The following negative statements will be proven in Section 3: No computable sequence of rationals which converges to a computable (nonrandom) real can dominate a computable sequence of rationals which converges to a noncomputable (random) real.

Since it is arguable whether it is justified to say that $(b_i)_i$ converges slower than $(a_i)_i$ when $(a_i)_i$ does not dominate $(b_i)_i$ (see Section 3 for a discussion), we shall also consider a stricter type of convergence, which we shall call *monotonic convergence*. Informally, a sequence $(a_i)_i$ converges monotonically if it has the following property: if for some index i the number a_i is already quite close to the limit α, then all the following numbers a_j for $j > i$ cannot be too far away from α either. In Section 4 we shall see that for monotonically convergent sequences we can prove in a much sharper sense that every computable sequence which converges monotonically to a random real converges slower than every computable sequence of rationals which converges monotonically to a computable real. Both the second result above (comparing nonrandom reals and random reals) and this result make essential use of a connection between approximability of a real and the information content of the prefixes of its binary expansion.

2 Prerequisites

In this section we introduce some general notation and basic notions from algorithmic information theory. By $\mathbf{N}$ and $\mathbf{R}$ we denote the set of nonnegative integers and the set of reals, respectively. If X and Y are sets, then $f : X \overset{o}{\to} Y$ denotes a possibly partial function defined on a subset of X. Let $\Sigma = \{0, 1\}$ denote the binary alphabet; Σ^* is the set of (finite) binary strings and Σ^ω is the set of infinite binary sequences. The length of a string x is denoted by $|x|$. For a sequence $\mathbf{x} = x_0 x_1 \cdots x_n \cdots \in \Sigma^\omega$ and an integer $n \geq 0$, $\mathbf{x}(n)$ denotes the initial segment of length $n+1$ of $\mathbf{x}$ and x_i denotes the ith digit of $\mathbf{x}$, i.e., $\mathbf{x}(n) = x_0 x_1 \cdots x_n$. Lower case letters c, d, e, k, l, m, n will denote nonnegative integers, lower case letters a, b will denote rationals, and x, y, z strings. By $\mathbf{x}, \mathbf{y}, \cdots$ we denote infinite sequences from Σ^ω; finally, we reserve α, β, γ for reals.

We call a partial recursive function $M : \Sigma^* \overset{o}{\to} \Sigma^*$ a *self–delimiting Turing machine* if its program set $\text{dom}\,(M) = \{x \in \Sigma^* \mid M(x) \text{ is defined}\}$ is prefix–free, i.e. a set of strings with the property that no string in it is a proper prefix of another string in it. The *program-size complexity* of a string $x \in \Sigma^*$ *relative to* M is $H_M(x) = \min\{|y| \mid y \in \Sigma^*,\ M(y) = x\}$, where $\min \emptyset = \infty$. It was shown by Chaitin [7] that there is a self-delimiting Turing machine U that is *universal* in the sense that, for every self-delimiting Turing machine M, there is a constant c_M (depending upon U and M) with the following property: if $x \in \text{dom}\,(M)$, then there is an $\tilde{x} \in \text{dom}\,(U)$ such that $U(\tilde{x}) = M(x)$ and $|\tilde{x}| \leq |x| + c_M$. Clearly, every universal machine produces every string. For two universal machines U and V, we have $H_U(x) = H_V(x) + O(1)$. In the following sections we shall therefore fix one universal machine U and write simply H instead of H_U and call $H(x)$ *the* program–size complexity of x.

Random sequences were originally defined by Martin-Löf [11] using constructive measure theory. In this paper we shall use the following complexity-theoretic characterization (see Chaitin [7]): An infinite sequence $\mathbf{x}$ is *random* if and only if there exists a constant $c > 0$ such that $H(\mathbf{x}(n)) > n - c$, for every integer $n \geq 0$.

We shall call a real number α *random* if its fractional part (that is, the real $\beta \in [0, 1)$ such that $\alpha - \beta$ is an integer) possesses a random binary expansion. A prominent example of a random real is Chaitin's Ω number, i.e. the halting probability of a universal self–delimiting Turing machine U: $\Omega_U = \sum_{x \in \text{dom}\,(U)} 2^{-|x|}$. A Chaitin Ω number can be approximated by a computable sequence of rationals, namely by the sequence of finite sums $(\sum_{i\leq n} 2^{-|x_i|})_n$ where $(x_i)_i$ is a fixed recursive injective enumeration of all strings in the program set dom (U). For more about Chaitin Ω numbers see Chaitin [7, 8], Solovay [13], Calude, Hertling, Khoussainov, Wang [3, 4], Calude and Nies [5].

3 Arbitrary Approximations

In this section we consider computable, converging sequences of rationals. We compare the possible rates of convergence for different classes of approximable numbers.

In the introduction we have defined computable reals as those reals which can be approximated by a computable sequence of rationals which converges computably. It is well-known that there are reals which can be approximated by a computable converging sequence of rationals, but not with a computable convergence rate. For example, if h is an injective, total recursive function which enumerates an r.e. set of nonnegative integers which is not recursive, then the sum $\sum_{k=0}^{\infty} 2^{-h(k)}$ is the limit of the computable sequence of partial sums $(\sum_{k=0}^{n} 2^{-h(k)})_n$, but it is not a computable real (Specker's construction [14]). A very interesting special class of numbers of this form are the Chaitin Ω numbers introduced in Section 2.

How fast can one approximate reals? First we look at computable reals. They can be approximated by a computable sequence of rationals which converges computably. By selecting computably an appropriate subsequence one can achieve almost arbitrarily fast convergence. But does every computable sequence of rationals which converges to a computable real converge computably? We show that this is not the case.

Theorem 3.1. *For every computable real α there is a computable sequence $(a_n)_n$ of rationals which converges to α, but which does not converge computably.*

Proof. Let $(a_n)_n$ be a computable sequence of rationals which converges to α computably, and let g be a total recursive function giving the convergence rate, i.e. $|a_m - \alpha| \leq 2^{-n}$ for all $m \geq g(n)$, for all n. Furthermore let $K \subseteq \mathbf{N}$ be a nonrecursive r.e. set, and let h be an injective, total recursive function enumerating K, i.e. $h(\mathbf{N}) = K$. We define a sequence $(b_n)_n$ of rationals by

$$b_n = a_n + 2^{-h(n)} ,$$

for all $n \in \mathbf{N}$. We prove three claims about the sequence $(b_n)_n$: 1. it is computable, 2. it converges to α, 3. it does not converge computably.

The first claim is clear because $(a_n)_n$ is a computable sequence of rationals and h a total recursive function. For the second claim we have to show that $\lim_{n\to\infty} 2^{-h(n)} = 0$ because $\lim_{n\to\infty} a_n = \alpha$ by assumption. Thus, the second claim is equivalent to

$$(\forall n)\ (\exists m)\ (\forall i \geq m)\ h(i) \geq n\, .$$

This follows from our assumption that h is injective and enumerates K: for each n there is a number m such that $K \cap \{0, 1, \ldots, n-1\} \subseteq \{h(0), h(1), \ldots, h(m-1)\}$. The injectivity of h implies $h(i) \geq n$ for all $i \geq m$. Finally we have to show the third claim. Assume that $(b_n)_n$ converges computably. Then there is a total recursive function f such that for all n

$$(\forall m \geq f(n))\ |\alpha - b_m| \leq 2^{-n}\, .$$

Fix a number n and let $m \geq \max\{g(n+2), f(n+2)\}$. By using the triangle inequality we obtain:

$$\begin{aligned} 2^{-h(m)} &\leq |2^{-h(m)} - (\alpha - a_m)| + |\alpha - a_m| \\ &= |b_m - \alpha| + |a_m - \alpha| \\ &\leq 2^{-(n+2)} + 2^{-(n+2)} = 2^{-(n+1)}. \end{aligned}$$

We conclude $h(m) \geq n+1$. Hence, for any number n:

$$n \in K \iff (\exists m < \max\{f(n+2), g(n+2)\})\ n = h(m).$$

This contradicts the assumption that K is not recursive. Hence, the sequence $(b_n)_n$ does not converge computably. This ends the proof of the theorem. □

The last theorem states that we can approximate every computable real noncomputably, that is, very slowly. Can we slow down the rate of convergence arbitrarily much? In this section we shall give a negative answer which is based on the domination relation introduced in Section 1. We note that the negation of the domination relation ("$(a_i)_i$ does not dominate $(b_i)_i$") can be formulated in the following two equivalent ways:

1. for every $c > 0$ there exists an i such that $|\beta - b_i| > c \cdot |\alpha - a_i|$.
2. for every $c > 0$ there exist infinitely many i such that $|\beta - b_i| > c \cdot |\alpha - a_i|$.

If this is the case then one might say that in some sense the sequence $(b_i)_i$ converges slower than the sequence $(a_i)_i$. But this formulation must be taken with care, since the inequality $|\beta - b_i| > c \cdot |\alpha - a_i|$ might be true only for a sparse set of indices where the terms $|\alpha - a_i|$ are especially small. In fact, the terms $\sup_{j \geq i} |\beta - b_j|$ and $\sup_{j \geq i} |\alpha - a_j|$ (which also seem to express what one would usually understand under the convergence rate) can be identical for all i even when neither $(a_i)_i$ dominates $(b_i)_i$ nor $(b_i)_i$ dominates $(a_i)_i$.

The following result states that no computable sequence $(a_i)_i$ of rationals which converges to a computable real can dominate a computable sequence of rationals converging to a noncomputable real. Hence, although we can have slow computable approximation of computable reals, we cannot slow it down arbitrarily.

Theorem 3.2. *Let $(a_n)_n$ be a computable sequence of rationals converging to a computable real α, and let $(b_n)_n$ be a computable sequence of rationals converging to a noncomputable real β. Then, for every $c > 0$ there are infinitely many i such that*

$$|\beta - b_i| > c \cdot |\alpha - a_i|.$$

Proof. For the sake of a contradiction we assume that there are constants $c, d \in \mathbf{N}$ such that

$$|\beta - b_i| \leq 2^c \cdot |\alpha - a_i|$$

for all $i \geq d$. Let $(\tilde{a}_i)_i$ be a computable sequence of rationals such that for all i

$$|\alpha - \tilde{a}_i| \leq 2^{-i}.$$

We define a computable function $h : \mathbf{N} \to \mathbf{N}$ by

$$h(i) = \min\{k \mid |\tilde{a}_k - a_k| \leq 2^{-i-c-1} \text{ and } k \geq \max\{i+c+1, d\}\}.$$

This function is well–defined because the sequences $(\tilde{a}_k)_k$ and $(a_k)_k$ tend to the same limit. We calculate for all i:

$$\begin{aligned}
|\beta - b_{h(i)}| &\leq 2^c \cdot |\alpha - a_{h(i)}| \\
&\leq 2^c \cdot (|\alpha - \tilde{a}_{h(i)}| + |\tilde{a}_{h(i)} - a_{h(i)}|) \\
&\leq 2^c \cdot (2^{-i-c-1} + 2^{-i-c-1}) \\
&= 2^{-i}.
\end{aligned}$$

Hence, the computable sequence $(b_{h(i)})_i$ converges computably. This contradicts the assumption that its limit β is a noncomputable real. □

We shall see that the last result is also true if we replace the computable real α by a nonrandom real α and the noncomputable real β by a random real β. In fact, the domination relation implies an estimate for the program–size complexity for the binary expansions of the reals. The following result was shown by Solovay [13] for increasing sequences of rationals, see also Calude, Hertling, Khoussainov, Wang [3, Theorem 4.5]. The proof can be carried over to arbitrary converging sequences.

Theorem 3.3. *Let $(a_i)_i$ and $(b_i)_i$ be converging sequences with $0.\mathbf{x} = \lim_{i\to\infty} a_i$ and $0.\mathbf{y} = \lim_{i\to\infty} b_i$. If $(a_i)_i$ dominates $(b_i)_i$, then there is a constant $c > 0$ with $H(\mathbf{y}(n)) \leq H(\mathbf{x}(n)) + c$ for all n.*

For the proof we use the following lemma. Its proof can be found in Calude, Hertling, Khoussainov, Wang [3].

Lemma 3.4. *For every positive integer c there exists a positive integer d_c such that for every $n \geq 1$ and for all strings $x, y \in \Sigma^n$ with $|0.x - 0.y| \leq c \cdot 2^{-n}$ we have*

$$|H(y) - H(x)| \leq d_c.$$

Proof of Theorem 3.3. For every n and large enough i we have $|0.\mathbf{x} - a_i| \leq 2^{-n-1}$ and hence, $|0.\mathbf{x}(n) - a_i| \leq |0.\mathbf{x}(n) - 0.\mathbf{x}| + |0.\mathbf{x} - a_i| \leq 2^{-n}$. Therefore, given $\mathbf{x}(n)$, we can compute an index i_n such that $|0.\mathbf{x}(n) - a_{i_n}| \leq 2^{-n}$. For this index i_n we have

$$|0.\mathbf{x} - a_{i_n}| \leq |0.\mathbf{x} - 0.\mathbf{x}(n)| + |0.\mathbf{x}(n) - a_{i_n}| \leq 2^{-n-1} + 2^{-n} = 3 \cdot 2^{-n-1}.$$

Let $c > 0$ be a constant such that $c \cdot |0.\mathbf{x} - a_i| \geq |0.\mathbf{y} - b_i|$ for all i. Let z_n be the string consisting of the first $n+1$ digits after the radix point of the binary expansion of b_{i_n} (containing infinitely many 1's). Then

$$\begin{aligned}
|0.\mathbf{y}(n) - 0.z_n| &\leq |0.\mathbf{y}(n) - 0.\mathbf{y}| + |0.\mathbf{y} - b_{i_n}| + |b_{i_n} - 0.z_n| \\
&\leq 2^{-n-1} + c \cdot |0.\mathbf{x} - a_{i_n}| + 2^{-n-1} \\
&\leq 2^{-n-1} + c \cdot 3 \cdot 2^{-n-1} + 2^{-n-1} \\
&= (3c+2) \cdot 2^{-n-1}.
\end{aligned}$$

Hence, by Lemma 3.4,

$$H(\mathbf{y}(n)) \leq H(z_n) + O(1) \leq H(\mathbf{x}(n)) + O(1).$$ □

Theorem 3.5. *Let $(a_n)_n$ be a computable sequence of rationals converging to a nonrandom real α, and let $(b_n)_n$ be a computable sequence of rationals converging to a random real β. Then, for every $c > 0$ there are infinitely many i such that*

$$|\beta - b_i| > c \cdot |\alpha - a_i| \,.$$

Proof. For the sake of a contradiction assume that the assertion is not true and that $(a_i)_i$ dominates $(b_i)_i$. Let $\alpha = 0.\mathbf{x}$ and $\beta = 0.\mathbf{y}$ (we can assume without loss of generality that α and β lie in the interval $[0, 1)$). Then, by Theorem 3.3, there is a constant c such that $H(\mathbf{y}(n)) \leq H(\mathbf{x}(n)) + c$ for all n. This implies that also $\mathbf{x}$ is random, i.e. α is random, a contradiction. □

4 Monotonic Approximations

In this section we analyze a restricted type of converging sequences: we consider sequences $(a_i)_i$ with limit α which converge monotonically in the sense that if for some index i the number a_i is already quite close to the limit α, then all the following numbers a_j for $j > i$ cannot be too far away from α either.

Definition 4.1. We say that a sequence $(a_i)_i$ of reals with limit α converges *monotonically* if there exists a constant $c > 0$ such that for all i and all $j \geq i$

$$c \cdot |\alpha - a_i| \geq |\alpha - a_j| \,.$$

For example, any converging and monotonic, i.e. either nondecreasing or nonincreasing sequence of reals converges monotonically: one can take the constant $c = 1$. For example, the Chaitin Ω numbers can be approximated by computable sequences of rationals which converge monotonically.

It turns out that for sequences which converge monotonically the rate of convergence is less variable. The following proposition contrasts with Theorem 3.1.

Proposition 4.2. *Every computable sequence of rationals which converges monotonically to a computable real converges computably.*

Proof. Let $(a_i)_i$ be a computable sequence of rationals which converges monotonically to a computable real α. Let $c \geq 0$ be a constant such that for all i and all $j \geq i$

$$2^c \cdot |\alpha - a_i| \geq |\alpha - a_j| \,.$$

Furthermore, let $(b_i)_i$ be a computable sequence of rationals with $|\alpha - b_i| \leq 2^{-i}$ for all i. For any i there exists a number k with $|\alpha - a_k| \leq 2^{-i-2-c}$. For this k we have

$$|a_k - b_{i+2+c}| \leq |a_k - \alpha| + |\alpha - b_{i+2+c}| \leq 2^{-i-2-c} + 2^{-i-2-c} = 2^{-i-1-c} \,.$$

Hence, we can define a computable function $h : \mathbf{N} \to \mathbf{N}$ by

$$h(i) = \min\{k \mid |a_k - b_{i+2+c}| \leq 2^{-i-1-c}\} \,.$$

In view of the monotonicity of $(a_i)_i$ we see for any i and any $j \geq h(i)$

$$|\alpha - a_j| \leq 2^c \cdot |\alpha - a_{h(i)}| \leq 2^c \cdot (|\alpha - b_{i+2+c}| + |b_{i+2+c} - a_{h(i)}|) \leq 2^c \cdot (2^{-i-2-c} + 2^{-i-1-c}) < 2^{-i} \,.$$

Hence, the sequence $(a_i)_i$ converges computably. □

In Section 3 we have considered arbitrary converging and computable sequences $(a_i)_i$ and $(b_i)_i$ and have explicitly formulated two gaps with respect to the convergence rates, one from computable to noncomputable numbers, and one from nonrandom to random numbers. Both results were based on the inequality $|\beta - b_i| > c \cdot |\alpha - a_i|$ holding for infinitely many i. While we had some doubts whether in this case one can really claim that $(b_i)_i$ converges slower than $(a_i)_i$, we shall see now that these doubts can be cast aside if we consider only monotonically converging sequences: then we can replace the quantifier "for infinitely many i" by the quantifier "for almost all i". Certainly in this case it is justified to say that $(b_i)_i$ converges slower than $(a_i)_i$.

Scholium 4.3. *Let $(a_i)_i$ be a computable sequence of rationals which converges monotonically to a computable number α, and let $(b_i)_i$ be a computable sequence of rationals which converges monotonically to a random real β. Then for every $c > 0$ there exists a $d > 0$ such that for all $i \geq d$*

$$|\beta - b_i| > c \cdot |\alpha - a_i| . \tag{1}$$

Lemma 3.4 and the program–size complexity are essential for the proof: Lemma 3.4 is used in the proof of Lemma 4.4, which is crucial for the proof of Scholium 4.3.

Lemma 4.4. *Let $(b_i)_i$ be a computable sequence of rationals which converges to a random real β. Then for every $d > 0$ and almost all i*

$$|\beta - b_i| > 2^{d-i} .$$

Proof. Let $d > 0$ be fixed. It is clear that we can without loss of generality assume that β and all rationals b_i lie in the interval $(0, 1)$. Let $0.\mathbf{y}$ be the binary expansion of β. For every i, let $z_i \in \Sigma^{i+1}$ be the string consisting of the first $i + 1$ digits after the radix point of the binary expansion of c_i (containing infinitely many 1's). Then

$$0 \leq c_i - 0.z_i \leq 2^{-i-1} .$$

Since the sequence $(z_i)_i$ is a computable sequence of strings there exists a constant e_1 such that for all i

$$H(z_i) \leq 2 \log i + e_1 . \tag{2}$$

For the sake of a contradiction let us assume that there are infinitely many i with $|\beta - b_i| \leq 2^{d-i}$. Then for all these i

$$\begin{aligned} |0.\mathbf{y}(i) - 0.z_i| &\leq |0.\mathbf{y}(i) - 0.\mathbf{y}| + |0.\mathbf{y} - b_i| + |b_i - 0.z_i| \\ &\leq 2^{-i-1} + 2^{d+1} \cdot 2^{-i-1} + 2^{-i-1} \\ &= (2 + 2^{d+1}) \cdot 2^{-i-1} . \end{aligned}$$

With Lemma 3.4 we conclude that there is a constant e_2 such that $H(\mathbf{y}(i)) \leq H(z_i) + e_2$ for all these i. Using (2) we obtain

$$H(\mathbf{y}(i)) \leq 2 \log i + e_1 + e_2$$

for infinitely many i. This contradicts the randomness of $\mathbf{y}$, i.e. the randomness of the real β. □

Proof of Scholium 4.3. Let $(a_i)_i$ and $(b_i)_i$ be as in the scholium and fix a number $c > 0$. We wish to show that (1) is true for almost all i.

First, we show that it is sufficient to prove this for $c = 1$. Indeed, since we can enlarge c, we can assume that c is a rational. Then we can prove the assertion for the sequence $(ca_i)_i$ instead of $(a_i)_i$ with the constant c in (1) replaced by 1. The sequence $(ca_i)_i$ is also a computable sequence of rationals and it converges monotonically to the computable real $c\alpha$.

Secondly, we show that we can restrict ourselves to the case that the sequence $(a_i)_i$ is of the form $a_i = 2^{-s(i)}$ where $s : \mathbf{N} \to \mathbf{N}$ is a computable, nondecreasing, unbounded function with $s(0) = 0$. Indeed, since we wish to show $|\beta - b_i| > |\alpha - a_i|$ only for almost all i, we can forget finitely many terms of both sequences $(a_i)_i$ and $(b_i)_i$ and assume that $|\alpha - a_i| \leq 1$ for all i. According to Proposition 4.2 the sequence $(a_i)_i$ converges computably to α. Hence, there is a computable function $g : \mathbf{N} \to \mathbf{N}$ with

$$|\alpha - a_i| \leq 2^{-j}$$

for all i, j with $i \geq g(j)$. We can additionally assume that g is increasing and, because of $|\alpha - a_i| \leq 1$ for all i, also that $g(0) = 0$. We define a computable, nondecreasing, unbounded function $s : \mathbf{N} \to \mathbf{N}$ by $s(0) = 0$ and

$$s(i) = \max\{j \mid g(j) \leq i\}$$

for $i > 0$. Then we observe $i \geq g(s(i))$ and hence $|\alpha - a_i| \leq 2^{-s(i)}$, for all i. Therefore, it is sufficient to prove that

$$|\beta - b_i| > 2^{-s(i)} \tag{3}$$

holds true for almost all i.

Thus, from now on we assume that $s : \mathbf{N} \to \mathbf{N}$ is a computable, nondecreasing, unbounded function with $s(0) = 0$ and we wish to show that (3) is true for almost all i. We define a computable nondecreasing function $f : \mathbf{N} \to \mathbf{N}$ by $f(i) = \max\{j \mid s(j) \leq i\}$, for all i. Then we have for all $k \geq 0$

$$f(s(k)) = \max\{j \mid s(j) \leq s(k)\} \geq k\,.$$

Finally we define a computable sequence $(\tilde{b}_i)_i$ by $\tilde{b}_i = b_{f(i)}$. Since the sequence $(b_i)_i$ converges monotonically there exists a constant $d \geq 0$ such that for all i, j with $j \geq i$

$$|\beta - b_j| \leq 2^d \cdot |\beta - b_i|\,.$$

By Lemma 4.4 there exists a constant e_1 such that $|\beta - \tilde{b}_j| > 2^{d-j}$ for all $j \geq e_1$. We set $e_2 = f(e_1) + 1$. Then $s(i) > e_1$ for all $i \geq e_2$. Because of $i \leq f(s(i))$ for all $i \geq 0$ we obtain for all $i \geq e_2$

$$|\beta - b_i| \geq 2^{-d} \cdot |\beta - b_{f(s(i))}| = 2^{-d} \cdot |\beta - \tilde{b}_{s(i)}| > 2^{-d} \cdot 2^{d-s(i)} = 2^{-s(i)}\,.$$

This ends the proof of Scholium 4.3. □

We conclude with some remarks on further interesting questions. The topic of this paper, the relation between the possible rates of convergence of computable sequences of rationals and the information contents of the prefixes of the binary expansions of their limits, should be analyzed further. The domination relation between sequences of rationals and the induced structure on the reals, induced by considering the limits, should be investigated more. For nondecreasing sequences of reals first results along this line can be found in Calude, Hertling, Khoussainov, Wang [3, 4]. Also, the properties of the class of all reals which can be approximated by a computable sequence of rationals converging monotonically, should be analyzed.

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