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The Constructive Implicit Function Theorem and Applications in Mechanics

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Abstract

We examine some ways of proving the Implicit Function Theorem and the Inverse Function Theorem within Bishop's constructive mathematics. Section 2 contains a new, entirely constructive proof of the Implicit Function Theorem. The paper ends with some comments on the application of the Implicit Function Theorem in classical mechanics.

1 Introduction

In this paper, which is written entirely within the framework of constructive mathematics **(BISH)** erected by the late Errett Bishop [2], we examine a standard proof of the Implicit Function Theorem and give a completely new proof. As far as understanding constructive mathematics goes, the reader need only be aware that when working constructively, we interpret "existence" strictly as "computability". To do so, we need to be careful about our logic. For example, when we prove a disjunction $P \vee Q$, we need to either produce a proof of P or produce a proof of Q; it is not enough, constructively, to show that $\neg(\neg P \wedge \neg Q)$. To understand this better, consider the case

$$\begin{aligned} P &\equiv \forall n\,(a_n = 0)\,, \\ Q &\equiv \exists n\,(a_n = 1)\,, \end{aligned}$$

where (a_n) is a binary sequence. A constructive proof of $P \vee Q$ will produce one of the following:

- a demonstration that $a_n = 0$ for all n;
- the computation of a positive integer n such that $a_n = 1$.

A constructive proof of $\neg(\neg P \wedge \neg Q)$ will simply show that it is impossible for both P and Q to be false, but, in the case where Q actually holds, it will not enable us to compute n with $a_n = 1$.

One consequence of this illustration is that constructive mathematics cannot use the **Law of Excluded Middle**, $P \vee \neg P$. Note that the exclusion of this law from constructive mathematics is a consequence of the interpretation of "existence"; it does *not* arise from some arbitrary, irrational distrust of the law itself.

The logic we use in constructive mathematics is not the familiar, **classical logic,** but one, called **intuitionistic logic**, that was first discussed in [12] and that has subsequently proved of great significance for logicians. In fact, it appears that constructive mathematics, in practice, is simply mathematics using intuitionistic logic (see [13], [14], and [5]). For more on intuitionistic logic, see [1], [7], [6], and [16].

In view of the nature of this special issue of *Chaos, Solitons & Fractals*, the reader might reasonably ask what is the connection with constructive mathematics and complexity. We believe that constructive mathematics is concerned with, *inter alia*, computability in principle (is the object computable even under the most ideal conditions?), whereas complexity deals with computability in practice (can the object be computed using a feasible amount of time, memory, or other resource?). It makes sense to investigate the former notion of computability before investing effort in the latter. In other words, a constructive development of a theory can be regarded as a sensible first step towards a development that distinguishes those situations that are amenable to feasible computation from those that are not.

2 The Implicit Function Theorem

In this section we illustrate the problems associated with constructivising one of the standard classical proofs of the Implicit Function Theorem. We use standard modern notations for derivatives, such as D for the derivative itself, and D_k for the kth partial derivative ($k = 1, 2$), of a mapping from a subset of $\mathbf{R}^m \times \mathbf{R}^n$ to $\mathbf{R}^p$.

For the most part we confine our attention to the following special case of the **Implicit Function Theorem.**

Theorem 1 *Let Φ be a differentiable mapping of a neighbourhood of $(x_0, y_0) \in \mathbf{R}^m \times \mathbf{R}^n$ into $\mathbf{R}^p$, let $\Phi(x_0, y_0) = 0$, and let*[1] $\det(D_2\Phi(x_0, y_0)) \neq 0$. *Then there exist $r > 0$ and a differentiable function $f : \overline{B}(x_0, r) \subset \mathbf{R}^m \to \mathbf{R}^n$ such that for each $x \in \overline{B}(x_0, r) \subset \mathbf{R}^m$, $(x, f(x))$ is the unique solution y of the equation $\Phi(x, y) = 0$ in some neighbourhood of (x_0, y_0).*

[1] a metric space the inequality $x \neq y$ means that the distance between x and y is positive. Note that the statement

$$\forall x \in \mathbf{R}\, (\text{not}\, (x = y) \Rightarrow x \neq y)$$

implies a principle—Markov's Principle—that represents an unbounded search and is not derivable with intuitionistic logic.

The general case of the theorem, as stated by Dieudonné ([9], (10.2.1)), requires a theory of differentiable functions on Banach spaces, where, in the infinite–dimensional case, there are problems with finding suitable constructive definitions of "differentiable on an open set". In fact, we shall simplify the exposition by taking a further special case, in which $n = p = 1$.

Although Dieudonné's proof of the Implicit Function Theorem is based on a constructively valid contraction mapping theorem, there are a several details that have to be tidied up before the proof can be constructivised. For example, in proving the uniqueness of the implicit function f, he uses the classical notion of connectedness in a standard way. His argument goes as follows: suppose there are two implicit functions f, g on $B \equiv \overline{B}(x_0, r)$, and show that

$$\{x \in B : f(x) = g(x)\},$$

which is certainly closed in B, is open in B; it follows (classically) that this set equals B. Unfortunately, the classical notion of connectedness used here by Dieudonné is nonconstructive, since it fails in the recursive model of constructive mathematics (there exists a recursive example of a proper subset of $[0,1]$ that is both open and closed in $[0,1]$ ([7], Ch. 3)). What is true constructively is that if S is a nonempty subset of B that is open, closed, and **located**—in the sense that

$$\rho(x, S) = \inf\{|x - s| : s \in S\}$$

exists for each $x \in B$—then $S = B$ [4].

To obtain a constructive analogue of Dieudonné's proof of uniqueness, we use Theorem (4.9), Chapter 4 of [3], to choose arbitrarily small $\varepsilon > 0$ such that

$$S = \{x \in B : |f(x) - g(x)| \leq \varepsilon\}$$

is compact, and therefore both closed and located; we then show that S is also open in B, so that $S = B$. Since $\varepsilon > 0$ is arbitrary, we conclude that $f(x) = g(x)$ for all $x \in B$.

This is a small illustration of the patching up that needs to be done to turn a classical proof of the Implicit Function Theorem into a constructive one. But there are other problems that we must address in this process, the first of which is to ensure that we get the right definitions. For example, in constructive mathematics for a real–valued function f to be **differentiable** on a compact (that is, totally bounded, complete) set $K \subset \mathbf{R}^N$ we require that it be uniformly differentiable on K; thus , given $\varepsilon > 0$, we must be able to find $\omega > 0$ such that

$$|f(x) - f(x') - Df(x')\,(x - x')| \leq \varepsilon\,\|x - x'\|$$

whenever $x = (x_1, x_2)$ and $x' = (x'_1, x'_2)$ belong to K and

$$\|x - x'\| = \max\{|x_1 - x'_1|, |x_2 - x'_2|\} \leq \omega.$$

For convenience, we usually write $\omega(\varepsilon)$ for ω, thereby making a notationally convenient (but mathematically unjustifiable[2]) use of the Axiom of Choice; we

[2] Axiom of Choice entails the Law of Excluded Middle [11].

then refer to the "function" ω as a **modulus of differentiability** for f on K. In turn, for the differentiability of a function f on an open set $\Omega \subset \mathbf{R}^N$ we require that f be (uniformly) differentiable on each compact set K that is well contained in Ω (that is, has Ω as a neighbourhood).

We make these requirements of differentiability because

- they are satisfied in the intuitionistic model of BISH [7],
- the usual notion of differentiability at a point is not computationally strong enough for practical purposes, and
- in practice, without additional hypotheses such as recursiveness, constructively defined differentiable functions always appear to be differentiable in our strong sense.

We make a similar requirement of a **continuous function** on Ω : it must be uniformly continuous on each compact set well contained in Ω. (Note that the classical uniform continuity theorem is not provable in BISH, since there is a recursive counterexample to it; see Chapter 6 of [7].) Of course, a function that is differentiable in our sense is continuous in our sense.

3 A New Proof of The Implicit Function Theorem

Rather than complete the patching–up of Dieudonné's proof of the Implicit Function Theorem, we present a new proof, one that depends only on elementary arguments about continuous functions. For this we need some lemmas, the first two of which we state without proof since their constructive proofs are, at most, minor adaptations of arguments used in Chapter 7 of [9].

Lemma 1 *Let f be differentiable on the compact interval $[a,b]$, and let $m \leq |f'(x)| \leq M$ for all $x \in [a,b]$. Then*

$$m(b-a) \leq |f(b) - f(a)| \leq M(b-a). \qquad \square$$

The second lemma is a version of **Rolle's Theorem.**

Lemma 2 *Let f be differentiable on the compact interval $[a,b]$. Then*

$$\inf_{a \leq x \leq b} |f'(x)| \leq (b-a)^{-1} |f(b) - f(a)|. \qquad \square$$

The key to our proof of the existence of an implicit function is provided by the following simple lemma.

Lemma 3 *Under the hypotheses of* Theorem 1*, there exist $r, s > 0$ such that*

$$|\Phi(x_0, y_0 \pm s)| \geq \tfrac{2}{3} |D_2\Phi(x_0, y_0)| \, s, \tag{1}$$

$$\inf_{\|h\| \leq r, \, |k| \leq s} |D_2\Phi(x_0 + h, \, y_0 + k)| > 0, \tag{2}$$

and

$$\inf_{\|h\| \leq r} |\Phi(x_0 + h, \, y_0 \pm s)| \geq \tfrac{1}{2} |D_2\Phi(x_0, y_0)| \, s > \sup_{\|h\| \leq r} |\Phi(x_0 + h, \, y_0)| \, . \tag{3}$$

Proof. Choose an open ball B, with centre (x_0, y_0) and radius R, such that

$$|D_2\Phi(x, y)| > \tfrac{1}{2} |D_2\Phi(x_0, y_0)|$$

for all $(x, y) \in B$. Since Φ is differentiable at (x_0, y_0), there exists $s \in (0, R)$ such that if $|y - y_0| \leq s$, then

$$|\Phi(x_0, y) - D_2\Phi(x_0, y_0)(y - y_0)| \leq \tfrac{1}{3} |D_2\Phi(x_0, y_0)(y - y_0)|$$

and therefore

$$|\Phi(x_0, y)| \geq \tfrac{2}{3} |D_2\Phi(x_0, y_0)(y - y_0)| \, .$$

In particular, we obtain inequality (1). Since $\Phi(x_0, y_0) = 0$ and Φ is continuous, we can now choose $r \in (0, R)$ such that inequalities in (3) hold. Since $r < R$, our choice of R ensures that (2) also holds. □

It is convenient to separate out the proof of the differentiability of the implicit function whose existence will be established later.

Lemma 4 *Let B be a compact ball in $\mathbf{R}^m$, J a compact interval in $\mathbf{R}$, and $\Phi : B \times J \to \mathbf{R}$ be a (uniformly) differentiable function such that*

$$0 < m = \inf_{B \times J} |D_2\Phi| \, .$$

Suppose that there exists a function $f : B \to J$ such that $\Phi(x, f(x)) = 0$ for all $x \in B$. Then f is uniformly differentiable on B, and

$$f'(\boldsymbol{\xi}) = -\tfrac{D_1\Phi(\boldsymbol{\xi}, f(\boldsymbol{\xi}))}{D_2\Phi(\boldsymbol{\xi}, f(\boldsymbol{\xi}))}$$

for each $\boldsymbol{\xi} \in B$.

Proof. Let ω be a modulus of differentiability for Φ on $B \times J$. Let $0 < \varepsilon < 1/2m$, and let x_1, x_2 be points of B such that

$$\|x_1 - x_2\| \leq \min \{\omega(\varepsilon), \, \omega(\omega(\varepsilon))\} \, .$$

Then

$$\begin{aligned}&\left|\Phi(x_1,f(x_1))-\Phi(x_2,f(x_2))-D_1\Phi\left(x_2,f(x_2)\right)(x_1-x_2)\right.\\&\quad\left.-D_2\Phi\left(x_2,f(x_2)\right)\left(f(x_1)-f(x_2)\right)\right|\\\leq\;&\varepsilon\left(\|x_1-x_2\|+|f(x_1)-f(x_2)|\right).\end{aligned}$$

But $\Phi(x_1,f(x_1))=0=\Phi(x_2,f(x_2))$ and $|D_2\Phi\left(x_2,f(x_2)\right)|\geq m$, so

$$\begin{aligned}&\left|\tfrac{D_1\Phi(x_2,f(x_2))}{D_2\Phi(x_2,f(x_2))}\left(x_1-x_2\right)+f(x_1)-f(x_2)\right|\\\leq\;&m^{-1}\varepsilon\left(\|x_1-x_2\|+|f(x_1)-f(x_2)|\right)\\\leq\;&\tfrac{1}{2}\left(\|x_1-x_2\|+|f(x_1)-f(x_2)|\right).\end{aligned}\tag{4}$$

Hence

$$|f(x_1)-f(x_2)|\leq 2\left|\tfrac{D_1\Phi(x_2,f(x_2))}{D_2\Phi(x_2,f(x_2))}\left(x_1-x_2\right)\right|+\|x_1-x_2\|.$$

Choosing a bound M for $|D_1\Phi|$ on the compact set $B\times J$, we see that

$$|f(x_1)-f(x_2)|\leq\left(2Mm^{-1}+1\right)\|x_1-x_2\|.$$

It follows from (4) that

$$\left|f(x_1)-f(x_2)+\tfrac{D_1\Phi(x_2,f(x_2))}{D_2\Phi(x_2,f(x_2))}\left(x_1-x_2\right)\right|\leq 2m^{-1}\left(Mm^{-1}+1\right)\|x_1-x_2\|\,\varepsilon.$$

Hence f is uniformly differentiable on $B\times J$, with

$$f'(\xi)=-\tfrac{D_1\Phi(\xi,f(\xi))}{D_2\Phi(\xi,f(\xi))}$$

for each $\xi\in I$. $\square$

Proof of the Implicit Function Theorem. Assume that the hypotheses of Theorem 1 are satisfied, and choose r,s as in Lemma 3. Let

$$K=\left\{(x,y):|x-x_0|\leq r,\ |y-y_0|\leq s\right\},$$

which is a compact set. Fix $\boldsymbol{\xi}$ with $|\boldsymbol{\xi}-x_0|\leq r$, and let

$$0<\varepsilon<m\equiv\inf_{\|h\|\leq r,\,|k|\leq s}\left|D_2\Phi\left(x_0+h,\,y_0+k\right)\right|.$$

Consider y,y^* such that $|y-y_0|\leq s$, $|y^*-y_0|\leq s$, and $|\Phi(\boldsymbol{\xi},y)-\Phi(\boldsymbol{\xi},y^*)|<\varepsilon^2$. If $|y-y^*|>\varepsilon$, then, by Lemma 2, there exists η between y and y^* such that

$$\left|D_2\Phi\left(\boldsymbol{\xi},\eta\right)\right|<|y-y^*|^{-1}\varepsilon^2<\varepsilon<m\leq\left|D_2\Phi\left(\boldsymbol{\xi},\eta\right)\right|,$$

a contradiction. Hence $|y-y^*|<2\varepsilon$. In particular, if $|\Phi(\boldsymbol{\xi},y)|<\varepsilon^2/2$ and $|\Phi(\boldsymbol{\xi},y^*)|<\varepsilon^2/2$, then $|y-y^*|<2\varepsilon$.

Next suppose that

$$0 < \gamma = \inf_{|y-y_0| \leq s} |\Phi(\boldsymbol{\xi}, y)|.$$

If $|y - y_0| \leq s$, then

$$\left|D_2\Phi^2(\boldsymbol{\xi}, y)\right| = 2\,|\Phi(\boldsymbol{\xi}, y)|\,|D_2\Phi(\boldsymbol{\xi}, y)| \geq 2\gamma m > 0.$$

Since the function $D_2\Phi^2(\boldsymbol{\xi}, \cdot)$ is continuous on the interval $J = [y_0 - s, y_0 + s]$, the Intermediate Value Theorem (see [3], Ch. 2, (4.8)) allows us to assume, without loss of generality, that $D_2\Phi^2(\boldsymbol{\xi}, y) \geq 2\gamma m$ for all $y \in J$. So $\Phi^2(\boldsymbol{\xi}, \cdot)$ is strictly increasing on J, and therefore $\Phi^2(\boldsymbol{\xi}, y_0) > \Phi^2(\boldsymbol{\xi}, y_0 - s)$; this is impossible, in view of our choice of s. Hence $\gamma = 0$ and for each n we can choose y_n such that $|y_n - y_0| \leq s$ and $|\Phi(\boldsymbol{\xi}, y_n)| < 1/2n^2$. The work in the first paragraph of this proof now shows that if $j, k > m^{-1}$, then $|y_j - y_k| < 2/n$; so (y_n) is a Cauchy sequence, and therefore converges to a limit y_∞, in the interval J. The same argument also shows that y_∞ is the unique solution y in J of the equation $\Phi(\boldsymbol{\xi}, y) = 0$, so we may define a function $f : [x_0 - r, x_0 + r] \to J$ by $f(\boldsymbol{\xi}) = y_\infty$. Reference to Lemma 4 completes the proof. $\square$

Note the following simple proof that the function f in our theorem is uniformly continuous on I. Let δ be a modulus of continuity for Φ on K. Given $\varepsilon > 0$, and points $\boldsymbol{\xi}, \boldsymbol{\xi}' \in I$ such that $\left|\boldsymbol{\xi} - \boldsymbol{\xi}'\right| \leq \omega(\varepsilon^2)$, we have

$$\left|\Phi(\boldsymbol{\xi}', f(\boldsymbol{\xi}))\right| = \left|\Phi(\boldsymbol{\xi}', f(\boldsymbol{\xi})) - \Phi(\boldsymbol{\xi}, f(\boldsymbol{\xi}))\right| \leq \varepsilon^2;$$

since also $\Phi(\boldsymbol{\xi}', f(\boldsymbol{\xi}')) = 0$, it follows from the first part of the proof of Theorem 1 that $\left|f(\boldsymbol{\xi}) - f(\boldsymbol{\xi}')\right| < 2\varepsilon$.

With classical logic we can simplify the proof of the existence of f by taking y_∞ to be a point where the continuous function $\Phi(\boldsymbol{\xi}, \cdot)$ attains its infimum on the compact interval J. This move is nonconstructive, since there exists a recursive example of a positive-valued uniformly continuous mapping on $[0, 1]$ whose infimum is 0; see Chapter 6 of [7].

Next we consider the relation between the Implicit Function Theorem and the following **Inverse Function Theorem**.

Theorem 2 *Let $U \subset \mathbf{R}^n$ be open and nonempty, a an element of U, and $f : U \longrightarrow \mathbf{R}^n$ a differentiable mapping on U such that* $\det Df(a) \neq 0$. *Then there exist open sets $V, W \subset \mathbf{R}^n$ such that $a \in V \subset U$, $b = f(a) \in W$, f is a differentiable homeomorphism of V onto W, and $f^{-1} : W \to V$ is differentiable.*

Minor adaptations of the classical argument found in [8] enable us to prove that the Inverse Function Theorem implies the Implicit Function Theorem. We show that the Inverse Function Theorem can be derived constructively from the Implicit Function Theorem, so that the two theorems are, in fact, equivalent constructively. To this end, assume the Implicit Function Theorem and the hypotheses of the Inverse Function Theorem. Let

$$U_1 = U \times \mathbf{R}^n \subset \mathbf{R}^{2n},$$

define a mapping $g : U_1 \to \mathbf{R}^n$ by

$$g(x, y) = f(x) - y,$$

and set $b = f(a)$. It is clear that

$$g(a, b) = 0, \quad (a, b) \in U_1, \text{ and } D_1 g(a, b) = Df(a) \neq 0.$$

Thus the Implicit Function Theorem, with the order of its variables reversed, applies to g. Hence there exist

- two open sets V_1, W such that

  $$(a, b) \in V_1 \subset U_1 \subset \mathbf{R}^{2n},$$

  and $b \in W \subset \mathbf{R}^n$;
- a differentiable function $h : W \to \mathbf{R}^n$ such that the following two conditions are equivalent:
  - $g(x, y) = 0$ for each $(x, y) \in V_1$;
  - $x = h(y)$ for each $y \in W$.

Note that $g(x, y) = 0$ if and only if $y = f(x)$. Now let

$$V = \{x \in h(W) : \exists y \in \mathbf{R}^n \, ((x, y) \in V_1)\}.$$

Since

$$(h \circ f)(x) = h(f(x)) = h(y) = x$$

for all $x \in V$, we see that f maps V onto W. Since f and h are differentiable, it follows that f is a differentiable homeomorphism of V onto W, with differentiable inverse h.

4 Some Applications to Physics

The Implicit Function Theorem is frequently used in mechanics—for example, in the construction of canonical transformations in analytical mechanics (see [10] and [15]).

For another example, take the motion of a point mass m, subject to a force law f generated by an infinitely differentiable potential V satisfying the condition

$$DV(\xi) \neq 0 \text{ for each } \xi \neq 0.$$

The existence of an infinitely differentiable inverse function is guaranteed constructively by our Inverse Function Theorem. This situation arises in the solution of certain Cauchy problems, such as the damped pendulum

$$m\ddot{x} + \lambda\dot{x} + k sin x = \gamma f(t) \quad (t \in \mathbf{R}^+), \tag{5}$$

where $\gamma \in \mathbf{R}$; λ, m, k are positive; $\lambda^2 \neq 4km$; and f is periodic and infinitely differentiable. If the oscillations are small, then the damped linear oscillator admits isochronous periodic motions with forcing term ([10], pp. 71-74). More precisely, *if f is infinitely differentiable with period $T > 0$, then, provided γ is small enough, there exists a periodic motion, with period T, satisfying* (5).

Together with (5) let us consider the linearised equation

$$m\ddot{x} + \lambda\dot{x} + kx = \gamma f,$$

which admits a periodic solution $\tilde{x}$ isochronous with f. We look for a periodic solution of (5) of the form

$$x(t) = \gamma\tilde{x}(t) + y(t) \quad \left(t \in \mathbf{R}^+\right)$$

with initial data

$$y(0) = \varepsilon, \quad \dot{y}(0) = \eta.$$

We set

$$x(T) \ = \ \gamma\tilde{x}(T) + a(\varepsilon, \eta, \gamma)$$

and

$$\dot{x}(T) \ = \ \gamma\dot{\tilde{x}}(T) + b(\varepsilon, \eta, \gamma).$$

Since $\tilde{x}(0) = \tilde{x}(T)$ and $\dot{\tilde{x}}(0) = \dot{\tilde{x}}(T)$ by the periodicity of $\tilde{x}$, the condition that (5) admits a periodic solution with period T can be restated as $a(\varepsilon, \eta, \gamma) = \varepsilon$ and $b(\varepsilon, \eta, \gamma) = \eta$. But the solvability of the equations

$$\begin{cases} a(\varepsilon, \eta, \gamma) & = & \varepsilon, \\ b(\varepsilon, \eta, \gamma) & = & \eta, \end{cases}$$

is equivalent to the existence of a periodic solution of (5) with period T. We now have the problem of expressing ε and η as functions of a "sufficiently small" γ. This is possible if the Jacobian determinant of partial derivatives with respect to ε and η is nonzero. The computation of these partial derivatives is based on the equations

$$\begin{aligned} a(\varepsilon, \eta, \gamma) & = y(T), \\ b(\varepsilon, \eta, \gamma) & = \dot{y}(T), \end{aligned}$$

where $y(t)$ is a solution of the Cauchy problem

$$m\ddot{y}(t) + \lambda\dot{y}(t) + ky(t) = k\left(\gamma\tilde{x}(t) + y(t) - \sin(\gamma\tilde{x}(t) + y(t))\right)$$

with initial conditions $y(0) = \varepsilon$, $\dot{y}(0) = \eta$. Finally, it is easy to see that the corresponding Jacobian determinant equals

$$(e^{a_+ T} - 1)(e^{a_- T} - 1) \neq 0,$$

where a_+ and a_- are, respectively, the positive and negative parts of a. So the Cauchy problem has a constructive solution given by our Implicit Function Theorem.

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