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# Feedback for Relations

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**Abstract**

In our previous papers [3,2] we have proved that there are nine types of finite relations which are closed under a natural definition of feedback. In this note we prove that this natural definition is the unique feedback which satisfies the axioms of a biflow over there usual composition and sum.

## 1 Preliminaries

### 1.1 Finite Relations

For each nonnegative integer $n$ we denote by $[n]$ the set $\{1, 2, \ldots, n\}$; in particular $[0] = \emptyset$.

Let $S$ be a set of sorts. For $a \in S^*$ we denote by $|a|$ the length of $a$ and for each $i \in [|a|]$ we denote by $a_i \in S$ the $i^{th}$ letter of a. As we use the additive notation for the concatenation of $S^*$ we write $a = a_1 + a_2 + \ldots + a_{|a|}$.

For $a, b \in S^*$ let

$$\mathsf{Rel}_S(a, b) = \{(a, f, b) \mid f \subseteq [|a|] \times [|b|], (i, j) \in f \text{ implies } a_i = b_j\}.$$

Throughout this paper we shall write shortly $f$ instead of $(a, f, b) \in \mathsf{Rel}_S(a, b)$ and we shall say that $f$ is an *$S$-sorted relation* from $a$ to $b$.

The basic operations of $\mathsf{Rel}_S$ are *composition* and *sum.* The composite $fg \in \mathsf{Rel}_S(a, c)$ is defined for each $f \in \mathsf{Rel}_S(a, b)$ and $g \in \mathsf{Rel}_S(b, c)$ by

$$fg = \{(i, j) \mid (\exists k)(i, k) \in f \text{ and } (k, j) \in g\}.$$

The sum $f + g \in \mathsf{Rel}_S(a + c, b + d)$ is defined for each $f \in \mathsf{Rel}_S(a, b)$ and $g \in \mathsf{Rel}_S(c, d)$ by

$$f + g = f \cup \{(|a| + i, |b| + j) \mid (i, j) \in g\}.$$

The following particular relations will be used in the sequel.

*Identity*

$$\mathsf{I}_a = \{(i, i) \mid i \in [|a|]\} \in \mathsf{Rel}_S(a, a).$$

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*Block transposition*

$$^{a}\mathsf{X}^{b} = \{(i, |b|+i) \mid i \in [|a|]\} \cup \{(|a|+j, j) \mid j \in [|b|]\} \in \mathsf{Rel}_S(a+b, b+a).$$

*Block identification*

$$\vee_a = \{(i,i) \mid i \in [|a|]\} \cup \{(|a|+i, i) \mid i \in [|a|]\} \in \mathsf{Rel}_S(a+a, a).$$

*Adding dummy elements to the target*

$$\top_a = \emptyset \in \mathsf{Rel}_S(\lambda, a), \text{ where } \lambda \text{ is the empty word.}$$

*Adding dummy elements to the source*

$$\perp^a = \emptyset \in \mathsf{Rel}_S(a, \lambda).$$

*Block ramification(fork)*

$$\wedge^a = \{(i,i) \mid i \in [|a|]\} \cup \{(i, |a|+i) \mid i \in [|a|]\} \in \mathsf{Rel}_S(a, a+a).$$

*Empty relation*

$$0^a_b = \emptyset \in \mathsf{Rel}_S(a, b).$$

For each $a \in S^*$ and each nonnegative integer $n$ we define inductively $\vee^n_a \in \mathsf{Rel}_S(na, a)$ and $\wedge^a_n \in \mathsf{Rel}_S(a, na)$ by the following equalities:

$$\begin{array}{rcl|rcl} \vee^0_a & = & \top_a & \wedge^a_0 & = & \perp^a \\ \vee^{n+1}_a & = & (\mathsf{I}_a + \vee^n_a)\vee_a & \wedge^a_{n+1} & = & \wedge^a(\mathsf{I}_a + \wedge^a_n) \end{array}$$

In [3] we have proved that each $f \in \mathsf{Rel}_S(a, b)$ may be represented as

$$f = \left(\sum_{j=1}^{|a|} \wedge^{a_j}_{m_j}\right) f_2 \left(\sum_{i=1}^{|b|} \vee^{n_i}_{b_i}\right), \tag{1}$$

where $f_2$ is a bijection.

We have defined 16 types of finite relations $xy$-$\mathsf{Rel}_S$ using two parameters, $x \in \{a, b, c, d\}$ and $y \in \{\alpha, \beta, \gamma, \delta\}$; the indices $m_j$ and $n_j$ used in ( 1) satisfy the following restrictions:

| $x$ | restrictions | $y$ | restrictions |
|---|---|---|---|
| $a$ | $m_j = 1$ | $\alpha$ | $n_i = 1$ |
| $b$ | $m_j \leq 1$ | $\beta$ | $n_i \leq 1$ |
| $c$ | $m_j \geq 1$ | $\gamma$ | $n_i \geq 1$ |
| $d$ | none | $\delta$ | none |

Hence by definition $xy$-$\mathsf{Rel}_S$ is the set of all relations having at least one representation of the form (1) such that the indices $m_j$ and $n_i$ satisfy the above restrictions. Some of these types of relations are well known, for example $a\delta$-$\mathsf{Rel}_S$ represents functions and $a\alpha$-$\mathsf{Rel}_S$ denotes bijections (therefore we shall write in the sequel $\mathsf{Bi}_S$ instead of $a\alpha$-$\mathsf{Rel}_S$).

All 16 types of relations are closed under composition and sum.

## 1.2 Algebraic Structures

We recall the definition of the algebraic structures which will be used subsequently in the paper. Let $B$ be a category whose objects form a monoid $(M, +, 0)$ and such that for each $a, b, c, d \in M$ a sum operation is given

$$+ : B(a, b) \times B(c, d) \to B(a + c, b + d).$$

$B$ is called a *strict monoidal category* (*smc* for short) if the following axioms are satisfied:

1. $f + (g + h) = (f + g) + h$
2. $f + \mathsf{I}_0 = f = \mathsf{I}_0 + f$
3. $\mathsf{I}_a + \mathsf{I}_b = \mathsf{I}_{a+b}$
4. $(f + g)(u + v) = fu + gv$

An smc-morphism is a functor which is a monoid morphism on objects and preserves the sum.

Suppose that for every $a, b \in M$ some distinguished morphisms ${}^a\mathsf{X}^b \in B(a + b, b + a)$ are given. An *smc* is called *symmetric* (*ssmc* for short) if the following axioms are satisfied:

5. ${}^a\mathsf{X}^c(g + f){}^d\mathsf{X}^b = f + g$ for $f : a \to b$ and $g : c \to d$
6. ${}^a\mathsf{X}^0 = \mathsf{I}_a$
7. ${}^a\mathsf{X}^{b+c} = ({}^a\mathsf{X}^b + \mathsf{I}_c)(\mathsf{I}_b + {}^a\mathsf{X}^c)$

An ssmc-morphism is a smc-morphism $H$ such that $H({}^a\mathsf{X}^b) = {}^{H(a)}\mathsf{X}^{H(b)}$.

Notice that $xy$-$\mathsf{Rel}_S$ is an ssmc for each $x \in \{a, b, c, d\}$ and $y \in \{\alpha, \beta, \gamma, \delta\}$.

The concept of an *$xy$-ssmc* depends on the two parameters: $x \in \{a, b, c, d\}$ and $y \in \{\alpha, \beta, \gamma, \delta\}$. For each $e \in M$ we will use the distinguished morphisms $\top_e \in B(0, e)$, $\perp^e \in B(e, 0)$, $\vee_e \in B(e+e, e)$ and $\wedge^e \in B(e, e+e)$. The following table shows, for every value of the parameters, which distinguished morphisms are involved:

| $x$ | distinguished morphism | $y$ | distinguished morphism |
|---|---|---|---|
| $a$ | none | $\alpha$ | none |
| $b$ | $\perp^e$ | $\beta$ | $\top_e$ |
| $c$ | $\wedge^e$ | $\gamma$ | $\vee_e$ |
| $d$ | $\perp^e$ and $\wedge^e$ | $\delta$ | $\top_e$ and $\vee_e$ |

The axioms which are satisfied by the distinguished morphisms are chosen from the next table for each $xy$ case according to the following rule: select all axioms in which only the distinguished morphisms corresponding to the $xy$ case appear.

| | | | |
|---|---|---|---|
| A. | $(\vee_a + \mathsf{I}_a)\vee_a = (\mathsf{I}_a + \vee_a)\vee_a$ | A°. | $\wedge^a(\wedge^a + \mathsf{I}_a) = \wedge^a(\mathsf{I}_a + \wedge^a)$ |
| B. | ${}^a\mathsf{X}^a\vee_a = \vee_a$ | B°. | $\wedge^{a\,a}\mathsf{X}^a = \wedge^a$ |
| C. | $(\top_a + \mathsf{I}_a)\vee_a = \mathsf{I}_a$ | C°. | $\wedge^a(\perp^a + \mathsf{I}_a) = \mathsf{I}_a$ |
| D. | $\vee_a\perp^a = \perp^a + \perp^a$ | D°. | $\top_a\wedge^a = \top_a + \top_a$ |

| | |
|---|---|
| E. | $\top_a\perp^a = \mathsf{I}_0$ |
| F. | $\vee_a\wedge^a = (\wedge^a + \wedge^a)(\mathsf{I}_a + {}^a\mathsf{X}^a + \mathsf{I}_a)(\vee_a + \vee_a)$ |
| G. | $\wedge^a\vee_a = \mathsf{I}_a$ |

| | | | |
|---|---|---|---|
| SV1. | $\top_0 = \mathsf{I}_0$ | SV1°. | $\perp^0 = \mathsf{I}_0$ |
| SV2. | $\top_{a+b} = \top_a + \top_b$ | SV2°. | $\perp^{a+b} = \perp^a + \perp^b$ |
| SV3. | $\vee_0 = \mathsf{I}_0$ | SV3°. | $\wedge^0 = \mathsf{I}_0$ |
| SV4. | $\vee_{a+b} = (\mathsf{I}_a + {}^b\mathsf{X}^a + \mathsf{I}_b)(\vee_a + \vee_b)$ | SV4°. | $\wedge^{a+b} = (\wedge^a + \wedge^b)(\mathsf{I}_a + {}^a\mathsf{X}^b + \mathsf{I}_b)$ |

An $xy$-morphism is a ssms-morphism which preserves the distinguished morphisms which are involved.

Notice that $xy$-$\mathsf{Rel}_S$ is an $xy$-ssms for each $x \in \{a,b,c,d\}$ and $y \in \{\alpha,\beta,\gamma,\delta\}$.

A *biflow* is an ssmc $B$ endowed for each $a,b,c \in M$ with an unary operation

$$\uparrow^a : B(a+b, a+c) \to B(b,c)$$

called (left) *feedback* which satisfies the following axioms:

F1. $f(\uparrow^a g)h = \uparrow^a[(\mathsf{I}_a + f)g(\mathsf{I}_a + h)]$
F2. $\uparrow^a f + \mathsf{I}_d = \uparrow^a(f + \mathsf{I}_d)$
F3. $\uparrow^{a+b}[f({}^b\mathsf{X}^a + \mathsf{I}_d)] = \uparrow^{b+a}[({}^b\mathsf{X}^a + \mathsf{I}_c)f]$ for $f : a+b+c \to b+a+d$
F4. $\uparrow^a\uparrow^b f = \uparrow^{b+a} f$
F5. $\uparrow^a \mathsf{I}_a = \mathsf{I}_0$
F6. $\uparrow^{a\,a}\mathsf{X}^a = \mathsf{I}_a$

A consequence of the above axioms is $\uparrow^0 f = f$.

### 1.3 Feedback and Relations

The feedback $\uparrow^s$ for $s \in S$ is defined in $\mathsf{Rel}_S$ for $f \in \mathsf{Rel}_S(s+a, s+b)$ by

$$\uparrow^s f = \{(i,j) \mid (1+i, 1+j) \in f \text{ or } \{(1+i,1),(1,1+j)\} \subseteq f\}$$

This definition may be completed by induction on the length starting with $\uparrow^\lambda f = f$ and using F4. The next fact comes from [2].

**Fact 1** $xy$-$\mathsf{Rel}_S$ is closed under $\uparrow$ if and only if $xy \in \{a\alpha, d\delta\}$ or $x = b$ or $y = \beta$.

The case $a\alpha$ was studied in [1]. The result in the next section covers the cases $a\beta, b\alpha, b\beta, b\gamma, c\beta, b\delta, d\beta$ and $d\delta$.

## 2 Main result

Let $xy \in \{a\alpha, d\delta\}$ or $x = b$ or $y = \beta$. Let $B$ be a biflow over an $xy$-*ssmc* which satisfies the following axioms:

| | | |
|---|---|---|
| $\uparrow^a \vee_a = \perp^a$ | only for | $xy \in \{b\gamma,\ b\delta,\ d\delta\}$ |
| $\uparrow^a \wedge^a = \top_a$ | only for | $xy \in \{c\beta,\ d\beta,\ d\delta\}$ |
| $\uparrow^a[(\wedge^a + \mathsf{I}_a)(\mathsf{I}_a + {}^a\mathsf{X}^a)(\vee_a + \mathsf{I}_a)] = \mathsf{I}_a$ | only for | $xy = d\delta$ |

**Fact 2** $xy$-$\mathsf{Rel}_S$ satisfies the above hypotheses.

**Theorem 3** Let $B$ be a biflow over an $xy$-*ssmc* which satisfies the above three axioms.

1) If $H : xy$-$\mathsf{Rel}_S \to B$ is an $xy$-*ssmc* morphism then $H$ preserves the feedback.

2) The biflow structure of $xy$-$\mathsf{Rel}_S$ is unique if its $xy$-ssmc structure is the above one.

**Proof** 1) As the monoid of the objects of $\mathsf{Rel}_S$ is a free one it suffices to show $H$ preserves $\uparrow^s$ for each $s \in S$.

Let $f \in xy\text{-}\mathsf{Rel}_S(s+a, s+b)$ with $s \in S$ and its standard writing (see [3]) :

$$f = \left( \wedge_{m_0}^{s} + \sum_{j \in [|a|]} \wedge_{m_j}^{a_j} \right) g \left( \vee_{s}^{n_0} + \sum_{i \in [|b|]} \vee_{b_i}^{n_i} \right) .$$

As

$$H\left(\uparrow^s f\right) = \left( \sum_{j \in [|a|]} \wedge_{m_j}^{H(a_j)} \right) H\left(\uparrow^s \left[ \left( \wedge_{m_0}^{s} + \mathsf{I}_{\sum m_j a_j} \right) g \left( \vee_{s}^{n_0} + \mathsf{I}_{\sum n_i b_i} \right) \right] \right) \left( \sum_{i \in [|b|]} \vee_{H(b_i)}^{n_i} \right)$$

and

$$\uparrow^{H(s)} H(f) = \left( \sum_{j \in [|a|]} \wedge_{m_j}^{H(a_j)} \right) \uparrow^{H(s)} H\left( \left( \wedge_{m_0}^{s} + \mathsf{I}_{\sum m_j a_j} \right) g \left( \vee_{s}^{n_0} + \mathsf{I}_{\sum n_i b_i} \right) \right) \left( \sum_{i \in [|b|]} \vee_{H(b_i)}^{n_i} \right)$$

it is sufficient to prove that $H$ preserves the feedback only for the following relations:

$$f = \left( \wedge_{m}^{s} + \mathsf{I}_a \right) g \left( \vee_{s}^{n} + \mathsf{I}_b \right)$$

where

- $g \in Bi_S(ms + a, ns + b)$ and
- $\left( \mathsf{I}_{ms} + \top_a \right) g \left( \mathsf{I}_{ns} + \perp^b \right) \in \{ 0_{ns}^{ms},\, \mathsf{I}_s + 0_{(n-1)s}^{(m-1)s} \}$.

The last property comes from the standard writing of $f$ (see[3]).

This fact is proved by cases.

1. Case $m = 0$ (only for $x \in \{b,\, d\}$):

$$\begin{array}{rcl} \uparrow^{H(s)} H(f) & = & \uparrow^{H(s)} \left[ \left( \perp^{H(s)} + \mathsf{I}_{H(a)} \right) H(g) \left( \vee_{H(s)}^{n} + \mathsf{I}_{H(b)} \right) \right] \\ & = & \uparrow^{0} \left[ H(g) \left( \vee_{H(s)}^{n} \perp^{H(s)} + \mathsf{I}_{H(b)} \right) \right] \\ & = & H\left( g \left( \perp^{ns} + \mathsf{I}_b \right) \right) \; = \; H\left( \uparrow^s f \right) \end{array}$$

2. Case $n = 0$ (only for $y \in \{\beta,\, \delta\}$) : dual.

3. Case $m \geq 1$, $n \geq 1$ and $\left( \mathsf{I}_{ms} + \top_a \right) g \left( \mathsf{I}_{ns} + \perp^b \right) = 0_{ns}^{ms}$.

As there exist $u \in Bi_S(a, ns + c)$ and $v \in Bi_S(ms + c, b)$ such that

$$g = \left( \mathsf{I}_{ms} + u \right) \left( {}^{ms}\mathsf{X}^{ns} + \mathsf{I}_c \right) \left( \mathsf{I}_{ns} + v \right),$$

we deduce:

$$\begin{array}{rcl} f & = & \left( \mathsf{I}_s + u \right) \left[ \left( \wedge_{m}^{s} + \mathsf{I}_{ns} \right) {}^{ms}\mathsf{X}^{ns} \left( \vee_{s}^{n} + \mathsf{I}_{ms} \right) + \mathsf{I}_c \right] \left( \mathsf{I}_s + v \right) \\ & = & \left( \mathsf{I}_s + u \right) \left[ \left( \mathsf{I}_s + \vee_{s}^{n} \right) {}^{s}\mathsf{X}^{s} \left( \mathsf{I}_s + \wedge_{m}^{s} \right) + \mathsf{I}_c \right] \left( \mathsf{I}_s + v \right) \\ & = & \left[ \mathsf{I}_s + u \left( \vee_{s}^{n} + \mathsf{I}_c \right) \right] \left( {}^{s}\mathsf{X}^{s} + \mathsf{I}_c \right) \left[ \mathsf{I}_s + \left( \wedge_{m}^{s} + \mathsf{I}_c \right) v \right] \end{array}$$

thus

$$\uparrow^{H(s)} H(f) = H\left( u \left( \vee_{s}^{n} + \mathsf{I}_c \right) \right) H\left( \left( \wedge_{m}^{s} + \mathsf{I}_c \right) v \right) = H\left( u \left( \vee_{s}^{n} \wedge_{m}^{s} + \mathsf{I}_c \right) v \right) = H\left( \uparrow^s f \right) .$$

4. Case $m \geq 1$, $n \geq 1$ and $(\mathsf{I}_{ms} + \top_a)\, g\, (\mathsf{I}_{ns} + \perp^b) = \mathsf{I}_s + 0^{(m-1)s}_{(n-1)s}$.

4a. Case $m = 1$ or $n = 1$.
Notice that $g = \mathsf{I}_s + h$ with $h \in Bi_S((m-1)s + a,\, (n-1)s + b)$.
If $m = 1$ and $n = 1$ then $f = \mathsf{I}_s + h$ hence $\uparrow^{H(s)} H(f) = H(h) = H(\uparrow^s f)$ .
If $m = 1$ and $n > 1$ (only for $y \in \{\gamma,\, \delta\}$) then

$$f = (\mathsf{I}_s + h)\left[(\mathsf{I}_s + \vee_s^{n-1})\vee_s + \mathsf{I}_b\right] = \left[\mathsf{I}_s + h\,(\vee_s^{n-1} + \mathsf{I}_b)\right](\vee_s + \mathsf{I}_b)$$

therefore

$$\begin{array}{rcl}\uparrow^{H(s)} H(f) & = & H\left(h\,(\vee_s^{n-1} + \mathsf{I}_b)\right)\left(\uparrow^{H(s)}\vee_{H(s)} + \mathsf{I}_{H(b)}\right)\\ & = & H\left(h\,(\vee_s^{n-1} + \mathsf{I}_b)\right)\left(\perp^{H(s)} + \mathsf{I}_{H(b)}\right)\\ & = & H\left(h\left(\perp^{(n-1)s} + \mathsf{I}_b\right)\right) = H\left(\uparrow^s f\right) .\end{array}$$

For $m > 1$ and $n = 1$ (only for $x \in \{c,\, d\}$) the proof is dual.

4b. Case $m > 1$ and $n > 1$ (only for $xy = d\delta$).
As there exist $u \in Bi_S(a,\, (n-1)s + c)$ and $v \in Bi_S((m-1)s + c,\, b)$ such that

$$g = (\mathsf{I}_{ms} + u)\left(\mathsf{I}_s + {}^{(m-1)s}\mathsf{X}^{(n-1)s} + \mathsf{I}_c\right)(\mathsf{I}_{ns} + v)$$

we deduce:

$$\begin{array}{rcl} f & = & (\mathsf{I}_s + u)\left[\left(\wedge^s\left(\mathsf{I}_s + \wedge^s_{m-1}\right) + \mathsf{I}_{(n-1)s}\right)\left(\mathsf{I}_s + {}^{(m-1)s}\mathsf{X}^{(n-1)s}\right)\right.\\ & & \left.\left((\mathsf{I}_s + \vee_s^{n-1})\vee_s + \mathsf{I}_{(m-1)s}\right) + \mathsf{I}_c\right](\mathsf{I}_s + v)\\ & = & (\mathsf{I}_s + u)\left[(\wedge^s + \vee_s^{n-1})\left(\mathsf{I}_s + {}^s\mathsf{X}^s\right)\left(\vee_s + \wedge^s_{m-1}\right) + \mathsf{I}_c\right](\mathsf{I}_s + v)\\ & = & \left(\mathsf{I}_s + u\,(\vee_s^{n-1} + \mathsf{I}_c)\right)\left[(\wedge^s + \mathsf{I}_s)\left(\mathsf{I}_s + {}^s\mathsf{X}^s\right)(\vee_s + \mathsf{I}_s) + \mathsf{I}_c\right]\left(\mathsf{I}_s + \left(\wedge^s_{m-1} + \mathsf{I}_c\right)v\right) .\end{array}$$

therefore

$$\begin{array}{rcl}\uparrow^{H(s)} H(f) & = & H\left(u\,(\vee_s^{n-1} + \mathsf{I}_c)\right)\left[\uparrow^{H(s)}\left(\left(\wedge^{H(s)} + \mathsf{I}_{H(s)}\right)\left(\mathsf{I}_{H(s)} + {}^{H(s)}\mathsf{X}^{H(s)}\right)\right.\right.\\ & & \left.\left.\left(\vee_{H(s)} + \mathsf{I}_{H(s)}\right)\right) + \mathsf{I}_{H(c)}\right] H\left(\left(\wedge^s_{m-1} + \mathsf{I}_c\right)v\right)\\ & = & H\left(u\left(\vee_s^{n-1}\wedge^s_{m-1} + \mathsf{I}_c\right)v\right) = H\left(\uparrow^s f\right) .\end{array}$$

2) Suppose ($xy$-$\mathsf{Rel}_S$, $\cdot$, $\mathsf{I}$, $+$, $\mathsf{X}$, $\Uparrow$) is a biflow. Notice that the three axioms from the beginning of the second section hold. The proof for the last one is done by induction on the length of $a$. For

$$h = \Uparrow^a\left[(\wedge^a + \mathsf{I}_a)\left(\mathsf{I}_a + {}^a\mathsf{X}^a\right)(\vee_a + \mathsf{I}_a)\right]$$

we deduce

$$\begin{array}{rcl} h & = & \Uparrow^{2a}\left[(\vee_a\wedge^a + \mathsf{I}_a)\left(\mathsf{I}_a + {}^a\mathsf{X}^a\right)\right]\\ & = & \Uparrow^a\Uparrow^a\left[\left((\wedge^a + \wedge^a)\left(\mathsf{I}_a + {}^a\mathsf{X}^a + \mathsf{I}_a\right)(\vee_a + \vee_a) + \mathsf{I}_a\right)\left(\mathsf{I}_a + {}^a\mathsf{X}^a\right)\right]\\ & = & \Uparrow^a\left[\left(\wedge^a\Uparrow^a\left[(\wedge^a + \mathsf{I}_{2a})\left(\mathsf{I}_a + {}^a\mathsf{X}^a + \mathsf{I}_a\right)(\vee_a + \mathsf{I}_{2a})\right]\vee_a + \mathsf{I}_a\right){}^a\mathsf{X}^a\right]\\ & = & \Uparrow^a\left[\left(\wedge^a\,(h + \mathsf{I}_a)\vee_a + \mathsf{I}_a\right){}^a\mathsf{X}^a\right]\\ & = & (\Uparrow^{a\,a}\mathsf{X}^a)(\wedge^a\,(h + \mathsf{I}_a)\vee_a)\\ & = & \wedge^a\,(h + \mathsf{I}_a)\vee_a.\end{array}$$

From the last equality we deduce $\mathsf{I}_a \subseteq h$ which assure the proof for $|a| \leq 1$. The proof for the inductive step for $s \in S$ is the following:

$$\Uparrow^{a+s}\left[(\wedge^{a+s} + \mathsf{I}_{a+s})\left(\mathsf{I}_{a+s} + {}^{a+s}\mathsf{X}^{a+s}\right)(\vee_{a+s} + \mathsf{I}_{a+s})\right]$$

$$= \Uparrow^{a+s}[(\wedge^a + \wedge^s + \mathsf{I}_{a+s})(\mathsf{I}_a + {}^a\mathsf{X}^s + \mathsf{I}_{s+a+s})(\mathsf{I}_{a+s} + {}^{a+s}\mathsf{X}^{a+s})(\mathsf{I}_a + {}^s\mathsf{X}^a + \mathsf{I}_{s+a+s})(\vee_a + \vee_s + \mathsf{I}_{a+s})]$$

$$= \Uparrow^s\Uparrow^a[(\wedge^a + \wedge^s + \mathsf{I}_{a+s})(\mathsf{I}_{2a} + {}^{2s}\mathsf{X}^a + \mathsf{I}_s)(\mathsf{I}_a + {}^a\mathsf{X}^a + \mathsf{I}_s + {}^s\mathsf{X}^s)(\mathsf{I}_{2a} + {}^a\mathsf{X}^{2s} + \mathsf{I}_s)(\vee_a + \vee_s + \mathsf{I}_{a+s})]$$

$$= \Uparrow^s[(\wedge^s + \mathsf{I}_{a+s})({}^{2s}\mathsf{X}^a + \mathsf{I}_s)(\mathsf{I}_{a+s} + {}^s\mathsf{X}^s)(h + \mathsf{I}_{s+s+s})({}^a\mathsf{X}^{2s} + \mathsf{I}_s)(\vee_s + \mathsf{I}_{a+s})]$$
$$= {}^a\mathsf{X}^s\{\Uparrow^s[(\wedge^s + {}^s\mathsf{X}^a)({}^{2s}\mathsf{X}^a + \mathsf{I}_s)(\mathsf{I}_{a+s} + {}^s\mathsf{X}^s)({}^a\mathsf{X}^{2s} + \mathsf{I}_s)(\vee_s + {}^a\mathsf{X}^s)]\}{}^s\mathsf{X}^a$$
$$= {}^a\mathsf{X}^s\{\Uparrow^s[(\wedge^s + \mathsf{I}_s)(\mathsf{I}_s + {}^s\mathsf{X}^s)(\vee_s + \mathsf{I}_s)] + \mathsf{I}_a\}{}^s\mathsf{X}^a = \mathsf{I}_{a+s}.$$

As $\mathbf{1}_{xy-\mathsf{Rel_S}} : xy\text{-}\mathsf{Rel}_S \to (xy\text{-}\mathsf{Rel}_S, \cdot, \mathsf{I}, +, \mathsf{X}, \Uparrow)$ is an $xy$-$ssmc$ morphism, we deduce from 1) it preserves the feedback thus $\uparrow = \Uparrow$ . ■

**References**


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