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Recursively Enumerable Reals and Chaitin Ω Numbers*

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Abstract

A real α is called recursively enumerable if it can be approximated by an increasing, recursive sequence of rationals. The halting probability of a universal self-delimiting Turing machine (Chaitin's Ω number, [10]) is a random r.e. real. Solovay's [25] Ω-like reals are also random r.e. reals. Solovay showed that any Chaitin Ω number is Ω-like. In this paper we show that the converse implication is true as well: any Ω-like real in the unit interval is the halting probability of a universal self-delimiting Turing machine.

Following Solovay [25] and Chaitin [11] we say that an r.e. real α *dominates* an r.e. real β if from a good approximation of α from below one can compute a good approximation of β from below. We shall study this relation and characterize it in terms of relations between r.e. sets. Ω-like numbers are the maximal r.e. real numbers with respect to this order, that is, from a good approximation to an Ω-like real one can compute a good approximation for *every* r.e. real. This property shows the strength of Ω for approximation purposes. However, the situation is radically different if one wishes to compute digits of the binary expansion of an r.e. real: one cannot compute with a total recursive function the first n digits of the r.e. real $0.\chi_K$ (the characteristic sequence of the halting problem) from the first $g(n)$ digits of Ω, for any total recursive function g.

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1 Introduction

Algorithmic information theory, as developed by Chaitin [9, 10, 12], Kolmogorov [17], Solomonoff [24], Martin-Löf [20], and others (see Calude [4]), gives a satisfactory description of the quantity of information of individual finite strings and infinite sequences. The same quantity of information may be organised in various ways; in order to quantify the degree of organisation of the information in a string or a sequence, Bennett [2], Juedes, Lathrop, and Lutz [14], and others, have considered the computational depth. Roughly speaking, the computational depth of an object is the amount of time required by an algorithm to derive the object from its shortest description. Bennett [2] showed that the characteristic sequence χ_K of the halting problem is strongly deep, while no random sequence is strongly deep. Investigating this matter further, Juedes, Lathrop, and Lutz [14] have considered the notion of "usefulness" of infinite sequences. A sequence $\mathbf{x}$ is useful if all recursive sequences can be computed with oracle access to $\mathbf{x}$ within a fixed recursive time bound. For example χ_K is useful, while no recursive or random sequence is useful.

It is well known that the halting probability of a universal self-delimiting Turing machine, called Chaitin Ω number (see Chaitin [10, 13], Rozenberg and Salomaa [21], Calude [4]), is random, but χ_K is not; Ω and χ_K contain the same quantity of information but codified in vastly different ways. As we noted before, χ_K is useful but Ω is not useful in the sense of Juedes, Lathrop, and Lutz [14]. However, when one is interested in approximating sequences[1] Ω is more "useful" than χ_K; it is one of the aims of this paper to give a mathematical sense to this statement.

R.e. reals[2] are extensively used in computable analysis, see Weihrauch [28] and Ko [16]. We will characterize r.e. reals in various ways. In order to compare the "usefulness" of r.e. reals for approximation purposes, Solovay [25] (see also Chaitin [11]) has introduced the following notion. A real β *dominates* a real α if there exists a partial recursive function f on rationals and a constant $c > 0$ such that if p is a rational number less than β, then $f(p)$ is (defined and) less than α, and the inequality

$$c(\beta - p) \geq \alpha - f(p)$$

holds. In this case we write $\alpha \leq_{dom} \beta$. Informally, a real β *dominates* a real α if from a good approximation of β from below one can compute a good approximation of α from below. The relation $\leq_{dom}$ is transitive and reflexive, hence it naturally defines a partially ordered set $\langle \mathbf{R}_{r.e.}; \leq_{dom} \rangle$ whose elements are the $=_{dom}$-equivalence classes of r.e. reals. This partially ordered set possesses natural properties as one should expect. It has a minimum element which is the equivalence class containing exactly all recursive reals; it has a maximum element which is the equivalence class containing all Chaitin Ω numbers. It is an upper semilattice. The least upper bound of any two classes containing r.e. reals α and β, respectively, is the class containing the r.e. real $\alpha + \beta$. This implies: $+$ is compatible with $\leq_{dom}$, that is if $\alpha_1 \leq_{dom} \beta_1$ and $\alpha_2 \leq_{dom} \beta_2$, then $\alpha_1 + \alpha_2 \leq_{dom} \beta_1 + \beta_2$. We also stress that there is a deep relationship between $\leq_{dom}$ and randomness. Indeed, if $\alpha \leq_{dom} \beta$, then β is "more random" than α in the sense that the Chaitin complexity of the first n digits of α does *not* exceed the Chaitin complexity of the first n digits of

[1] As in constructive mathematics, see Bridges and Richman [3], Weihrauch [28] and Ko [16], and many other areas.

[2] A real α is called r.e. if it can be approximated by a recursive increasing sequence of rationals.

β by more than a constant. In this respect, the partially ordered set $\langle \mathbf{R}_{r.e.}, \leq_{dom} \rangle$ can be thought of as the world where effective objects (r.e. reals) are compared according to their degree of randomness. The more random an effective object is, the closer it is to the Chaitin Ω numbers; the less random an effective object is, the closer it is to a recursive real. We study this relation $\leq_{dom}$ further and characterize it in terms of certain reducibilities between r.e. sets.

Solovay [25] (see also Chaitin [11]) called an r.e. real Ω-*like* if it dominates every r.e. real. He showed that every Chaitin Ω number is Ω-like. In this paper we prove the converse implication by showing that any Ω-like real in the unit interval is the halting probability of a universal self-delimiting Turing machine. This shows the strength of all Ω's for approximation purposes: from a good approximation of Ω one can obtain a good approximation of any r.e. real, and no other r.e. reals have this property. Consequently, compared with a non-Ω-like r.e. real number, any number Ω either contains more information or at least the information contained in Ω is structured in a more useful way. However, the situation is radically different if we do not wish just to compute an arbitrary rational approximation of an r.e. real but to compute digits of its binary representation: we cannot compute with a total recursive function the first n digits of the r.e. real $0.\chi_K$ (the characteristic sequence of the halting problem) from the first $g(n)$ digits of Ω, for any total recursive function g.

We give a brief summary of the paper. The next section introduces some basic notations. In Section 3 we define the program size complexity of strings, we define Chaitin Ω numbers, and state some basic known results. We give a short proof of the well-known result that Chaitin Ω numbers are random. In Section 4, we introduce r.e. reals and give several characterizations of r.e. reals. In this section we also introduce the domination relation and prove some basic and important facts about this relation. In Section 5, we introduce the partially ordered set $\langle \mathbf{R}_{r.e.}, \leq_{dom} \rangle$ and exhibit a relationship between this partially ordered sets and Turing reducibility. We also give a characterization of $\leq_{dom}$ in terms of certain reducibilities between sets of strings. In the next section, we prove that any Ω-like real is in fact the halting probability of some universal Turing machine. We also consider the question whether Ω-like reals are also good for computing the digits of the binary representation of r.e. reals. The last section contains some open problems and comments.

2 Notation

By $\mathbf{N}, \mathbf{Q}$ and $\mathbf{R}$ we denote the set of positive integers, the set of rational numbers, and the set of real numbers, respectively. A sequence $q_0, q_1, q_2, \ldots$ of numbers (integers, rationals, or reals) is said to be increasing (non-decreasing) if $q_i < q_{i+1}$ (if $q_i \leq q_{i+1}$) for all i. If f and g are natural number functions, the formula $f(n) \leq g(n) + O(1)$ means that there is a constant $c > 0$ with $f(n) \leq g(n) + c$, for all n. If X and Y are sets, then $f : X \stackrel{o}{\to} Y$ denotes a possibly partial function defined on a subset of X.

Let $\Sigma = \{0, 1\}$ denote the binary alphabet, Σ^* is the set of (finite) binary strings, Σ^n is the set of binary strings of length n, and Σ^ω the set of infinite binary sequences. The length of a string x is denoted by $|x|$; λ is the empty string. Let $<$ be the quasi-lexicographical order on Σ^* and let $string_n$ ($n \geq 0$) be the nth string under this ordering. For strings $x, y \in \Sigma^*$, xy is the concatenation of x and y. For a sequence $\mathbf{x} = x_0 x_1 \cdots x_n \cdots \in \Sigma^\omega$ and an integer number $n \geq 0$, $\mathbf{x}(n)$ denotes the initial segment of length $n+1$ of $\mathbf{x}$ and x_i denotes the ith digit of $\mathbf{x}$, i.e., $\mathbf{x}(n) = x_0 x_1 \cdots x_n$. Lower case letters k, l, m, n will denote positive integers, and x, y, z strings. By $\mathbf{x}, \mathbf{y}, \cdots$ we denote infinite sequences from Σ^ω; finally, we reserve α, β, γ for reals. A subset of Σ^* is called a language. Capital letters are used to denote languages. For languages A and B, $A \subseteq B$ denotes that A is a subset of B. We fix a standard recursive pairing function $\lambda k, y < k, y >$ defined on $\mathbf{N} \times \Sigma^*$ with values in Σ^*. For a set $A \subseteq \Sigma^*$ let $A_k = \{x \mid < k, x > \in A\}$. For a language A, χ_A denotes the infinite characteristic sequence of A, that is, $(\chi_A)_n = 1$ if $string_n \in A$ and $(\chi_A)_n = 0$ otherwise. For $A \subseteq \Sigma^*$, $A\Sigma^\omega$ denotes the set of sequences $\{w\mathbf{x} \mid w \in A, \mathbf{x} \in \Sigma^\omega\}$.

We assume that the reader is familiar with Turing machine computations, including oracle computations. We use K to denote the halting problem, that is, $string_n \in K$ if and only if the nth Turing machine halts on the input $string_n$. We say that a language A is Turing reducible to a language B, and we write $A \leq_T B$, if there is an oracle Turing machine M such that $M^B(string_n) = (\chi_A)_n$, for all $x \in \Sigma^*$. Let D be a total standard notation of all finite sets of words in Σ^*. We say that a language A is truth-table reducible to a language B, and we write $A \leq_{tt} B$, if there are two total recursive functions $f : \Sigma^* \to \mathbf{N}$ and $g : \Sigma^* \to \Sigma^*$ such that $x \in A$ if and only if $\chi_B(f(x)) \in D_{g(x)}$ (compare Soare [23]). For further notation we refer to Calude [4].

3 Complexity and Randomness

In this section, we review some fundamentals of algorithmic information theory that we will use in this paper. We are especially concerned with self-delimiting (Chaitin/program-size) complexity and algorithmic randomness. Program-size complexity is a technical improvement of the original formulation of the descriptive complexity that was developed by Chaitin [10]; the advantage of the self-delimiting version is that it gives a precise characterization of algorithmic probability and random sequences.

Program-size complexity employs a slightly restricted model of deterministic Turing machine computation. A self-delimiting Turing machine M has a program tape, an output tape, and a work tape. Only 0's, 1's and blanks can ever appear on a tape. The program tape and the output tape are infinite to the right, while the worktape is infinite in both directions. Each tape has a scanning head. The program and output tape heads cannot move left, but the worktape head can move in both directions. The program tape is read-only, the output tape is write-only, and the worktape is read/write.

A self-delimiting Turing machine M starts in the initial state with a program $x \in \Sigma^*$ on its program tape, the output tape blank, and the worktape blank. The left-most cell of the program tape is blank and the program tape head initially scans this cell. The program x lies immediately to the right of this cell and the rest of the program tape is blank. The output tape head initially scans the left-most cell of the output tape.

During each cycle of operation the machine may halt, move the read head of the program tape one cell to the right, move the read/write head of the worktape one cell to the left or to the right, and move the write head of the output tape one cell to the right. The machine changes state: the action performed and the next state are both functions of the present state and the contents of the two cells being scanned by the program tape head and by the worktape head.

If, after finitely many steps, M halts with the program tape head scanning the last bit of x, then the computation is a success, and we write $M(x) < \infty$; the output of the computation is the string $M(x) \in \Sigma^*$ that has been written on the output tape. Otherwise, the computation is a failure, we write $M(x) = \infty$, and there is no output.

In view of the above definition, a successful computation must end with the program tape head scanning the last bit of the program. Since the program tape head is read-only and cannot move left, this implies that for every self-delimiting Turing machine M the program set

$$PROG_M = \{x \in \Sigma^* \mid M(x) < \infty\}$$

is an *instantaneous code*, i.e., a set of strings with the property that no string in it is a proper prefix of another. Conversely, every prefix-free r.e. set set of words is the domain of some self-delimiting Turing machine. It follows by Kraft's inequality that, for every self-delimiting Turing machine M,

$$\Omega_M = \sum_{x \in PROG_M} 2^{-|x|} \leq 1.$$

The number Ω_M is called the halting probability of M. In what follows we will omit the adjective "self-delimiting", since this is the only type of Turing machine considered in this paper.

Definition 3.1 Let M be a Turing machine. The *program-size complexity* of the string $x \in \Sigma^*$ (relatively to M) is $H_M(x) = \min\{|y| \mid y \in \Sigma^*,\ M(y) = x\}$, where $\min \emptyset = \infty$.

It was shown by Chaitin [10] (see Calude [4]) that there is a self-delimiting Turing machine U that is *universal*, in the sense that, for every self-delimiting Turing machine M, there is a constant c_M (depending upon M) with the following property: if $M(x) < \infty$, then there is an $x' \in \Sigma^*$ such that $U(x') = M(x)$ and $|x'| \leq |x| + c_M$. Clearly, every universal machine produces every string; denote by x^* the *canonical program* of x, i.e., $x^* = \min\{y \in \Sigma^* \mid U(y) = x\}$, where the minimum is taken according to the quasi-lexicographical order. For two universal machines U and V, we see $H_U(x) = H_V(x) + O(1)$. The halting probability Ω_U of a universal machine U is called *Chaitin Ω number*; for more about Ω_U see Bennett [1], Calude, Salomaa [5], Calude, Meyerstein [7]. In the rest of the paper, unless stated otherwise, we will use a fixed universal machine U and will omit the subscript U in $H_U(x)$ and Ω_U. We will also abuse our notation by identifying the real number Ω with the infinite binary sequence which corresponds to Ω (i.e., the infinite[3] binary expansion of Ω without "0.").

In the study of algorithmic information theory, we are often required to construct a Turing machine which satisfies certain properties. The following extension (Chaitin [10]; see also Calude, Grozea [6]) of Kraft's inequality is very useful for this purpose:

Theorem 3.2 (Kraft-Chaitin) *Given a recursive list of "requirements" $< s_i, n_i >$ $(i \geq 0, s_i \in \Sigma^*, n_i \in \mathbf{N})$ such that $\sum_i 2^{-n_i} \leq 1$, we can effectively construct a self-delimiting Turing machine M and a recursive one-to-one enumeration $x_0, x_1, x_2, \ldots$ of words x_i of length n_i such that $M(x_i) = s_i$ for all i and $M(x) = \infty$ if $x \notin \{x_i \mid i \in \mathbf{N}\}$.*

Notice that the halting probability of the machine M constructed above is $\Omega_M = \sum_i 2^{-n_i}$.

We conclude this section with a brief discussion of (algorithmically) random infinite binary sequences.[4] Random sequences were originally defined by Martin-Löf [20]

[3] This expansion is unique since by Theorem 3.4, Ω is random and, hence, irrational.

[4] The interested reader is referred to Calude [4] and Wang [27] for more details.

using constructive measure theory. Complexity-theoretic characterizations of random sequences have been obtained by Chaitin [10] (see also Levin [19], Schnorr [22]).

We use Chaitin's [10] characterization: an infinite sequence $\mathbf{x}$ is *random* if there is a constant c such that $H(\mathbf{x}(n)) > n - c$, for every integer $n > 0$. A slightly different characterization is contained in the next theorem. Martin-Löf's definition is based on randomness tests. A Martin-Löf test is an r.e. set $A \subseteq \Sigma^*$ satisfying the following measure-theoretical condition:

$$\mu(A_i \Sigma^\omega) \leq 2^{-i},$$

for all $i \in \mathbf{N}$.[5] Here μ denotes the usual product measure on Σ^ω, given by $\mu(\{w\}\Sigma^\omega) = 2^{-|w|}$, for $w \in \Sigma^*$.

Theorem 3.3 (Chaitin [10]) *Let $\mathbf{x} \in \Sigma^\omega$. The following statements are equivalent:*

1. *The sequence $\mathbf{x}$ is random.*
2. *We have: $\lim_{n \to \infty} H(\mathbf{x}(n)) - n = \infty$.*
3. *For every Martin-Löf test A, $\mathbf{x} \notin \bigcap_{i \geq 0} (A_i \Sigma^\omega)$.*

Theorem 3.4 (Chaitin [10]) *For every universal machine U, the halting probability Ω_U is random.*

Proof. Let f be a recursive one-to-one function which enumerates $PROG_U$, the domain of U. Let $\omega_k = \sum_{j=0}^{k} 2^{-|f(j)|}$. Clearly, (ω_k) is an increasing sequence of rationals converging to Ω_U. Consider the binary expansion of $\Omega_U = 0.\Omega_0\Omega_1\cdots$.

We define a (self-delimiting) Turing machine M as follows: on input $x \in \Sigma^*$ compute $y = U(x)$ and the smallest number (if it exists) t with $\omega_t \geq 0.y$. Let $M(x)$ be the first (in quasi-lexicographical order) word not belonging to the set $\{U(f(0)), U(f(1)), \ldots, U(f(t))\}$ if both y and t exist, and $M(x) = \infty$ if $U(x) = \infty$ or t does not exist.

If $x \in PROG_M$ and x' is a word with $U(x) = U(x')$, then $M(x) = M(x')$. Applying this to an arbitrary $x \in PROG_M$ and the canonical program $x' = (U(x))^*$ of $U(x)$ yields

$$H_M(M(x)) \leq |x'| = H_U(U(x)) \,. \tag{1}$$

Furthermore, by the universality of U there is a constant $c > 0$ with

$$H_U(M(x)) \leq H_M(M(x)) + c \tag{2}$$

for all $x \in PROG_M$. Now, fix a number n and assume that x is a word with $U(x) = \Omega_0\Omega_1\cdots\Omega_{n-1}$. Then $M(x) < \infty$. Let t be the smallest number (computed in the second step of M) with $\omega_t \geq 0.\Omega_0\Omega_1\cdots\Omega_{n-1}$. We have

$$0.\Omega_0\Omega_1\cdots\Omega_{n-1} \leq \omega_t < \omega_t + \sum_{s=t+1}^{\infty} 2^{-|f(s)|} = \Omega_U \leq 0.\Omega_0\Omega_1\cdots\Omega_{n-1} + 2^{-n} \,.$$

[5] See Calude [4] for a detailed motivation.

Hence, $\sum_{s=t+1}^{\infty} 2^{-|f(s)|} \leq 2^{-n}$. This implies $|f(s)| \geq n$, for every $s \geq t+1$. ¿From the construction of M we conclude that $H_U(M(x)) \geq n$. Using (2) and (1) we obtain

$$\begin{aligned} n &\leq H_U(M(x)) \\ &\leq H_M(M(x)) + c \\ &\leq H_U(U(x)) + c \\ &= H_U(\Omega_0\Omega_1 \cdots \Omega_{n-1}) + c\,, \end{aligned}$$

which, by virtue of Theorem 3.3, proves that the sequence $\Omega_0\Omega_1\cdots$ is random. □

4 R.E. Reals and Domination

In this section we study properties of r.e. reals. A real α is called *r.e.* if the set $\{p \in \mathbf{Q} \mid p < \alpha\}$ of rational numbers less than α is r.e.[6] It is the aim of this section to compare the information contents of r.e. reals. We start with several characterizations of r.e. reals.

For a set $X \subseteq \mathbf{N}$ we define the number

$$2^{-X-1} = \sum_{n \in X} 2^{-n-1}\,.$$

This number lies in the interval $[0,1]$. If we disregard all finite sets X, which lead to rational numbers 2^{-X-1}, we get a bijection $X \mapsto 2^{-X-1}$ between the class of infinite subsets of $\mathbf{N}$ and the real numbers in the interval $(0,1]$. If $0.\mathbf{y}$ is the binary expansion of a real α with infinitely many ones, then $\alpha = 2^{-X_\alpha-1}$ where $X_\alpha = \{i \mid y_i = 1\}$. Clearly, if X_α is r.e., then the number $2^{-X_\alpha-1}$ is r.e., but the converse is not true as the Chaitin Ω numbers show. First we characterize r.e. reals α in terms of the sets X_α; then we characterize r.e. reals in terms of prefix-free sets of words.[7] For a prefix-free set $A \subseteq \Sigma^*$ we define a real number by

$$2^{-A} = \sum_{x \in A} 2^{-|x|}$$

which, due to Kraft's inequality, lies in the interval $[0,1]$.

Theorem 4.1 *For a real $\alpha \in (0,1]$ the following conditions are equivalent:*

1. *The number α is r.e.*
2. *There is a recursive, non-decreasing sequence of rationals $(a_n)_{n\geq 0}$ which converges to α.*
3. *There is a recursive, increasing sequence of rationals $(a_n)_{n\geq 0}$ which converges to α.*
4. *There is a prefix-free r.e. set $A \subseteq \Sigma^*$ with $\alpha = 2^{-A}$.*
5. *There is a total recursive function $f : \mathbf{N}^2 \to \{0,1\}$ such that*
   (a) *If for some k, n we have $f(k,n) = 1$ and $f(k,n+1) = 0$ then there is an $l < k$ with $f(l,n) = 0$ and $f(l,n+1) = 1$.*

[6] Note that the property of being r.e. depends only on the fractional part of the real number.
[7] Note that the prefix-free r.e. sets $A \subseteq \Sigma^*$ are exactly the domains of self-delimiting Turing machines.

(b) *We have:* $k \in X_\alpha \iff \lim_{n\to\infty} f(k,n) = 1$.

Proof. It is obvious that conditions 1., 2. and 3. are equivalent and that 4. implies 3.

"3. $\Rightarrow$ 4.": We can assume that $0 < a_j < \alpha \leq 1$, for all j. Using the recursive sequence (a_j) of rationals one can construct two recursive sequences (n_i) and (k_j) of positive integers such that $k_j < k_{j+1}$ and $\sum_{i=0}^{k_j} 2^{-n_i} < a_j < 2^{-j} + \sum_{i=0}^{k_j} 2^{-n_i}$ for all j. Obviously $\sum_{i=0}^{\infty} 2^{-n_i} = \alpha$. By the Kraft-Chaitin Theorem (Theorem 3.2) there are a one-to-one recursive sequence (x_i) of words with $|x_i| = n_i$, for all i, and a self-delimiting Turing machine whose domain A is the set $\{x_i \mid i \in \mathbf{N}\}$. We obtain $\alpha = 2^{-A}$.

"5. $\Rightarrow$ 2.": Note that the assumption (a) implies that for each k the sequence $f(k,0), f(k,1), f(k,2), \ldots$ changes its value at most $2^{k+1} - 1$ times (proof by induction over k). Hence the limit $\lim_{n\to\infty} f(k,n)$ exists. From here one we write $f_{k,n}$ for $f(k,n)$. We claim that (a) implies:

$$0.f_{0,n}f_{1,n}\ldots f_{m,n} \leq 0.f_{0,n+1}f_{1,n+1}\ldots f_{m,n+1}, \tag{3}$$

for all m, n. Assume that (3) is not true for some m and some n. Fix this number n and choose m minimal such that (3) is not true. Then, because of $0.f_{0,n}f_{1,n}\ldots f_{m-1,n} \leq 0.f_{0,n+1}f_{1,n+1}\ldots f_{m-1,n+1}$ we must have $f_{m,n} = 1$ and $f_{m,n+1} = 0$. By (a) there is a number $l < m$ with $f_{l,n} = 0$ and $f_{l,n+1} = 1$. Using $0.f_{0,n}f_{1,n}\ldots f_{l-1,n} \leq 0.f_{0,n+1}f_{1,n+1}\ldots f_{l-1,n+1}$ we obtain

$$\begin{aligned}
0.f_{0,n}f_{1,n}\ldots f_{m,n} &= 0.f_{0,n}f_{1,n}\ldots f_{l-1,n}0f_{l+1,n}\ldots f_{m,n}\\
&\leq 0.f_{0,n}f_{1,n}\ldots f_{l-1,n}1\\
&\leq 0.f_{0,n+1}f_{1,n+1}\ldots f_{l-1,n+1}1\\
&\leq 0.f_{0,n+1}f_{1,n+1}\ldots f_{l-1,n+1}1f_{l+1,n+1}\ldots f_{m,n+1}\\
&= 0.f_{0,n+1}f_{1,n+1}\ldots f_{m,n+1}\,.
\end{aligned}$$

Contradiction! Thus, (3) is true for all m, n.

Define a recursive sequence of rationals a_n by $a_n = 0.f_{0,n}f_{1,n}\ldots f_{n,n}$. Then, by (3) $a_n \leq a_{n+1}$ for all n. Let $0.\mathbf{y} = 0.y_0y_1y_2\ldots$ be the binary expansion of α which contains infinitely many ones. By (b), for each K there is a number N_K with $y_k = f_{k,n}$ for all $k \leq K$ and $n \geq N_K$. Hence, $|a_n - \alpha| \leq 2^{-K}$ for all $n \geq \max\{K, N_K\}$. We conclude $\lim_{n\to\infty} = \alpha$. Hence (a_n) is a non-decreasing recursive sequence of rationals converging to α. Thus, α is r.e.

"3. $\Rightarrow$ 5.": Again we can assume that $0 < a_n < \alpha \leq 1$, for all n. Define f such that $0.f_{0,n}f_{1,n}f_{2,n}\ldots$ is the binary expansion of a_k containing infinitely many ones, for each k. Then f is recursive. From $a_n < a_{n+1}$ it follows that f satisfies (a). The equivalence $k \in X_\alpha \iff \lim_{n\to\infty} f(k,n) = 1$ follows from $\lim_{n\to\infty} a_n = \alpha$ and from $a_n < \alpha$ for all n. □

In order to compare the information contents of r.e. reals, Solovay [25] has introduced the following definition.

Definition 4.2 (Solovay [25] and Chaitin [11]) The real α is said to *dominate* the real β if there are a partial recursive function $f : \mathbf{Q} \overset{o}{\to} \mathbf{Q}$ and a constant $c > 0$ with the property that if p is a rational number less than α, then $f(p)$ is (defined and) less than β, and it satisfies the inequality

$$c(\alpha - p) \geq \beta - f(p)\,.$$

In this case we write $\alpha \geq_{dom} \beta$ or $\beta \leq_{dom} \alpha$.

Roughly speaking, a real α dominates a real β if from any good approximation to α from below (say, from a rational number $p < \alpha$ with $\alpha - p < 2^{-n}$) one can effectively obtain a good approximation to β from below (a rational number $f(p) < \beta$ with $\beta - f(p) < 2^{-n+\text{constant}}$). For r.e. reals this can also be expressed as follows.

Lemma 4.3 *An r.e. real α dominates an r.e. real β if and only if there are recursive, non-decreasing sequences (a_i) and (b_i) of rationals and a constant c with $\lim_n a_n = \alpha$, $\lim_n b_n = \beta$, and $c(\alpha - a_n) \geq \beta - b_n$, for all n.*

Proof. If α dominates β and (a_n) is an increasing, recursive sequence of rationals with limit α, we set $b_n = \max\{f(a_0), \ldots, f(a_n)\}$ where f is the function showing $\alpha \geq_{dom} \beta$.

On the other hand, assume that (a_n) and (b_n) are non-decreasing recursive sequences tending to α and β, respectively, and that c is a constant with $c(\alpha - a_n) \geq \beta - b_n$, for all n. We define a partial recursive function $f : \mathbf{Q} \stackrel{o}{\to} \mathbf{Q}$ as follows. Given $p \in \mathbf{Q}$, compute the smallest i such that $a_i \geq p$. If such an i has been found, set $f(p) = b_i$. If $p < \alpha$, then $f(p)$ is defined and is smaller than β. It is clear that this function f show $\beta \leq_{dom} \alpha$. $\Box$

Lemma 4.4 *For every positive integer c we can effectively construct a positive integer N_c such that for every $n \geq 1$ and all strings $x, y \in \Sigma^n$ with $|0.x - 0.y| \leq c \cdot 2^{-n}$ we have*

$$|H(y) - H(x)| \leq N_c.$$

Proof. Given $x \in \Sigma^n$ we notice that the number of strings $y \in \Sigma^n$ such that $|0.y - 0.x| \leq c \cdot 2^{-n}$ is at most $2c + 1$. Indeed, for every $0 \leq i \leq 2^n - 1$, there are at most $2c+1$ positive integers j in the interval $[0, 2^n - 1]$ such that $|i - j| \leq c$. To compute such a y it is sufficient to know the canonical program x^* of x and the position of y in the small set described above. This position can be described by a word of length $2 + [\log c]$. Consequently, there is a constant N_1 depending only on c with $H(y) \leq H(x) + N_1$. By symmetry there is a constant N_2 with $H(x) \leq H(y) + N_2$. Taking the maximum N_c of these two constants gives the assertion. $\Box$

Theorem 4.5 (Solovay [25]) *Let $\mathbf{x}, \mathbf{y} \in \Sigma^\omega$ be two infinite binary sequences such that both $0.\mathbf{x}$ and $0.\mathbf{y}$ are r.e. reals and $0.\mathbf{x} \geq_{dom} 0.\mathbf{y}$. Then*

$$H(\mathbf{y}(n)) \leq H(\mathbf{x}(n)) + O(1).$$

Proof. In view of the fact that $0.\mathbf{x} \geq_{dom} 0.\mathbf{y}$, there is a constant $c \in \mathbf{N}$ such that, for every $n \in \mathbf{N}$, given $\mathbf{x}(n)$, we can find, in an effective manner, a rational $p_n \leq 0.\mathbf{y}$ satisfying

$$\frac{c}{2^{n+1}} \geq c(0.\mathbf{x} - 0.\mathbf{x}(n)) \geq 0.\mathbf{y} - p_n > 0.$$

Let z_{p_n} be the first $n+1$ digits of the binary expansion of p_n. Then

$$|0.\mathbf{y}(n) - 0.z_{p_n}| \leq \frac{c+1}{2^{n+1}}.$$

Hence, by Lemma 4.4,

$$H(\mathbf{y}(n)) \leq H(z_{p_n}) + O(1) \leq H(\mathbf{x}(n)) + O(1). \qquad \Box$$

Next, we prove a few results which will be useful in discussing the lattice structure of r.e. reals under $\leq_{dom}$.

Lemma 4.6 *Let α, β and γ be r.e. reals. Then the following conditions hold:*

1. *The relation $\geq_{dom}$ is reflexive and transitive.*
2. *For every α, β one has $\alpha + \beta \geq_{dom} \alpha$.*
3. *If $\gamma \geq_{dom} \alpha$ and $\gamma \geq_{dom} \beta$, then $\gamma \geq_{dom} \alpha + \beta$.*
4. *For every non-negative α and positive β one has $\alpha \cdot \beta \geq_{dom} \alpha$.*
5. *If α and β are non-negative, and $\gamma \geq_{dom} \alpha$ and $\gamma \geq_{dom} \beta$, then $\gamma \geq_{dom} \alpha \cdot \beta$.*

Proof. 1. This is straightforward from the definition.

2. For each rational number $p < \alpha + \beta$, we can compute two rational numbers p_1, p_2 such that $p_1 < \alpha$, $p_2 < \beta$ and $p_1 + p_2 \geq p$ because α and β are r.e. reals. Now $\alpha + \beta - p \geq \alpha + \beta - p_1 - p_2 > \alpha - p_1$. Hence $\alpha + \beta \geq_{dom} \alpha$.

3. Let c be a constant such that for each rational number $p < \gamma$ we can find – in an effective manner – two rational numbers $p_1 < \alpha$ and $p_2 < \beta$ satisfying $c(\gamma - p) \geq \alpha - p_1$ and $c(\gamma - p) \geq \beta - p_2$. Then $2c(\gamma - p) \geq \alpha - p_1 + \beta - p_2 = \alpha + \beta - (p_1 + p_2)$.

4. Given a rational $p < \alpha\beta$ we can compute two rationals $p_1 < \alpha$ and $p_2 < \beta$ such that $p_1 p_2 \geq p$. For $c = 1/\beta$ we obtain:

$$c(\alpha\beta - p) \geq c(\alpha\beta - p_1 p_2) \geq c(\alpha\beta - p_1\beta) = \alpha - p_1 .$$

5. Assume that c is a constant such that, given a rational $p < \gamma$, we can find rationals $p_1 < \alpha$ and $p_2 < \beta$ satisfying $c(\gamma - p) \geq \alpha - p_1$ and $c(\gamma - p) \geq \beta - p_2$. With $\tilde{c} = c \cdot (\alpha + \beta)$ we obtain:

$$\begin{aligned} \alpha\beta - p_1 p_2 &= \alpha(\beta - p_2) + p_2(\alpha - p_1) \\ &\leq (\alpha + p_2)c(\gamma - p) \\ &\leq (\alpha + \beta)c(\gamma - p) \\ &= \tilde{c}(\gamma - p) . \end{aligned}$$

□

Corollary 4.7 *The sum of a random r.e. real and an r.e. real is a random r.e. real. The product of a positive random r.e real with a positive r.e. real is a random r.e. real.*

Proof. This follows from Lemma 4.6 and Theorem 4.5. □

Corollary 4.8 *The class of random r.e. reals is closed under addition. The class of positive random r.e. reals is closed under multiplication.*

The last corollary contrasts with the fact that addition and multiplication do not preserve randomness. For example, if α is a random number, then $1 - \alpha$ is random as well, but $\alpha + (1 - \alpha) = 1$ is not random.

5 More about Domination

In the following we discuss the lattice structure of the r.e. reals under the domination relation.

For every infinite sequence $\mathbf{x} \in \Sigma^\omega$ such that $0.\mathbf{x}$ is an r.e. real, let $A_\mathbf{x} = \{v \in \Sigma^* \mid 0.v \leq 0.\mathbf{x}\}$ and $A^{\#}_\mathbf{x} = \{string_n \mid x_n = 1\}$. Then, obviously, $A_\mathbf{x}$ is an r.e set which is Turing equivalent to $A^{\#}_\mathbf{x}$.[8] In the following, we establish the relationship between

[8] Note that $A^{\#}_\mathbf{x}$ is not necessarily an r.e. set.

domination and Turing reducibility.

Lemma 5.1 *Let $\mathbf{x}, \mathbf{y} \in \Sigma^\omega$ be two infinite binary sequences such that both $0.\mathbf{x}$ and $0.\mathbf{y}$ are r.e. reals and $0.\mathbf{x} \geq_{dom} 0.\mathbf{y}$. Then $A_{\mathbf{y}} \leq_T A_{\mathbf{x}}$.*

Proof. Without loss of generality, we may assume that

$$\mathbf{y} \notin \{x0000\cdots, x1111\cdots \mid x \in \Sigma^*\}. \tag{4}$$

Let $f : \Sigma^* \overset{o}{\to} \Sigma^*$ be a partial recursive function and $c \in \mathbf{N}$ a constant satisfying the following inequality for all $n > 0$:

$$0 < 0.\mathbf{y} - 0.f(\mathbf{x}(n-1)) \leq \frac{c}{2^n}.$$

Given a word z we wish to decide whether $z \in A_{\mathbf{y}}$. Using the oracle $A^{\#}_{\mathbf{x}}$ we compute the least $i \geq 0$ such that either

$$0.f(\mathbf{x}(i-1)) \geq 0.z \quad \text{or} \quad 0.z - 0.f(\mathbf{x}(i-1)) > \frac{c}{2^i}.$$

Such an i must exist in view of (4). Finally, if $0.f(\mathbf{x}(i-1)) \geq 0.z$, then $z \in A_{\mathbf{y}}$; otherwise $z \notin A_{\mathbf{y}}$. □

Does the converse of Lemma 5.1 hold true? A negative answer will be given in Corollary 6.11.

For two reals α and β, $\alpha =_{dom} \beta$ denotes the conjunction $\alpha \geq_{dom} \beta$ and $\beta \geq_{dom} \alpha$. For a real α, let $[\alpha] = \{\beta \in \mathbf{R} \mid \alpha =_{dom} \beta\}$; $\mathbf{R}_{r.e.} = \{[\alpha] \mid \alpha \text{ is an r.e. real}\}$. Finally, let $\langle RE\,; \leq_T\rangle$ denote the upper semi-lattice structure of the class of r.e. sets under the Turing reducibility.

Definition 5.2 A *strong homomorphism* from a partially ordered set $(X, \leq)$ to another partially ordered set $(Y, \leq)$ is a mapping $h : X \to Y$ such that

1. For all $x, x' \in X$, if $x \leq x'$, then $h(x) \leq h(x')$.
2. For all $y, y' \in Y$, if $y \leq y'$, then there exist x, x' in X such that $x \leq x'$ and $h(x) = y$, $h(x') = y'$.

Theorem 5.3 *The structure $\langle \mathbf{R}_{r.e.}\,; \leq_{dom}\rangle$ is an upper semi-lattice. There is a strong homomorphism from $\langle \mathbf{R}_{r.e.}\,; \leq_{dom}\rangle$ onto $\langle RE\,; \leq_T\rangle$.*

Proof. By Lemma 4.6 the structure $\langle \mathbf{R}_{r.e.}\,; \leq_{dom}\rangle$ is an upper semi-lattice. Every $=_{dom}$-equivalence class of r.e. reals contains an r.e. real of the form $0.\mathbf{x}$. Lemma 5.1 shows that by $0.\mathbf{x} \mapsto A_{\mathbf{x}}$ one defines a mapping from $\langle \mathbf{R}_{r.e.}\,; \leq_{dom}\rangle$ to $\langle RE\,; \leq_T\rangle$, which satisfies the first condition in the definition of a strong homomorphism. We have to show that this mapping satisfies also the second condition. Let $B, C \subseteq \Sigma^*$ be two r.e. sets with $C \leq_T B$. We have to show that there are two r.e. reals $0.\mathbf{x}$ and $0.\mathbf{y}$ with the following three properties: (I) $0.\mathbf{x}$ dominates $0.\mathbf{y}$, (II) $A_{\mathbf{x}}$ is Turing equivalent to B, and (III) $A_{\mathbf{y}}$ is Turing equivalent to C.

We can assume that the set B and C have the form $B = \{string_n \mid n \in \tilde{B}\}$ and $C = \{string_n \mid n \in \tilde{C}\}$ where $\tilde{B}$ is an r.e. set of odd natural numbers and $\tilde{C}$ is an r.e. set of even natural numbers. Then the set $D = B \cup C$ is Turing equivalent to B. We define two sequences $\mathbf{x}, \mathbf{y} \in \Sigma^\omega$ by $\mathbf{x} = \chi_D$ and $\mathbf{y} = \chi_C$. The real numbers $0.\mathbf{x}$ and $0.\mathbf{y}$ are r.e. They have the properties (II) and (III) because $A_\mathbf{x}$ is Turing equivalent to $A^\#_\mathbf{x} = D$, which is Turing equivalent to B, and $A_\mathbf{y}$ is Turing equivalent to $A^\#_\mathbf{y} = C$. It is left to show that $0.\mathbf{x}$ dominates $0.\mathbf{y}$. Let

$$b_0, b_1, b_2, \ldots \quad \text{and} \quad c_0, c_1, c_2, \ldots$$

be one-to-one recursive enumerations of $\tilde{B}$ and of $\tilde{C}$. The rational sequences

$$\left(\sum_{i=0}^{n}(2^{-b_i} + 2^{-c_i})\right)_{n\geq 0} \quad \text{and} \quad \left(\sum_{i=0}^{n} 2^{-c_i}\right)_{n\geq 0}$$

are increasing, recursive, converge to $0.\mathbf{x}$ and to $0.\mathbf{y}$, respectively, and satisfy the inequality:

$$0.\mathbf{x} - \sum_{i=0}^{n}(2^{-b_i} + 2^{-c_i}) \geq 0.\mathbf{y} - \sum_{i=0}^{n} 2^{-c_i}\,.$$

Hence, by Lemma 4.3, the number $0.\mathbf{x}$ dominates $0.\mathbf{y}$. □

We continue with the characterization of the domination relation between r.e. real numbers in terms of prefix-free r.e. sets of words. We consider only infinite prefix-free r.e. sets. By *R.E.* we denote the class of all infinite prefix-free r.e. subsets of Σ^*. First, we consider a relation between r.e. sets which is very close to the domination relation, but will turn out to be not equivalent.

Definition 5.4 Let $A, B \in R.E.$ The set A *strongly simulates* B (shortly, $B \leq_{ss} A$) if there is a partial recursive function $f : \Sigma^* \overset{o}{\to} \Sigma^*$ which satisfies the following conditions:

1. $A \subseteq dom(f)$ and $B = f(A)$,
2. there is a constant $c > 0$ such that $|x| \leq |f(x)| + c$, for all $x \in A$.

If $A \leq_{ss} B$ and $B \leq_{ss} A$, then we say that A and B are $\sim_{ss}$*-equivalent.*

The following lemma follows immediately from the definition.

Lemma 5.5 *The relation $\leq_{ss}$ is reflexive and transitive.*

Hence, the relation $\leq_{ss}$ defines a partially ordered set $\langle R.E._{\cdot ss}, \leq_{ss}\rangle$ where $R.E._{\cdot ss}$ is the set of $\sim_{ss}$-equivalence classes of $R.E.$ Our next goal is to see how the strong simulation relation $\leq_{ss}$ and $\leq_{dom}$ are related.

Lemma 5.6 *If A, B are infinite prefix-free r.e. sets and $B \leq_{ss} A$, then 2^{-A} dominates 2^{-B}.*

Proof. Let (x_i) be a one-to-one recursive enumeration of A. Let f be a function and $c > 0$ be a constant as in Definition 5.4. For each n and each $y \in B \setminus \{f(x_0), \ldots, f(x_n)\}$ there is a word $x \in A \setminus \{x_0, \ldots, x_n\}$ with $y = f(x)$ and $|x| \leq |f(x)| + c$. Hence,

$$\begin{aligned} 2^{-B} - 2^{-\{f(x_0),\ldots,f(x_n)\}} &= 2^{-(B\setminus\{f(x_0),\ldots,f(x_n)\})} \\ &\leq 2^c \cdot 2^{-(A\setminus\{x_0,\ldots,x_n\})} \\ &= 2^c \cdot (2^{-A} - 2^{-\{x_0,\ldots,x_n\}})\,. \end{aligned}$$

We conclude that 2^{-A} dominates 2^{-B}. $\square$

The next result shows that in some sense the converse implication in Lemma 5.6 is true as well. It will also be important in the following section.

Theorem 5.7 *Let α be an r.e. real in the interval $(0,1]$, and B be an infinite prefix-free r.e. set. If α dominates 2^{-B}, then there is an infinite prefix-free r.e. set A with $\alpha = 2^{-A}$ and $B \leq_{ss} A$.*

Proof. We assume that α dominates 2^{-B} and wish to construct an infinite prefix-free r.e. set A with $\alpha = 2^{-A}$. Let (y_i) be a one-to-one recursive enumeration of B and (a_n) be an increasing recursive sequence of rationals converging to α. In view of the domination property of α, there are an increasing, total recursive function $f : \mathbf{N} \to \mathbf{N}$ and a constant $c \in \mathbf{N}$ such that, for each $n \in \mathbf{N}$,

$$2^c \cdot (\alpha - a_n) \geq 2^{-B} - \sum_{i=0}^{f(n)} 2^{-|y_i|} . \tag{5}$$

Without loss of generality, we may assume that

$$a_0 \geq \sum_{i=0}^{f(0)} 2^{-|y_i|-c} \tag{6}$$

(otherwise we may take a large enough c). We construct a recursive sequence $(n_i)_{i\geq 0}$ of numbers and a recursive double sequence $(m_{i,j})_{i,j\geq 0}$ of elements in $\mathbf{N} \cup \{\infty\}$. These numbers n_i and the numbers $m_{i,j} \neq \infty$ will be the lengths of the words in the set A which we wish to construct. The numbers n_i serve in order to guarantee that $B \leq_{ss} A$. The numbers $m_{i,j}$ are used "to fill" the set A up in order to get exactly $\alpha = 2^{-A}$. This will follow directly from Equation (7) below.

Construction of (n_i): We define $n_i = |y_i| + c$, for all i.

Begin of construction of $(m_{i,j})$.

Stage 0. Let $m_{i,j} = \infty$, for all $i \leq f(0)$ and $j \in \mathbf{N}$.

Stage s $(s \geq 1)$. If

$$a_s \leq \sum_{i=0}^{f(s)} 2^{-n_i} + \sum_{i=0}^{f(s-1)} \sum_{j=0}^{\infty} 2^{-m_{i,j}} ,$$

then let $m_{i,j} = \infty$, for all i with $f(s-1) < i \leq f(s)$ and $j \in \mathbf{N}$. Otherwise, let $m_{i,j} = \infty$, for all i with $f(s-1) < i < f(s)$ and $j \in \mathbf{N}$, and let $(m_{f(s),j})_{j\in\mathbf{N}}$ be recursively defined in such a way that

$$\sum_{j=0}^{\infty} 2^{-m_{f(s),j}} = a_s - \left(\sum_{i=0}^{f(s)} 2^{-n_i} + \sum_{i=0}^{f(s-1)} \sum_{j=0}^{\infty} 2^{-m_{i,j}} \right) .$$

End of construction of $(m_{i,j})$.

First, we prove the following equation:

$$\alpha = \sum_{i=0}^{\infty}\left(2^{-n_i} + \sum_{j=0}^{\infty} 2^{-m_{i,j}}\right). \tag{7}$$

For the proof, we distinguish the following two cases.

Case 1. There are infinitely many stages s such that $a_s = \sum_{i=0}^{f(s)}\left(2^{-n_i} + \sum_{j=0}^{\infty} 2^{-m_{i,j}}\right)$. For this case, it is straightforward that the equation (7) holds.

Case 2. The inequality $a_s < \sum_{i=0}^{f(s)}\left(2^{-n_i} + \sum_{j=0}^{\infty} 2^{-m_{i,j}}\right)$ holds true for almost all $s \in \mathbf{N}$. On the one hand,

$$\alpha = \lim_{s\to\infty} a_s \le \sum_{i=0}^{\infty}\left(2^{-n_i} + \sum_{j=0}^{\infty} 2^{-m_{i,j}}\right). \tag{8}$$

For the inverse estimate, define s_0 to be the largest stage such that $a_{s_0} = \sum_{i=0}^{f(s_0)}\left(2^{-n_i} + \sum_{j=0}^{\infty} 2^{-m_{i,j}}\right)$. Such a stage s_0 exists because of (6). By (5) we have

$$\alpha - a_{s_0} \ge \sum_{i=f(s_0)+1}^{\infty} 2^{-|y_i|-c}.$$

Hence, by the construction,

$$\alpha \ge \sum_{i=0}^{\infty}\left(2^{-n_i} + \sum_{j=0}^{\infty} 2^{-m_{i,j}}\right). \tag{9}$$

By combining (8) and (9) we obtain the equality (7) also in this case.

Let $h : \mathbf{N} \to \{(i,j) \in \mathbf{N}^2 \mid m_{i,j} \neq \infty\}$ be a recursive bijection and define a recursive sequence (m'_i) of numbers by $m'_i = m_{h(i)}$. Using this sequence we define (n'_i) by $n'_{2i} = n_i$ and $n'_{2i+1} = m'_i$. By Theorem 3.2 and (7), combined with $0 < \alpha < 1$, we can construct a one-to-one recursive sequence (x_i) of words with $|x_i| = n'_i$ such that the set $\{x_i \mid i \in \mathbf{N}\}$ is prefix-free. We set $A = \{x_i \mid i \in \mathbf{N}\}$ and, using (7), obtain

$$2^{-A} = \sum_{i=0}^{\infty} 2^{-n'_i} = \sum_{i=0}^{\infty} 2^{-n_i} + \sum_{i=0}^{\infty} 2^{-m'_i} = \alpha\,.$$

Finally we define a recursive function $g : A \to B$ by $g(x_{2i}) = y_i$ and such that $|g(x_{2i+1})| \ge |x_{2i+1}|$, for all i. This is possible because B is infinite. Obviously, $g(A) = B$, and $|x| \le |g(x)| + c$, for all $x \in A$. This shows $B \le_{ss} A$. □

Theorem 5.8 *The mapping h defined by $h(A) = 2^{-A}$ is a strong homomorphism from $\langle R.E._{ss}, \le_{ss}\rangle$ onto $\langle \mathbf{R}_{r.e.}, \le_{dom}\rangle$.*

Proof. By Lemma 5.6 and Theorem 5.7. □

The next result shows that h cannot be one-to-one.

Theorem 5.9 *There exist infinite prefix-free r.e. sets A and B with $2^{-A} = 2^{-B} = 1$ but $A \not\le_{ss} B$ and $B \not\le_{ss} A$.*

Proof. We define two sequences (n_i) and (m_i) of natural numbers by

$$n_0 = 0, \quad m_i = 2^{n_i}, \quad \text{and} \quad n_{i+1} = 2^{m_i},$$

for all i. Let D_i be the set of all words of length $n_i + 1$ with prefix 0^i1, and let E_i be a set of all words of length $m_i + 1$ with prefix 0^i1. Then $|D_i| = 2^{n_i - i}$ and $|E_i| = 2^{m_i - i}$. Define $A = \bigcup_i D_i$ and $B = \bigcup_i E_i$. Both sets A and B are obviously recursive and prefix-free and satisfy $2^{-A} = 2^{-B} = 1$. We have to show that neither A strongly simulates B nor B strongly simulates A. If A would strongly simulate B there would be a surjective mapping from A to B satisfying the second condition in Definition 5.4, that is a mapping which does not map long words to short words. We show that this is impossible. Namely, A contains at most $\sum_{j=0}^{i} 2^{n_j - j} \leq 2^{n_i}$ words of length less than $n_{i+1} + 1 = 2^{2^{n_i}} + 1$ while B contains $2^{m_i - i} = 2^{2^{n_i} - i}$ words of length $m_i + 1 = 2^{n_i} + 1$. For large n_i — and the sequence n_i is unbounded — this contradicts $B \leq_{ss} A$. On the other hand, A contains $2^{n_{i+1} - i - 1} = 2^{2^{m_i} - i - 1}$ words of length $n_{i+1} + 1 = 2^{m_i} + 1$ while B contains at most $\sum_{j=0}^{i} 2^{m_j - j} \leq 2^{m_i}$ words of length less than $m_{i+1} + 1 = 2^{2^{m_i}} + 1$. This rules out the relation $A \leq_{ss} B$. □

However, by relaxing the strong simulation relation one can characterize the domination relation by a simulation relation between prefix-free r.e. sets. A sequence $D_0, D_1, D_2, \ldots$ of finite sets in Σ^* is called a *strong array* (Soare [23]) if there is a total recursive function g such that with respect to a standard bijection ν from $\mathbf{N}$ onto the set of all finite subsets of Σ^* we have $D_i = \nu(g(i))$ for all i.

Definition 5.10 An equivalence relation on an infinite, r.e. set A is called an *effective, finite partition* if each equivalence class is finite, and there is a strong array $D_0, D_1, D_2, \ldots$ of finite, pairwise disjoint sets such that each D_i is an equivalence class.

An example of an effective, finite partition is the partition whose equivalence classes contain only one element: if $a_0, a_1, \ldots$ is a one-to-one enumeration of A one sets $D_i = \{a_i\}$.

Definition 5.11 Let A and B be infinite, prefix-free, r.e. sets. We say that A *simulates* B if there are two effective, finite partitions (D_i) of A and (E_i) of B, respectively, and a constant $c > 0$ such that for all i:

$$c \cdot (2^{-D_i}) \geq 2^{-E_i}.$$

We are ready to characterize the $\leq_{dom}$-relation in terms of the simulation relation between sets. We remark that the following theorem is true also if the definition of an effective finite partition is changed so that the sets D_i may either be empty or equivalence classes.

Theorem 5.12 *Let $A, B \subseteq \Sigma^*$ be infinite prefix-free r.e. sets. Then A simulates B if and only if 2^{-A} dominates 2^{-B}.*

Proof. Assume that A simulates B. Let $D_0, D_1, D_2, \ldots$ be an effective list of equivalence classes on A, and $E_0, E_1, E_2, \ldots$ be a corresponding effective list of equivalence classes on B, $c > 0$ an appropriate constant. The rational sequences

$$\left(\sum_{i \leq n} 2^{-D_i}\right)_{n \geq 0} \quad \text{and} \quad \left(\sum_{i \leq n} 2^{-E_i}\right)_{n \geq 0}$$

are recursive, non-decreasing, and converge to 2^{-A} and to 2^{-B}, respectively. Furthermore we have

$$2^{-B} - \sum_{i\le n} 2^{-E_i} = \sum_{i>n} 2^{-E_i} \le c\cdot(\sum_{i>n} 2^{-D_i}) = c\cdot(2^{-A} - \sum_{i\le n} 2^{-D_i}).$$

This shows that 2^{-A} dominates 2^{-B}.

Now assume that 2^{-A} dominates 2^{-B}. That is, there are a constant $c > 0$ and a partial recursive function $f : \mathbf{Q} \overset{o}{\to} \mathbf{Q}$ such that for all rationals $q < 2^{-A}$ the number $f(q)$ is defined, smaller than 2^{-B} and satisfies the inequality $c(2^{-A} - q) \ge 2^{-B} - f(q)$. Since we can increase c we can assume that actually $c\cdot 2^{-A} > 2^{-B}$ and $c(2^{-A}-q) > 2^{-B} - f(q)$, for all rationals $q < 2^{-A}$. Let

$$x_0, x_1, x_2, \ldots \quad \text{and} \quad y_0, y_1, y_2, \ldots$$

be one-to-one recursive enumerations of A and B, respectively. Using f we see that there is a total recursive, increasing function $g : \mathbf{N} \to \mathbf{N}$ satisfying the inequality:

$$c\cdot(2^{-A} - \sum_{i\le m} 2^{-|x_i|}) > 2^{-B} - \sum_{i\le g(m)} 2^{-|y_i|},$$

for all m. We define a total recursive, increasing function $h : \mathbf{N}\cup\{-1\} \to \mathbf{N}\cup\{-1\}$ recursively by $h(-1) = -1$ and

$$h(n+1) = \min\{k > h(n) \mid c\cdot(\sum_{h(n)<i\le k} 2^{-|x_i|}) \ge \sum_{g(h(n))<i\le g(k)} 2^{-|y_i|}\},$$

for all $n \ge -1$ (where we assume $g(-1) = -1$). The function h is well-defined since for each $m \ge -1$ we have

$$c\cdot(\sum_{m<i} 2^{-|x_i|}) > \sum_{g(m)<i} 2^{-|y_i|}.$$

We set

$$D_i = \{x_{h(i-1)+1}, \ldots, x_{h(i)}\} \quad \text{and} \quad E_i = \{y_{g(h(i-1))+1}, \ldots, y_{g(h(i))}\}.$$

Then the sequence (D_i) is an effective finite partition of A, the sequence (E_i) is an effective finite partition of B, and we have

$$c\cdot 2^{-D_i} = c\cdot \sum_{h(i-1)<j\le h(i)} 2^{-|x_j|} \ge \sum_{gh(i-1)<j\le gh(i)} 2^{-|y_j|} = 2^{-E_i},$$

which shows that A simulates B. □

6 Random R.E. Reals and Ω-like Reals

In this section, we study random r.e. reals and Ω-like reals, which were introduced by Solovay [25]. Chaitin [11] has given a slightly different definition. We show that Chaitin Ω numbers, Solovay's Ω-like reals and Chaitin's Ω-like reals are all the same. Then we answer the question raised after Lemma 5.1. Furthermore we give an example of a random number α in Δ_2 such that neither α nor $1-\alpha$ is an r.e. real. Finally we address the question whether Ω is also maximally useful if one whishes to compute not only an approximation of an r.e. real but the digits of its binary representation. We start with Chaitin's definition of Ω-like reals.

Definition 6.1 (Chaitin [11]) An r.e. real α is called *Ω-like* if it dominates all r.e. reals.

Solovay's original manuscript [25] contains the following definition.

Definition 6.2 (Solovay [25]) A recursive, increasing, and converging sequence (a_i) of rationals (a_i) is called *universal* if for every recursive, increasing and converging sequence (b_i) of rationals there exists a number $c > 0$ such that

$$c \cdot (\alpha - a_n) \geq \beta - b_n$$

for all n, where $\alpha = \lim_n a_n$ and $\beta = \lim_n b_n$.

Solovay called a real α Ω-like if it is the limit of a universal recursive, increasing sequence of rationals. We shall see that both definitions are equivalent. One implication is very easy.

Lemma 6.3 *If a real α is the limit of a universal recursive, increasing sequence of rationals, then it is Ω-like.*

Proof. This follows immediately from Theorem 4.1 and Lemma 4.3. □

By modifying slightly a proof of Solovay [25] we obtain the following

Theorem 6.4 *Let U be a universal machine. Every recursive, increasing sequence of rationals converging to Ω_U is universal.*

Proof. Let (a_n) be an increasing, recursive sequence of rationals with limit Ω_U, and let (b_n) be an increasing, recursive, converging sequence of rationals. Set $\beta = \lim_{n\to\infty} b_n$. We have to show that there is a constant $c > 0$ with $c(\Omega_U - a_n) \geq \beta - b_n$ for all n. Without loss of generality, we may assume that $0 < b_n < \beta < 1$, for all $n \in \mathbf{N}$.

Let (x_i) be a one-to-one, recursive enumeration of $PROG_U$, and $\Omega_{U,n} = \sum_{i=0}^{n} 2^{-|x_i|}$. We define a total recursive, increasing function $g : \mathbf{N} \cup \{-1\} \to \mathbf{N} \cup \{-1\}$ by $g(-1) = -1$ and

$$g(n) = \min\{j > g(n-1) \mid \Omega_{U,j} \geq a_n\}\,.$$

The sequence $(\Omega_{U,g(n)})$ is an increasing, recursive sequence with limit Ω_U. In view of the inequality

$$\Omega_U - a_n \geq \Omega_U - \Omega_{U,g(n)}$$

it is sufficient to prove that there is a constant $c > 0$ with $c(\Omega_U - \Omega_{U,g(n)}) \geq \beta - b_n$ for all n.

For each $i \in \mathbf{N}$, let y_i be the first string (with respect to the quasi-lexicographical ordering) which is not in the set $\{U(x_j) \mid j \leq g(i)\} \cup \{y_j \mid j < i\}$. Furthermore, put $n_i = [-\log(b_{i+1} - b_i)] + 1$. Since $\sum_{i=0}^{\infty} 2^{-n_i} \leq \beta - b_0 < 1$, by Theorem 3.2 we can construct a Turing machine M such that, for every $i \in \mathbf{N}$, there is a string $u_i \in \Sigma^{n_i}$ satisfying $M(u_i) = y_i$. Hence, there is a constant c_M such that $H_U(y_i) \leq n_i + c_M$. In view of the choice of y_i, there is a string $x'_i \in PROG_U \setminus \{x_j \mid j \leq g(i)\}$ such that $|x'_i| \leq n_i + c_M$ and

$U(x_i') = y_i$. For different i and j we have $y_i \neq y_j$, whence $x_i' \neq x_j'$. We obtain

$$\begin{aligned}
\Omega_U - \Omega_{U,g(n)} &= \textstyle\sum_{i=g(n)+1}^{\infty} 2^{-|x_i|} \\
&\geq \textstyle\sum_{i=n}^{\infty} 2^{-|x_i'|} \\
&\geq \textstyle\sum_{i=n}^{\infty} 2^{-n_i - c_M} \\
&\geq 2^{-c_M - 1} \textstyle\sum_{i=n}^{\infty} (b_{i+1} - b_i) \\
&= 2^{-c_M - 1}(\beta - b_n),
\end{aligned}$$

which proves the assertion. □

Thus, every Chaitin Ω number is Ω-like in Solovay's sense. The converse of Theorem 6.4 holds true even for Ω-like numbers in Chaitin's sense.

Theorem 6.5 *Let $0 < \alpha < 1$ be an Ω-like real. Then there exists a universal machine U such that $\Omega_U = \alpha$.*

Proof. Let V be a universal machine. Since α is Ω-like it dominates 2^{-PROG_V}. By Theorem 5.7 there exist a prefix-free r.e. set A with $2^{-A} = \alpha$, a recursive function $f : A \to PROG_V$ with $A = dom(f)$ and $f(A) = PROG_V$, and a constant $c > 0$ with $|x| \leq |f(x)| + c$, for all $x \in A$. We define a machine U by $U(x) = V(f(x))$. The universality of V implies that also U is universal. We have $\alpha = 2^{-A} = 2^{-PROG_U} = \Omega_U$. □

The following theorem summarizes our description of Ω-like numbers.

Theorem 6.6 *Let $0 < \alpha < 1$ be an r.e. real. The following conditions are equivalent:*

1. *For some universal Turing machine U, $\alpha = \Omega_U$.*
2. *The real α is Ω-like.*
3. *There exists a universal recursive, increasing sequence of rationals converging to α.*
4. *Every recursive, increasing sequence of rationals with limit α is universal.*

Proof. This follows from Lemma 6.3, Theorem 6.4, and Theorem 6.5. □

The following result was proved by Solovay [25] for Ω-like numbers.

Corollary 6.7 *Let U and V be two universal machines. Then $H(\Omega_U(n)) = H(\Omega_V(n)) + O(1)$, for all $n \in \mathbf{N}$.*

Proof. This follows from Theorem 4.5 and Theorem 6.6. □

In analogy with Corollary 4.7 we obtain

Corollary 6.8 *The fractional part of the sum of an Ω number and an r.e. real is an Ω number. The fractional part of the product of an Ω number with a positive r.e. real is an Ω number. Especially, the fractional parts of the sum and product of two Ω numbers are again Ω numbers.*

Proof. This follows from Lemma 4.6 and Theorem 6.6. □

Corollary 6.9 (Solovay [25]) *Every Ω-like real is random.*

Proof. This follows from Theorem 3.4 and Theorem 6.5. □

Corollary 6.10 *The real $0.\chi_K$ is not Ω-like.*

Proof. It is well known that χ_K is not random, whence the corollary follows from Corollary 6.9. □

Now we can answer the question raised after Lemma 5.1. The sets A_Ω and A_{χ_K} are defined as before Lemma 5.1.

Corollary 6.11 *The following statements hold:*

1. $0.\chi_K \not\geq_{dom} \Omega$,
2. $A_\Omega =_T A_{\chi_K} =_T K$.

Proof. The first claim follows from Corollary 6.10. $A_\Omega \leq_T A_{\chi_K} =_T K$ is clear and $A_{\chi_K} \leq_T A_\Omega$ follows from Lemma 5.1. □

Let Δ_2 be the class of sets which are recursive in K. Then all Ω-like reals[9] are in Δ_2. Now one may ask whether there exists a random real in Δ_2 which is not in the set $\{\alpha, 1-\alpha \mid \alpha \text{ is } \Omega\text{-like}\}$? We give a positive answer to this question.

Theorem 6.12 *There is a random sequence $\mathbf{y}$ with $A^{\#}_{\mathbf{y}} \in \Delta_2$ such that neither $0.\mathbf{y}$ nor $1 - 0.\mathbf{y}$ is an r.e. real.*

Proof. Let $\mathbf{x} = x_0x_1x_2\ldots$ be an infinite binary sequence such that $0.\mathbf{x}$ is Ω-like. We define an infinite binary sequence $\mathbf{y} = y_0y_1y_2\ldots$ by letting

$$y_i = \begin{cases} x_i, & \text{if } i \leq 1, \\ x_{i+3^n}, & \text{if } 3^n < i \leq 2\cdot 3^n, \\ x_{i-3^n}, & \text{if } 2\cdot 3^n < i \leq 3^{n+1}. \end{cases}$$

The sequence $\mathbf{y}$ is obtained by recursively re-ordering the digits of the sequence $\mathbf{x}$. Hence, also $\mathbf{y}$ is a random sequence in Δ_2. Next we show that neither $0.\mathbf{y}$ nor $1 - 0.\mathbf{y}$ is an r.e. real. In fact, we show more:

$$0.\mathbf{x} \not\geq_{dom} 0.\mathbf{y} \quad \text{and} \quad 0.\mathbf{x} \not\geq_{dom} 1 - 0.\mathbf{y}\,. \tag{10}$$

By symmetry, it suffices to show that $0.\mathbf{x}$ does not dominate $0.\mathbf{y}$. For the sake of a contradiction, assume that $0.\mathbf{x} \geq_{dom} 0.\mathbf{y}$. Then, by Theorem 4.5,

$$H(\mathbf{y}(2\cdot 3^n)) \leq H(\mathbf{x}(2\cdot 3^n)) + O(1),$$

and hence, by the definition of $\mathbf{y}$ we obtain

$$H(\mathbf{x}(3^{n+1})) \leq H(\mathbf{y}(3^{n+1})) + O(1) \leq H(\mathbf{x}(2\cdot 3^n)) + O(1),$$

[9] Note that here we identify a real $0.\mathbf{x}$ with the set $A^{\#}_{\mathbf{x}}$.

for all $n \in \mathbf{N}$. That is,

$$H(\mathbf{x}(3^{n+1})) \leq 2 \cdot 3^n + H(string_{2\cdot 3^n}) + O(1).$$

Since $\lim_{n\to\infty}(3^{n+1} - 2 \cdot 3^n - H(string_{2\cdot 3^n})) = \infty$ the sequence $\mathbf{x}$ is not random by Theorem 3.3. This contradicts the fact that $0.\mathbf{x}$ is Ω-like. We have proved (10). By Definition 6.1 we conclude that neither $0.\mathbf{y}$ nor $1 - 0.\mathbf{y}$ is an r.e. real. $\square$

In a sense, compared with a non-Ω-like r.e. real, an Ω-like real number either contains more information or at least its information is structured in a more useful way because we can find a good approximation from below to any r.e. real from a good approximation from below to any fixed Ω-like real. Sometimes we wish to compute not just an arbitrary approximation (say, of precision 2^{-n}) from below to an r.e. real, instead, we wish to compute a special approximation, namely the first n digits of its binary expansion. Is the information in Ω organized in such a way as to guarantee that for any r.e. real α there exists a total recursive function $g : \mathbf{N} \to \mathbf{N}$ (depending upon α) such that from the first $g(n)$ digits of Ω we can actually compute the first n digits of α? We show that the answer to this question is negative if one demands that the computation is done by a total recursive function.

For two infinite sequences $\mathbf{x}, \mathbf{y} \in \Sigma^\omega$ we write $0.\mathbf{x} \leq_{tt} 0.\mathbf{y}$ in case $A^{\#}_{\mathbf{x}} \leq_{tt} A^{\#}_{\mathbf{y}}$. It is easy to see that this can also be expressed as follows: $0.\mathbf{x} \leq_{tt} 0.\mathbf{y}$ if and only if there are two total recursive functions $g : \mathbf{N} \to \mathbf{N}$ and $F : \Sigma^* \to \Sigma^*$ with $\mathbf{x}(n) = F(\mathbf{y}(g(n)))$ for all n. This preorder $\leq_{tt}$ has a maximum among the r.e. reals, but this maximum is not Ω, as no random r.e. real is maximal.

Theorem 6.13 *The following statements hold:*

1. *For every r.e. real α, $\alpha \leq_{tt} 0.\chi_K$.*

2. *$0.\chi_K \not\leq_{tt} \Omega$.*

Proof. For the first assertion observe that for an arbitrary r.e. real $0.\mathbf{x}$ the set $A_{\mathbf{x}}$ is r.e., whence $A_{\mathbf{x}} \leq_1 K$ (i.e. there is a recursive one-to-one function g with $A_{\mathbf{x}} = g^{-1}(K)$). Since $A^{\#}_{\mathbf{x}} \leq_{tt} A_{\mathbf{x}}$ is obvious we obtain $A^{\#}_{\mathbf{x}} \leq_{tt} K$. The second assertion follows from the following result by Bennett [2], (proven indirectly in Juedes, Lathrop, Lutz [14]) stating that

for every language $A \subseteq \Sigma^*$ with $K \leq_{tt} A$ the sequence χ_A is not random,

and from the fact that Ω is random (Theorem 3.4). $\square$

We remark that a direct proof of the cited fact by Bennett has been given by Calude and Nies [8], who also prove that Ω is wtt-complete (for the definition of wtt-reduction the reader is referred to Soare [23]). This last fact shows that for any r.e. real $0.\mathbf{x}$ there exist a total recursive function $g : \mathbf{N} \to \mathbf{N}$ and a partial recursive function $F : \Sigma^* \stackrel{o}{\to} \Sigma^*$ with $\mathbf{x}(n) = F(\Omega(g(n)))$ for all n (use again $A^{\#}_{\mathbf{x}} \leq_{tt} A_{\mathbf{x}}$).

7 Open Problems

We close our paper with some open problems and comments on some of them.

1. Does there exist a random r.e. real which is not Ω–like?

   Comment. Kucera [18] (see also Kautz [15]) has observed that $0'$ is the only r.e. degree which contains random sets. But Corollary 6.11 shows that $0'$ splits into different $=_{dom}$-equivalence classes. We believe that the question is important because any answer, whether it is positive or negative, leads to further interesting investigations. If the answer is yes, then one can look at the world of all random r.e. reals and investigate the nature of this world in terms of $\leq_{dom}$. If the answer is no, then one can analyze the world of all r.e. reals asking how "close" an r.e. real is to the class of random r.e. reals, which would simply be the Chaitin Ω numbers.

2. Let A be a universal Martin-Löf test. Is $\alpha = \sum_n \mu(A_n\Sigma^\omega)$ Ω-like? Weaker version: Is α random?

3. Further study the first order theory of $\langle \mathbf{R_{r.e.}}; \leq_{dom}\rangle$.

4. Do there exist prefix free r.e. sets A and B in the same 1-degree (m-degree) such that the r.e. reals 2^{-A} and 2^{-B} do not dominate each other?

   Comment. If such r.e. sets exist, then this will show that recursive isomorphism types do not preserve the domination relation. This would show that the relation $\leq_{dom}$ (or equivalently the simulation relation between prefix free sets) is quite different from known recursion–theoretic reducibilities.

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