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Games with Unknown Past

Bakhadyr Khoussainov
Department of Computer Science
University of Auckland

Alexander Yakhnis
Pioneer Technologies
Albuquerque, NM

Vladimir Yakhnis
Rockwell Science Center
Thousand Oaks, CA

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Bakhadyr Khoussainov
The University of Auckland, Auckland, New Zealand
Cornell University, Ithaca, New York 14850, USA
e-mail: bmk@cs.auckland.ac.nz

Alexander Yakhnis
Pioneer Technologies,
10300 Chapala PL NE, Albuquerque, NM 87111,
e-mail: AYakhnis@aol.com;

Vladimir Yakhnis
Rockwell Science Center,
1049 Camino Dos Rios, Thousand Oaks, CA 91358,
e-mail: vryakhni@scimail.risc.rockwell.com

Abstract

We define a new type of two player game occurring on a tree. The tree may have no root and may have arbitrary degrees of nodes. These games extend the class of games considered by Gurevich-Harrington in [5]. We prove that in the game one of the players has a winning strategy which depends on **finite bounded information about the past** part of a play and on future of each play that is **isomorphism types** of tree nodes. This result extends further the Gurevich-Harrington (GH) determinacy theorem from [5].

Keywords: Games, Gurevich–Harrington Games, Forgetful Determinacy Theorem, Strategies

1 Introduction

Using game-theoretic methods Gurevich and Harrington gave an elegant and new proof of the decidability of second order monadic theory of two successors known as $S2S$ [5]. Possible applications of their proof to the theoretical computer science and logic have attracted the attention of many mathematicians and computer scientists [Monk [6], McNaughton [7], Nerode - Yakhnis [11] [10] [12], Thomas [17], Zeitman [22], Yakhnis - Yakhnis [19]]. The games and new ideas presented in their paper undoubtedly constitutes an area for research and applications. For instance, Nerode, A. Yakhnis and V. Yakhnis have developed theoretical foundations for concurrency based on GH game-theoretic methods. They showed that concurrent programs can be viewed as winning strategies in GH type games. They aslo showed that GH games can be used as theoretical tools for program development and verification in compiler theory, operating systems design and verification, and hybrid systems theory.

This paper uses ideas of the original Gurevich-Harrington paper [5] in order to develop an extension of Gurevich-Harrington game-theoretic methods. We think that game-theoretic ideas developed by Gurevich-Harrington can be applied in a wide range of areas: logic, theory of concurrent and parallel computations, logic programming, real-time computing sytsems, artificial intelligence, robotics, operating systems design and verification, and hybrid systems theory, etc. Therefore, a goal of our extension is to expand potential applications of GH games.

Gurevich-Harrington games have several features. Each game generates a structure, a tree or a graph [Gurevich - Harrington [5], McNaughton [7], Zeitman [22],Yakhnis - Yakhnis[19] [20]], which has a fixed element, called **initial position**. Each play of the game begins from this **initial position**. For example, in games occuring on trees these elements are the roots of trees. The structures generated by games are **strongly locally finite**, that is the number of neighbours of every element is bounded by an $n \in \omega$. In addition, each player of the game has a finite alphabet from which the player picks elements and makes moves. This is **finiteness** of the game alphabet in [Gurevich - Harrington]. Each move of any player is **identified** with the choice of a letter from the alphabet. We omit all the above restrictions in our games. Namely, our game structures need not have initial elements. The game structures are not supposed to be locally finite. We do not necessarliy identify a move of a player with a choice of a letter from the alphabet. Each player of our games potentially has infinitely many choices to make moves.

One of our other intensions is to model proccesses which can be characterized as proccesses with **unknown past**. An example of a such proccess is a human–computer interaction: a user (computer) beginning to interact with a computer (user) does not neccessarily know the past history of the computer (user). We would like to point out that it is not a new idea to ivestigate proccessses with unknown past. For example, automata-theoretic treatment of procedures with unknown past has also been developed in [Nivat-Perrin [13], Perrin-Shupp [14], Semenov [15]]. The approach taken in these papers is motivated by problems from ergodic theory and symbolic dynamics [Nivat-Perrin [13], Perrin-Shupp [14]]. Another example is that investigations in modal and temporal logics with past tense temoral operators [21]. We hope that our generalization of Gurevich–Harrington games is appropriate to develop a game-theoretic approach for investigating proccesses with unknown past.

Now we give a brief summary of the paper. Section 2 is quite technical containing many definitions. In Section 2 we define our games. The underlying structure in which a game occurs is a tree $\mathcal{T}$ labelled with symbols from alphabet Σ. Each $\sigma \in \Sigma$ defines a unary predicate P_σ on $\mathcal{T}$ as follows: $x \in P_\sigma$ iff x is labelled with σ. Thus, in fact, the game occurs on the tree expanded by unary predicate symbols P_σ. In this section we technically modify the notions of games, winning conditions, moves of the players, and strategies. For example, we introduce the notions of a winning condition, apparent and oriented moves, plays, arena, and who wins a game. We also define apparent strategies, oriented strategies, tactics, and show the connections between them. We prove a technical theorem about the duality between arenas and opposing strategies. **Games with unknown past** are those which occur on trees with no root. A definition of the winner of a game with unkown past is unsymmetrical. This is caused by the assumption that the game does not have a beginning. However, for Gurevich - Harrington games our definition is equivalent to theirs. We also give examples to illustrate the notions of the section.

In section 3 similar to [Gurevich-Harrington [5]] we define the notion of rank for the players. For a given set C in a tree, we define a special game $G(C)$. Informally, this is a game in which one of the players targets the set C while the other player tries to avoid C. We show that one of the players wins the game $G(C)$. This is a generalization of Theorem 2 from [Gurevich - Harrington]. We then give sufficient model-theoretic conditions on arena and the set C for a winner to have apparent or oriented winning strategies.

In section 4 we investigate congruence relations of the games. A **congruence** of a game G is an equivalence E in the arena such that if $(x, y) \in E$, then **residual games** after x and y are isomorphic, and for any automorphism α of the residual arena defined by x, $(x, \alpha(x)) \in E$. The set of all congruences of a game forms a lattice with the maximal element. In this section we extend Gurevich - Harrington's **Sewing Lemma** by proving that if for every position x from a set X a player has a winning strategy to win the game after x, then the player has a uniform strategy winning all games which begin from all positions x in X.

In section 5 we prove the uniform determinacy theorem (UDT) for the games with unknown past. We base the proof of the theorem on ideas from [Gurevich - Harrington [5]]. The theorem states that if a winning condition W is given as a boolean combination of sets $[C_1], \ldots, [C_n]$, where $[C_i]$ consists of all paths in the game tree which intersect C_i infinitely many times, then one of the players has a winning oriented strategy which depends on **finite bounded information about the past** part of a play and on **isomorphism types** of tree nodes. A partial case of this theorem is the Gurevich - Harrington **determinacy theorem** [Gurevich-Harrington [5]].

We use common notations. For instance, if f is a mapping, then $dom(f)$ is the domain and $range(f)$ is the range of f. For a set S $card(S)$ is the number of elements in S. For a pair $z = (x, y)$, let $lz = x$ and $rz = y$. For $a = (a_1, \ldots, a_m)$, we put $\pi_i(a) = a_i$. If $\mathcal{T}$ is a model, then T denotes the domain of $\mathcal{T}$. While wrting the paper (we began writing the paper in 1992) it seemed that basic notions from model theory were convinient and transparent for our generalization of GH games. Therefore we used elementary notions from model theory such as for example, automorphisms, isomorphisms, submodels, and elementary equivalence. Al these notions can be found in any elementary book on model theory.

2 Games and Strategies

We define games over algebraic structures called trees, which may not have roots as opposed to ordinary trees. A **tree** is a partially ordered set $\mathcal{T}$ such that

1. For any $x \in T$, the set $Pred(x) = \{y|y \leq x\}$ is linearly ordered.

2. Any two elements have a common ancestor below each of them, that is $\forall xy \exists z(z \leq x \bigwedge z \leq y)$

3. The partial order is discrete, that is for all x, y if $x \leq y$, then there is finitely many elements z such that $x < z < y$.

Let $\mathcal{T}$ be a tree. From the definition it follows that for any $x \in \mathcal{T}$, $Pred(x) = \emptyset$ or $card(Pred(x)) = 1$. We also define $Suc(x) = \{y|x < y \bigwedge \neg\exists z(x < z < y)\}$ the set of all immediate successors of x. Since T is a tree there exist disjoint sets T_0 and T_1 be subsets of T such that $T_0 \bigcup T_1 = T$ and for any $x \in T$, if $x \in T_\epsilon$ then $Suc(x) \subset T_{1-\epsilon}$, where $\epsilon - 1 = 0$ if $\epsilon = 1$, and $\epsilon - 1 = 1$ if $\epsilon = 0$. We say that the set T_ϵ is the **set of nodes** for the player ϵ.

Let Σ be an alphabet of any cardinality. A mapping $v : \mathcal{T} \to \Sigma$ is called a **valuation** of $\mathcal{T}$. If v is a valuation, then we call the pair $(\mathcal{T}, v)$ a **valuated tree**. The valuated tree $(\mathcal{T}, v)$ can be considered as a model $\mathcal{T}_v$ of the enriched signature $(\leq, T_0, T_1, P_a; a \in \Sigma)$, which expands the language of $\mathcal{T}$, as follows. The domain of $\mathcal{T}_v$ is T; The relation $\leq$ in $\mathcal{T}_v$ is the same as $\leq$ in $\mathcal{T}$; For any $a \in \Sigma$, $\mathcal{T}_v \models P_a(x)$ if and only if $v(x) = a$.

Suppose that $\mathcal{T}'$ is any model of the language $(\leq, T_0, T_1, P_a; a \in \Sigma)$ and is an expansion of $\mathcal{T}$ such that

$$(\star) \quad \mathcal{T}' \models \forall x(\exists a \in \Sigma)(P_a(x) \bigwedge (P_a(x) \bigwedge P_b(x) \to a = b))$$

Then $\mathcal{T}'$ defines a valuation $v : T \to \Sigma$ such that $\mathcal{T}' = T_v$. Below we consider only structures $\mathcal{T}'$ which satisfy $(\star)$.

Our games occur on valuted trees $\mathcal{T}_v$. One needs to be careful in defining games on the valutaed trees. In GH games each move of any player is identified with a choice of a letter from an alphabet while in our games we can not do this directly. Therefore below we first give basic definitions and simple results of technical nature.

Let $P(\mathcal{T}_v)$ be the set of all paths on T_v, that is all maximal linearly ordered subsets of $\mathcal{T}$.

Definition 2.1 *A partial mapping $f : Z \to T \times \Sigma$ is* **labelled prepath** *on $\mathcal{T}_v$ if $range(lf)$ is a path, f is an isomorphism from $(dom(f), \leq)$ into $(range(lf), \leq)$, and for any $x \in dom(f)$, $x + 1 \in dom(f)$,* If f is a lebelled prepath, then the partial function $g : Z \rightarrow \Sigma$ defined by $g(i) = rf(i)$ is called **apparent prepath**. We say that labelled (apparent) prepaths f_1 and f_2 are equivalent if there exists an $i \in Z$ such that $f_1(n) = f_2(n+i)$ for all $n \in dom(f_1)$. Each class of equivalent labelled prepaths (apparent prepaths) we call **path** (**apparent path**). If μ is a labelled path, then let $app(\mu)$ be an apparent path defined by μ. We need these definition simply because of the following reasons. First it is not true that any mapping $g : Z \rightarrow \Sigma$ can be identified with a path. Second, even if g is an apparent path, then g does not always determines a unique path in $\mathcal{T}$.

Let $LbP(\mathcal{T}_v)$ be the set of all labelled paths and let and $App(T_v)$ be the set of all apparent paths on $\mathcal{T}_v$, respectively.

Definition 2.2 *A set W of lebelled paths is called* **a winning condition** *or* **a winning set** *for Player 0.*

Let $W \subseteq LbP(\mathcal{T}_v)$. Define an operator Cl over the subsets of $LbP(\mathcal{T}_v)$:

$$Cl(W) = \{\beta \mid \exists \alpha(\alpha \in W \& app(\alpha) = app(\beta))\}.$$

If $Cl(W) = W$, then we say that W is **closed**. From this definition we obtain

Proposition 2.1 *The operator Cl has the following properties:*

1. *For any $W \subseteq P(\mathcal{T}_v)$ we have $W \subseteq Cl(W)$,*
2. *If $W_1 \subseteq W_2 \subseteq P(\mathcal{T}_v)$, then $Cl(W_1) \subseteq Cl(W_2)$,*
3. *$Cl(Cl(W_1)) = Cl(W_1)$.* $\Box$

Another property of the closed winning sets (conditions) is given in the next proposition:

Proposition 2.2 *The collection of all closed winning sets forms a boolean algebra. Moreover, this collection can be taken as a topology of clopen sets over the set $LbP(\mathcal{T}_v)$.* $\Box$

Let p be any node of $\mathcal{T}_v$. We define the maximal **oriented move** and the maximal **non-oriented move** as follows:

$$OM(p) = \{(y,b) | y \in Suc(p) \bigwedge \mathcal{T}_v \models P_b(y)\}$$

and

$$NM(p) = \{y | \exists b \in \Sigma (y \in Suc(p) \bigwedge T_v \models P_b(y))\}$$

It is clear that $NM(p) = Suc(p)$ and $NM(p) = l(OM(p))$. Now suppose that p is a position for Player ϵ. Player ϵ can make one of the following actions at node p: he can choose a subset from $OM(p)$; he can choose a subset from $NM(p)$; finally he can choose a subset from Σ. In the first case we say that the move is **oriented**, in the second case **nonoriented**, and in the last case **apparent**. We say that a nonoriented move $S \subset NM(p)$ is **closed** if for all x, y, b the conditions $y \in S, x \in Suc(p)$, and $\mathcal{T}_v \models P_b(x) \& P_b(y)$ imply that $x \in S$. The notions of a move (oriented, non-oriented, apparent and closed) allow us to consider three types of plays (oriented, non-oriented, and closed) which begin at a node of T. Thus, let $p \in T$. A submodel P of $\mathcal{T}_v$ is an **oriented play from** p if P satisfies the following conditions:

1. p is the minimal element of P.
2. P is closed with respect to predecessors of $P \setminus \{p\}$.
3. For any $x \in P$ there exists a $y \in P$ such that $y \in Suc(x)$.

The reduction of any oriented play P to the language $(\leq)$ is **a non-oriented play**. A non-oriented play P is **closed** if for all x, y, z, b the conditions $x, z \in P$, $y, z \in Suc(x)$, and $\mathcal{T}_v \models P_b(y) \bigwedge P_b(z)$ imply that $y \in P$.

Example. Let α be a labelled path. Then for any $i \in Z$, the sequences $\alpha(i), \alpha(i+1), \ldots$ and $l\alpha(i), l\alpha(i+1), \ldots$ are oriented and non-oriented plays, respectively.

Let α be a labelled path. The sequence $\alpha(i), \alpha(i+1), \ldots$ is the **right ray from** $l\alpha(i)$, and the sequence $\ldots, \alpha(i-2), \alpha(i-1), \alpha(i)$ is a **left ray from** $l\alpha(i)$. It is obvious that any $p \in T$ defines the unique left ray $R_l(p)$. Let $R_r(p)$ be the set of all right rays from p. If W is a winning condition, then let $W_p = \{(\alpha(i), \alpha(i+1), \ldots) | \alpha \in W, l\alpha(i) = p\}$.

The games we define occur on arenas: A submodel $\mathcal{A}$ of $\mathcal{T}_v$ is an **oriented arena** if A is closed under predecessors, and for any $x \in A$ there exists $y \in A$ such that $y \in Suc(x)$. We call the reduction of any oriented arena to the language $\{\leq\}$ a **nonoriented arena**. A non-oriented arena $\mathcal{A}$ is **closed** if for all x, y, z, b the conditions $x, z \in A$, $y, z \in Suc(x)$, and $\mathcal{T}_v \models P_b(y) \bigwedge P_b(z)$

imply that $y \in A$. We would like to remark that the notions of play and arena are based on different intuitions. Plays correspond to an idea of a sequence of moves of players. Arenas reflect space and time where the plays take place. However, technically according to the given definitions, every play at p in $\mathcal{T}_v$ can be regarded as arena in the corresponding subtree of $\mathcal{T}_v$ with the root p. Now we can give our definition for games.

Definition 2.3 *Let $\mathcal{A}$ be an oriented arena and W be a winning condtions for* 0*. Then the triple $G = (\mathcal{A}, W, 0)$ is a* **game**.

Now our next goal is to define who wins the game G. We need to define the notion of strategy. A nondeterministic **apparent strategy** for ϵ is a function $f : T_\epsilon \to P(\Sigma)$ such that for any $x \in T_\epsilon$ there exists $y \in Suc(x)$ and $b \in f(x)$ for which $\mathcal{T}_v \models P_b(y)$. A function $t : T_\epsilon \to P(T_{1-\epsilon})$ is a nondeterministic **tactic** for ϵ if for any $x \in T_\epsilon$ we have $t(x) \subset NM(x)$. An **oriented strategy** for ϵ is a function $h : T_\epsilon \to P(T_\epsilon \times \Sigma)$ such that for any $x \in T_\epsilon$ the set $h(x)$ is a subset of $OM(x)$.

Example. Let f be an apparent strategy and let t be a tactic for ϵ. Then the following function (t, f) is an oriented strategy for ϵ

$$(t, f)(x) = \{(y, b) | y \in t(x) \& b \in f(x) \& \mathcal{T}_v \models P_b(y)\}.$$

Proposition 2.3 *For any oriented strategy h there exist an apparent strategy f and a tactic t such that $h = (t, f)$.*

Proof. Define f and t by $f(x) = r(h(x)) = \{b \mid \exists (y, b) \in h(x)\}$ and $t(x) = l(h(x)) = \{y \mid \exists (y, b) \in h(x)\}$. From the definition of (t, f) it follows easily that $h(x) \subseteq (t, f)(x)$. To see that $(t, f)(x) \subseteq h(x)$ consider (y, b) from $(t, f)(x)$. We must have for some $y' \in Suc(x)$, $(y', b) \in h(x)$; and for some $b' \in \Sigma$ we have $(y, b') \in h(x)$, $P_b(y)$ by the definitions of $f, t, (f, t)$ respectively. Since $P_{b'}(y)$, it follows $b' = b$. Thus $(y, b) \in h(x)$. Hence $(t, f)(x) \subseteq h(x)$. We conclude $h(x) = (t, f)(x)$. $\square$

Let t be a tactic and f be an apparent strategy for a player ϵ over $\mathcal{T}_v$. For this player we define an apparent strategy ap^t:

$$ap^t(x) = \{a \mid \exists y (y \in t(x) \wedge P_a(y))\}$$

and a tactic t^f:

$$ta^f(x) = \{y \mid \exists a(a \in f(x) \wedge P_a(y))\}.$$

Proposition 2.4 *Let t be a tactic and f be an apparent strategy for ϵ. Then $t \subseteq ta^{(ap^t)}$ and $f = ap^{(ta^f)}$. Moreover, $t = ta^{(ap^t)}$ if each non-oriented move of ϵ according to t is closed.*

Proof. The move $ta^{(ap^t)}(x)$ is a collection of all successors of x covered by labels which already cover the members of $t(x)$. The move $ap^{(ta^f)}(x)$ is a collection of labels which cover the members of the move $(ta^f)(x)$. But the latter members are all the successors of x covered by labels from $f(x)$. □

Now we investigate duality between arenas and pairs of opposing strategies. Let $\mathcal{A}$ be an arena in a valuated tree $\mathcal{T}_v$. We define an oriented strategy $f^0_{\mathcal{A}}$ for player 0 and an oriented strategy $g^1_{\mathcal{A}}$ for player 1 over $\mathcal{T}_v$. Put $\mathcal{A}_0 = \mathcal{A} \cap T_0$ and $\mathcal{A}_1 = \mathcal{A} \cap T_1$. Define

$f^0_{\mathcal{A}}(x) = \emptyset$ if $x \in T_0 \setminus \mathcal{A}$, and $f^0_{\mathcal{A}}(x) = \{(y,b) \mid y \in Suc(x) \wedge P_b(y)\}$ if $x \in \mathcal{A}_0$.

$g^1_{\mathcal{A}}(x) = \emptyset$ if $x \in T_1 \setminus \mathcal{A}$, and $g^1_{\mathcal{A}}(x) = \{(y,b) \mid y \in Suc(x) \wedge P_b(y)\}$ if $x \in \mathcal{A}_1$.

Let f, g be oriented strategies for players $0, 1$, respectively, in a valuated tree $\mathcal{T}_v$. Let $p \in T_\epsilon$ for any fixed player $\epsilon \in \{0,1\}$. We construct a maximal subset $A(p,f,g)$ of T which includes p and no points to the left of p and which is generated by the strategies. Informally $A(p,f,g)$ contains all plays if the players follows the strategies f and g.

Stage 0. Put $\mathcal{A}_0 = \{p\}$

Stage $2k+1$. Suppose $\mathcal{A}_{2k}$ has been defined. Let $\mathcal{A}_{2k+1}$ to be $\bigcup_{x \in \mathcal{A}_{2k} \cap T_\epsilon} lf(x)$ if $\epsilon = 0$; Put $\mathcal{A}_{2k+1} = \bigcup_{x \in \mathcal{A}_{2k} \cap T_\epsilon} lg(x)$ if $\epsilon = 1$.

Stage $2k+2$. Suppose $\mathcal{A}_{2k}$ has been defined. Define $\mathcal{A}_{2k+2}$ similar as in the previous stage replacing ϵ with $1-\epsilon$ and g with f.

Finally, define $\mathcal{A}(p,f,g) = \bigcup_{i \in \omega} \mathcal{A}_i$.

Given two opposing strategies as in the above, for any left ray α we consider the set of all nodes p on it for which the set $\mathcal{A}(p,f,g)$ is a non-oriented play. Let us call this set S_α. Define

$$\mathcal{A}(\alpha,f,g) = \begin{cases} \emptyset & \text{if } S_\alpha = \emptyset, \\ l\alpha \bigcup_{p \in S_\alpha} \mathcal{A}(p,f,g) & \text{otherwise} \end{cases}$$

The proof of the lemma below follows from the definitions above.

Lemma 2.1 *The submodel $\mathcal{A}(\alpha, f, g)$ of the model $\mathcal{T}_v$ is an oriented arena. Moreover, for any $x \in A(\alpha, f, g)$ the following properties hold: If $x \in l\alpha$, then $f^0_{\mathcal{A}(\alpha,f,g)}(x) = f(x) \cup (Suc(x) \cap \alpha)$ and $g^1_{\mathcal{A}(\alpha,f,g)}(x) = g(x) \cup (Suc(x) \cap \alpha)$. If $x \notin l\alpha$, then $f^0_{\mathcal{A}(\alpha,f,g)}(x) = f(x)$ and $g^1_{\mathcal{A}(\alpha,f,g)}(x) = g(x)$.* $\Box$

Definition 2.4 *Let f, g be strategies for Players $0, 1$ on $\mathcal{T}_v$ and let α_p be the left ray determined by a p on $\mathcal{T}_v$. Then The left ray α_p is* **consistent** *with the pair of strategies if for every node $x < p$ on α,*

$$x \in T_0 \rightarrow Suc(x) \cap \alpha \subseteq f(x)$$

and

$$x \in T_1 \rightarrow Suc(x) \cap \alpha \subseteq g(x).$$

A right ray $\alpha \in R_r(p)$ is **consistent** *with the pair of strategies if $\alpha \subseteq A(p, f, g)$. A labelled path α on $\mathcal{T}_v$ is* **consistent** *with the pair of opposing strategies if there is a node $p \in l\alpha$ such that α_p and the right ray determined by the path and p are both consistent with the pair of strategies.*

Theorem 2.1 *(***Duality of arenas and pairs of opposing strategies***)*

1. *Let f_0, f_1 be opposing oriented strategies on $\mathcal{T}_v$ for Players $0, 1$ respectively. Let α be a left ray on $\mathcal{T}_v$. Then for any $\epsilon \in \{0, 1\}$ and any $x \in \mathcal{A}(\alpha, f, g) \cap T_\epsilon$*

   $$f_\epsilon(x) = f^\epsilon_{\mathcal{A}(\alpha,f,g)}(x)$$

   if and only if $\alpha = S_\alpha$ and α is consistent with f_0, f_1.

2. *Let $\mathcal{A}$ be an arena in $\mathcal{T}_v$. For any left ray α in $\mathcal{T}_v$*

   $$A = A(\alpha, f^0_{\mathcal{A}}, f^1_{\mathcal{A}})$$

   if and only if $\alpha \subseteq A$.

Proof 1. Suppose α is consistent with f_0, f_1. The preceding lemma and the definition of consistency of α imply that $f_\epsilon(x) = f^\epsilon_{\mathcal{A}(\alpha,f,g)}(x)$.

Conversely, suppose that the above equality holds for every point from $A(\alpha, f, g)$ for appropriate ϵ. Using the equality for the nodes on alpha, we get that α is consistent with f_0, f_1.

(ii) If $A = A(\alpha, f^0_{\mathcal{A}}, f^0_{\mathcal{A}})$, then since α is a subset of $A(\alpha, f^0_{\mathcal{A}}, f^1_{\mathcal{A}})$, by the definition of this set, α is a subset of A.

Suppose now that $\alpha \subseteq A$. It is clear that $A(\alpha, f^0_{\mathcal{A}}, f^1_{\mathcal{A}}) \subseteq A$. Consider any $x \in A$. Pick some point p on α. By the definition of our trees $\mathcal{T}_v$ there is a point $q \in T$ such that $q \leq x, q \leq p$. It follows that $q \in \alpha$. It follows further that $x \in A(q, f^0_{\mathcal{A}}, f^1_{\mathcal{A}})$, but the latter set lies in $A(\alpha, f^0_{\mathcal{A}}, f^0_{\mathcal{A}})$. So does x. Hence the desired equality follows. □

Definition 2.5 *Let $G = (\mathcal{A}, W, 0)$ be a game.*

1. The player 0 **wins** *the game G if there exits an oriented strategy f for* 0 *and a left ray α such that for every oriented strategy g for* 1 *and for any node $p \in l\alpha$, in every play from p all right rays from p defined by f and g belong to the W_p.*

2. Player 0 **strongly wins** *the game G if there exits an apparent strategy f for* 0 *and a left ray α such that for every oriented strategy g for* 1 *and for any element $p \in l\alpha$, in every play from p all right rays from p defined by f and g belong to W_p.*

Note that the definition of a winner is not symmetric with respect to the players. Indeed, Player 1 **wins** the game G if there exits an oriented strategy g for 1 such that for any left ray α and for every oriented strategy f for 0 there exists $p \in \alpha$ such that every play defined by g and f from the node p has no right ray from p which belongs to W_p.

Proposition 2.5 *1. If the player ϵ strongly wins a game G, then this player wins G. Moreover, an apparant strategy f for player ϵ wins a game G if and anly if $h = (f, ta^f)$ is a winning oriented strategy .*

2. There exists a game G such that Player 0 *wins G but does not wins G strongly.*

3. Let $\mathcal{T}_v$ be a model such that

$$\mathcal{T}_v \models \forall x y_1 y_2 (y_1, y_2 \in Suc(x) \& (P_b(y_1) \& P_b(y_2) \to y_1 = y_2)$$

Then 0 *wins a game on $\mathcal{T}_v$ G if and only if* 0 *strongly wins G.*

Proof. Part 1 follows from the definition. Let $T = \{a, b\}^\star$ and let $\Sigma = \{\sigma\}$. Let $f \in W$, if and only if,

$$f(i) = \begin{cases} \emptyset & \text{if } i \leq 0, \\ (b, \sigma) & \text{if } i = 1, \\ (\delta, \sigma) & \text{if } i > 1, \text{ where } \delta \in \{a, b\} \end{cases}$$

Suppose that $\emptyset \in T_0$. Consider game $G = (T, W, 0)$. From the definition of G it follows that the player 0 has an oriented winning strategy. But this player does not possess an apparent winning strategy. To prove 3 note that for every oriented strategy h in this case

$$h = (ap(h), ta^{ap(h)}).$$

Thus by 1) an oriented strategy wins the game for a player iff so does an apparent strategy for the player. □

3 Gurevich-Harrington's Rank

Let $\epsilon \in \{0, 1\}$. We fix a subset C of T. Define a sequence $C_0(\epsilon), C_1(\epsilon), \ldots$ of subsets of T by induction. Let $C_0(\epsilon) = C$. Suppose that $C_n(\epsilon)$ has been defined. Then $x \in C_{n+1}$ if and only if the following conditions hold:

1. $x \notin C_n(\epsilon)$.

2. If $x \in T_{\epsilon-1}$, then for any $y \in Suc(x)$, we have $y \in C_n(\epsilon)$.

3. If $x \in T_\epsilon$, there exists a $y \in Suc(x)$ such that $y \in C_n(\epsilon)$.

Thus, $C_i(\epsilon) \bigcap C_j(\epsilon) = \emptyset$ for all $i \neq j$. Define $C(\epsilon) = \bigcup_i C_i(\epsilon)$ and $C^1(\epsilon) = C(\epsilon) \setminus C$. Define the function $rank(\epsilon, C)$:

$$rank(\epsilon, C)(x) = \begin{cases} undefined & \text{if } x \notin C(\epsilon) \\ i & \text{if } x \in C_i(\epsilon) \end{cases}$$

Let $x \in T$, and let $C \subset T$. Consider a submodel of $\mathcal{T}_v$ with domain $\{y \in T \mid y > x\}$. We call this submodel a residual model at x and denote it by $(\mathcal{T}_v)_x$ or $Res(x)$. Let $\mathcal{T}'_v = (\mathcal{T}_v, C)$ be an expansion of $\mathcal{T}_v$ by a unary predicate symbol interpreted as C. Consider the submodel $(Res(x), C)$. Define an equivalence relation E on T by

$$(x, y) \in E \leftrightarrow (Res(x), C) \cong (Res(y), C)$$

Definition 3.1 *1. An oriented strategy h* **respects an equivalence relation** *η on T if for all $(x, y) \in \eta$ and $a \in \Sigma$ the cardinalities of the sets $h_a(x) = \{z \mid (z, a) \in h(x)\}$ and $h_a(y) = \{z' \mid (z', a) \in h(y)\}$ are equal.*
2. An apparent strategy f **respects an equivalence relation** *η on T if for all $(x, y) \in \eta$, $f(x) = f(y)$.*

Theorem 3.1 *Let $C \subset T$ and let $W = \{\alpha | \exists i(l\alpha(i) \in C)\}$. Consider a game $G(C) = (\mathcal{T}_v, W, 0)$. In this game one of the players has a winning oriented strategy which respects the equivalence relation E.*

Proof. Consider the set $C(0)$. We have two cases.

Case 1. There exists a left ray α such that $l\alpha \subset C(0)$. In this case define the follwing oriented strategy called **decrease rank**:

If $x \notin C(0)$ or $x \in C$, then $decr^0_C(x) = \{(y, b) | y \in Suc(x) \bigwedge P_b(y)\}$.

Suppose that $x \in C_i$ for some $i > 0$. Then

$$decr^0_C(x) = \{(y, b) | y \in Suc(x) \bigwedge P_b(y) \bigwedge y \in C_{i-1}(0)\}$$

By the definitions of the equivalence relation E and the strategy $decr^0_C$, we obtain that f respects E. Since $l\alpha \subset C(0)$, the strategy f is winning for the player 0.

Case 2. Suppose that the previous case does not hold. Define the following oriented strategy for the player 1 called $avoid^1_C$.

If $x \in C(0)$, then $avoid^1_C(x) = \{(y, b) | y \in Suc(x) \bigwedge P_b(y)\}$.

Suppose that $x \notin C(0)$. Then

$$avoid^1_C(x) = \{(y, b) | y \in Suc(x) \bigwedge P_b(y) \bigwedge y \notin C(0)\}$$

By the definitions of $avoid^1_C$ and E, we obtain that g respects E. Since for any left ray α there exists $i \in Z$ such that $l\alpha(i) \notin C(0)$, we get that $avoid_C$ is a winning oriented strategy for the player 1. □

This theorem suggests the following two questions:

1. *Is it possible to weaken the definition of E to obtain similar theorem?*
2. *When one of the players strongly wins the game $G(C)$?*

Our next results answers these questions. On the set T define an equivalence E': $(x, y) \in E'$ if and only if the models $(Res(x), C, C(0), C(1))$ and $(Res(y), C, C(0), C(1))$ are elementary equivalent in the language expanded by $C(0), C(1)$, and C in the respective resudual models.

Corollary 3.1 *Let T be a tree, $C \subset T$, and let $W = \{\alpha | \exists i(l\alpha(i) \in C)\}$. Consider a game $G(C) = (\mathcal{T}_v, W, 0)$. In this game one of the players has a winning oriented strategy which respects the equivalence relation E'.*

Proof. Note that the sets $C_i(0)$, $i \in \omega$, are definable in the language $(\leq, T_0, T_1, C, P_a; a \in \Sigma)$. Indeed, $x \in C_0(0) \Leftrightarrow C(x)$. Suppose that $C_n(0)$ has been defined, then C_{n+1} is defined by

$$x \in C_{n+1}(0) \Leftrightarrow \neg C_n(x) \wedge (T_0(x) \Rightarrow \exists y(Suc(x, y) \wedge C_n(y))) \wedge (T_1(x) \Rightarrow \forall y(Suc(x, y) \Rightarrow C_n(y)))$$

Case 1. There exists a left ray α such that $l\alpha \subset C(0)$. In this case consider the oriented strategy $decr_C^0$ defined in the previous theorem. We need to show only that $decr_C^0$ respects E'. Suppose $xE'y$. It follows that models $(Res(x), C, C(0), C(1))$ and $(Res(y), C, C(0), C(1))$ are elementary equivalent. We have to show that for every $a \in \Sigma$, $Card(decr_{C,a}^0(x)) = Card(decr_{C,a}^0(y))$, where $decr_{C,a}^0(x) = \{(z, a) | (z, a) \in decr_C^0(x)\}$.

Assume first that $decr_{C,a}^0(x)$ is finite. Assume that $x \in C^1(0)$. This implies that for some $i > 0$, $x \in C_i(0)$. Because $C_{i-1}(0)$ is definable in the language of the model, it easily follows that $y \in C_i(0)$. It is easy to see that $Card(decr_{C,a}^0(x)) = Card(Suc(x) \cap C_{i-1}(0)) \cap P_a$. Since the latter set is finite, let n be a number of elements in it. Using the fact that this set has exactly n elements is expressible as a first order sentence in the language of the model $(Res(x), C, C(0), C(1))$, we obtain that the same sentence holds in the model $(Res(y), C, C(0), C(1))$.

Assume now that $decr_{C,a}^0(x)$ is infinite. Then for every $n > 0$ there is a sentence in the language of $(Res(x), C, C(0), C(1))$ saying that $decr_{C,a}^0(x)$ has more than n elements. Such a sentence holds in this model. Because of elementary equivalence it holds in $(Res(y), C, C(0), C(1))$, where it says that $decr_{C,a}^0(y)$ has more than n elements. It follows that $Card(decr_{C,a}^0(x)) = Card(decr_{C,a}^0(y))$.

Similar proof can be given for the case $x \notin C(0)$.

Case 2. For each left ray α, the set $l\alpha$ is not a subset of $C(0)$. In this case the strategy $avoid_C^1$ wins the game as shown in the preceding theorem.

The strategy $avoid^1_C$ respects the equivalence E' by a similar reasoning to the previous case. The corollary is proved.

Definition 3.2 *Let $C \subset T$. The set C is* **invariant** *if for any $x \in T$ the model $(Res(x), C)$ has the following property:*

Let $x_1, x_2 \in T$, $x_1, x_2 \in Suc(x)$ and let $b \in \Sigma$ be such that $\mathcal{T}_v \models \bigwedge_i P_b(x_i)$. Then there exists an automorphism α of $(Res(x), C)$ such that $\alpha(x_1) = x_2$

Theorem 3.2 *Let $C \subset T$ be an invariant set. Let $W = \{\alpha | \exists i(l\alpha(i) \in C)\}$. Consider the game $G(C) = (\mathcal{T}_v, W, 0)$. In this game one of the players has a winning apparent strategy which respects the equivalence relation E.*

Proof. We have two cases.
Case 1. There exists a left ray α such that $l\alpha \subset C(0)$. In this case define the following apparent strategy f:

If $x \notin C(0)$ or $x \in C$, then $f(x) = \{b | \exists y(y \in Suc(x) \wedge P_b(y))\}$.

Suppose that $x \in C_i$ for some $i > 0$. Then

$$f(x) = \{b | \exists y(y \in Suc(x) \bigwedge P_b(y) \bigwedge y \in C_{i-1}(0))\}$$

Since C is invariant it can be easily concluded that every move according to f is closed. It follows $decr^0_C = (f, ta^f)$. Hence f wins the game for this case. The strategy $decr^0_C$ respects E by one of the preceding theorems. This implies that so does f. Indeed, suppose xEy. Then for every $a \in \Sigma$, $card(decr^0_{C,a}(x)) = card(decr^0_{C,a}(y))$. It follows that $a \in f(x)$ if and only if $a \in f(y)$ for every $a \in \Sigma$. Thus $f(x) = f(y)$.
Case 2. Suppose that the previous case does not hold. Define the following apparent strategy g for the player 1.

If $x \in C(0)$, then $g(x) = \{b | \exists y(y \in Suc(x) \wedge P_b(y))\}$.

Suppose that $x \notin C(0)$. Then

$$g(x) = \{b | \exists y(y \in Suc(x) \bigwedge P_b(y) \bigwedge y \notin C(0))\}$$

Again, we get that g is a winning apparent strategy for the player 1 which respects E. The proof is similar to the case 1 and is by reduction to the oriented winning strategy $avoid_C$ for the player 1. In particular, every move according to g is closed. These considerations prove the theorem.

4 Congruence Relations Over Game Trees

Let $G = (A, W, 0)$ be a game, and let $x \in A$. We define a model G_x as follows: the domain A_x of G_x is $\{y|x < y \in A\}$; predicates P_a, $\leq$, T_0, T_1 are the restrictions of corresponding predicates in $\mathcal{T}_v$; Let $\alpha \in W$. Then we define a unary predicate which we also denote by α as $A_x \bigcap l\alpha$.

Definition 4.1 *An equivalence relation η is a* **congruence** *if the following two properties hold:*

1. *If $(x, y) \in \eta$, then models G_x and G_y are isomorphic.*
2. *If $(x, y) \in \eta$, $x \leq z$, then for any isomorphism $\beta : G_x \to G_y$ the pair $(z, \beta(z))$ belongs to η.*

For example, it is not hard to see that the equivalence E defined in the previous section for the game $G(C)$ is a congruence.

Lemma 4.1 *Let η be a congruence of the game $G = (A, W, 0)$. Then for any $x \in A$ the set*

$$\{(y, p)| \textit{ there exists an automorphism } \alpha \textit{ of } G_x \textit{ such that } \alpha(z) = p\}$$

is a subset of η.

Proof. Indeed, sinse η is an equivalence $(x, x) \in \eta$. Let α be an isomorphism from G_x to G_x. By the defiinition if $x \leq y$, then we obtain that $(y, \alpha(y)) \in \eta$. □

Proposition 4.1 *Let $G = (A, W, 0)$ be a game. Then the set of all congruences $Cong(G)$ of the game G is a lattice with* 1.

Proof. Indeed, it is easy to see that $Cong(G)$ is closed under intersection of congruences. Note that $1 = \{(x, y)|G_x \cong G_y\}$. Hence if $\eta_1, \eta_2 \in Cong(G)$, then there exists $sup\{\eta_1, \eta_2\}$. □

Let H be an oriented starategy for player ϵ. For any x we define a set H_x as the set of all z which are **consistent** with H after x.

Definition 4.2 *The starategy H strictly respects the congruence η if for all $(x, y) \in \eta$, every isomorphism between the models G_x and G_y is an isomorphism between the models (G_x, H_x) and (G_y, H_y).*

It foolows from the definition that if H strictly respects η, then by Lemma 4.1 above for any x the set H_x is invariant with respect to the automorphisms group of G_x.

Theorem 4.1 *Let $G = (A, W, 0)$ be a game, and let η be a congruence.*

1. *If an oriented strategy H strictly respects η, then H respects η.*
2. *If an apparent strategy H strictly respects η, then H respects η.*
3. *Let $C \subset A$. Consider the game $G(C)$ from the previous setion. Then one of the players has a winning strategy in the game $G(C)$ which strictly respects the congruence E.*

Proof. The proof of the theorem follows from the proof of the theorems of the previous section and the definitions of apparent and oriented strategies which respect equivelences. □

Proposition 4.2 *Let G be a game, and let η be a congruence. Consider a game $G_x(0) = (A_x, W_x, 0)$. Suppose that H and F be strategies for ϵ and $1 - \epsilon$ in the game $G_x(0)$, respectively, which strictly respect η on A_x. Let $(x, y) \in \eta$ and let $\beta : G_x \to G_y$ be an isomorphism. Define H^β and F^β by*

$$H^\beta(z) = \beta(H(\beta^{-1}(z))),$$

$$F^\beta(z) = \beta(F(\beta^{-1}(z))).$$

Then:

1. *The strategies H^β and F^β srictly respect η on A_y.*
2. *If H wins the game $G_x(0)$ against F, then H^β wins the game $G_y(0)$ against F^β.*
3. *If H wins the game $G_x(0)$, then H^β wins the game $G_y(0)$.*

Proof. 1. Let $(z_1', z_2') \in \eta$ and $z_1', z_2' \in A_y$. Then there exist $z_1, z_2 \in A_x$ such that $\beta(z_1) = z_1'$ and $\beta(z_2) = z_2'$. Since η is a congruence the models G_{z_1} and $G_{z_1'}$ are isomorphic. The strategy H respects η on A_x. Hence the models (G_{z_1}, H_{z_1}) and (G_{z_2}, H_{z_2}) are also isomorphic. By the defintion of H^β, it follows that the models $(G_{z_1'}, H^\beta_{z_1'})$ and $(G_{z_2'}, H^\beta_{z_2'})$ are also isiomorphic.

Using the previous part of the proposition and the defintions one can easily obtain the proof of parts 2) and 3).□

Lemma 4.2 *(***Sewing Lemma***) Let $G = (A, W, 0)$ be a game, let η be a congruence, and let H be a an oriented strategy for player $1-\epsilon$ which strictly respects η. Define the set D:*

$D = \{p|$ *in the game $G_p(0)$ player ϵ has an oriented strategy F^p which wins $G_p(0)$ against H and strictly respets η in A_p*$\}$

Then there exists an oriented strategy F for player ϵ in the game G such that F strictly respects η, and for any $p \in D(\epsilon)$ the strategy F wins $G_p(0)$ against H.

Proof. We can suppose that the set D is well ordered. For each $x \in D$ we define $F(x)$.

Case 1. $x \notin D(\epsilon)$. In this case, $F(x) = \{(y, b) | y \in Suc(x) \& P_b(y)\}$.

Case 2. $x \in D(\epsilon) \setminus D$. Let $x \in D_i(\epsilon)$. In this case,

$$F(x) = decr_D(x) = \{(y, b) | P_b(y) \& y \in Suc(x) \& y \in D_{i-1}(\epsilon)\}$$

Case 3. $x \in D$. Define the following two elements y_x, z_x as follows.

$$y_x = \mu y\{y | y \in D \text{ and there exists } z \geq y \text{ such that } (z, x) \in \eta \text{ and } z \text{ is consistent with } F^y \text{ and } H\}$$

$$z_x = \mu z\{z \in D | z \geq y_x,\ (z, x) \in \eta, \text{ and } z \text{ is consistent with } F^{y_x} \text{ and } H\}$$

Let β_x be an isomorphism from G_x to G_{z_x}. Then

$$F(x) = \beta_x^{-1} F^{y_x}(z_x)$$

Note that if $(x, p) \in \eta$ and $x \in D$, then $y_x = y_p$ and $z_x = z_p$. Moreover, $F(x)$ does not depend on the choise of β_x. Indeed, let β be another isomorphism, and let $u \in \beta_x^{-1} F^{y_x}(z_x)$. Then by lemma 3.1 $(\beta_x(u), \beta(u)) \in \eta$. Since F^{y_x} strictly respects η, we obtain that $\beta(u) \in F^{y_x}(z_x)$.

Now we prove that F strictly respects η.

Let $(x, p) \in \eta$. Then the models G_x and G_p are isomrphic. Let β be any isomorphism from G_x to G_p. By induction we prove that β is an isomorphism between (G_x, F_x) and (G_p, F_p).

In the first inductive step we have nothing to prove, since $x \in F_x$ and $p \in F_p$ and $\beta(x)$ is not defined. Put $Suc_0(x) = \{x\}$.

Suppose that

$$s \in Suc_{n+1}(x) = \bigcup_{r \in Suc_n(x)} Suc(r)$$

By inductive assumption we suppose that for each $z \in Suc_n(x)$, $\beta(z) \in F_p$ if and only if $z \in F_p$.

Case 1. Suppose that $s \in T_{1-\epsilon}$. Then $t = \beta(s) \in T_{1-\epsilon}$. Hence, it is obvious that $\beta(Suc(s)) \subset Suc(t)$.

Case 2. Suppose that $s \in T_\epsilon$ and $s \notin D(\epsilon)$. Then we have $\beta F(s) = \beta Suc(s) = Suc(t)$.

Case 3. Suppose that $s \in T_\epsilon \bigcap D$. In this case

$$F(s) = \beta_s^{-1} F^{y_s}(z_s) \quad \text{and} \quad F(t) = \beta_t^{-1} F^{y_s}(z_s)$$

Let $u \in \beta F(s)$. Then by lemma 3.1 we obtain $(\beta_s \beta^{-1} u, \beta_t u) \in \eta$. Since $\beta_s \beta^{-1} u \in F^{y_s}(z_s)$, we obtain that $\beta_t(u) \in F^{y_x}(z_x)$. It follows that $u \in F(t)$.

Case 4. Let $s \in T_\epsilon$ and $t \in D(\epsilon) \setminus D$. Then $F(t) = Decr_D^\epsilon(t)$. Then by the defintion of $Decr_D$ and the condition that $\beta : G_t \to G_s$ is an isomorphism, it follows that $\beta Decr_D^\epsilon(t) = Decr_D^\epsilon(s)$.

Thus we have proved that F strictly respects η.

We need to show that for any $p \in D$ the strategy F wins $G_p(0)$ against H. Let $(p, a)(p_0, a_0)(p_1, a_1)\ldots$ be a path consistent with F and H. We have the sequence $y_p, y_{p_0}, \ldots$ corresponding to the path. We reindex this sequence by $y, y_1, y_2, \ldots$. Note that there exists i such that $y_i = y_{i+1} = y_{i+2} = \ldots$. Since F^{y_i} strictly respects η, we obtain that the sequence

$$(p_i, a_i)(p_{i+1}, a_{i+1})\ldots$$

is consistent with $\beta_{p_i}^{-1} F^{y_i} \beta_{p_i}$ and H. It follows that the sequence

$$(p_i, a_i)(p_{i+1}, a_{i+1})\ldots$$

belongs to W_{p_i}. Hence $(p,a)(p_0,a_0)(p_1,a_1)\ldots \in W_p$. $\Box$

Let A be an arena, and let $C \subset A$. We define C_{inf} as the set of all lebelled paths α such that $card(l\alpha \bigcap C) = \omega$.

Theorem 4.2 *Let $C \subset A$. Consider the game $G = (A, C_{inf}, 0)$. Let E be the maximal congruence relation on G, namely*

$$E = \{(x,y) | G_x \cong G_y\}.$$

In this game one of the players has a winning oriented strategy which srictly respects E.

Proof. Define the set D by

$$D = \{p \in A | \text{ the player 1 has an oriented strategy winning the game } G_p(0) \text{ and srictly respecting } E\}$$

We have two cases.

Case 1. Suppose that for any left ray α the set $l\alpha \bigcap D$ is not empty. Then by the previous lemma there exists an oriented strategy H which respects E and which wins the game $G_p(0)$ for any $p \in D$. Hence H is an oriented strategy for player 1 which wins the game G and respects η.

Case 2. Suppose that there exists a left ray α such that $D \bigcap l\alpha = \emptyset$. By the lemma above $D = D(1)$. We seek a needed strategy for player 0 in the refinemet of the strategy $Avoid^0_D$. First, note that $A \setminus D \subset C(0)$. Indeed, otherwise applying the oriented strategy $Avoid^0_C$ to an element $p \in (A \setminus D) \setminus C(0)$ we would obtain that $p \in D$.

Let $p \in A \setminus D$. Then $Derc^0_C(p) \bigcap C(0) \bigcap (A \setminus D) \neq \emptyset$. Indeed, otherwise we again could find an oriented starategy which would win the game G_p and would strictly respect η.

Define the following strategy for player 0:

$$F(x) = \begin{cases} \{(y,b) | y \in Suc(x) \& P_b(y)\} & \text{if } x \in D, \\ Decr^0_C(x) \bigcap (A \setminus D) \bigcap C(0) & \text{if } x \in A \setminus D \end{cases}$$

The strategy F is a desired oriented strategy for player 0 which wins the game and strictly respects η. $\Box$

5 Uniform Determinacy Theorem

Let A be an arena in $\mathcal{T}_v$, S be a finite set of "colors", $C = (C_s, s \in S)$ be a list of subsets of the arena colored by a corresponding color s. We permit the members of C to have nonempty intersections. We define the notions of display and the latest appearance record as follows. We linearly order the set of colors S. A **display** is a word over S which does not have the same color repeated. Denote by $Display(S)$ the set of all such words.

Definition 5.1 *For any two nodes $x, y \in A, x \leq y$ define the latest appearance record of colors (LAR) as follows. Let $d \in Display(S)$. We define a function $LAR(x, d, y)$ by:*

$$LAR(x, d, x) = Delete(d \cdot l(x)),$$

where $\cdot$ is a concatenation of words, and $l(x)$ is a word whose letters are all of the colors from $\{s \in S \mid x \in C_s\}$ written in their linear order in S, Delete is an operation that deletes from two concatenated words the letters of the first word that also appear in the second word. Suppose $LAR(x, d, t)$ is defined, then for every $y \in Suc(t)$

$$LAR(x, d, y) = Delete(LAR(x, d, t) \cdot l(y)).$$

We cover any arena by a disjoint collection of rooted trees called **sectors** and define a congruence over the arena by means of this cover as follows.

Fix a node $p \in A$. Consider a left ray $\xi \subseteq A$ which ends at p. For every $j \in \xi$ define the sectors $Sect_j$ as follows:

$$Sect_j = \{x \in A \mid x \geq j \wedge \forall j' \in \xi (j < j' \rightarrow \neg(j' \leq x \bigvee x \leq j')\}.$$

For every node p as above and any display $d \in Display(S)$, we define a congruence over the arena A as follows. First abbreviate $ELAR$ as a binary relation generated over the arena by the sectors and LAR:

$$ELAR_1(x, y) \Leftrightarrow \exists j \in \xi (x, y \in Sect_j \& LAR(j, d, x) = LAR(j, d, y)).$$

We sometimes write $ELAR_1(x, y, d, p)$ to display the parameters d, p which gave rise to $ELAR_1(x, y)$ at hand.

The congruence is defined by

$$E_1^{d,p} = \{(x,y) \mid x,y \in A \& (A,C)_x \cong (A,C)_y \& ELAR_1(x,y)\}.$$

One can easily verify that the above binary relation is an equivalence. We abbreviate by $Bool(C)$ the collection of all sets of labelled paths from $LbP(A)$ which are boolean combinations of the sets $([C^s], s \in S)$.

Theorem 5.1 *(***Uniform Determinacy Theorem***) Consider an arena A, a finite set of colors S, a list C of colored subsets of A, a set of paths $W \in Bool(C)$ and a game $G = (A, W, 0)$. Fix a node $p \in A$ and a display $d \in Didsplay(S)$ and a congruence $E_1^{d,p}$ over the arena A. Then one of the players $\epsilon \in \{0,1\}$ wins the game G and has a winning strategy which strictly respects $E_1^{d,p}$.*

Proof. We begin by considering the case of only one color: $Card(S) = 1$. The list C denotes just one subset of the arena called C also. It follows that either $W = [C]$ or $W = [C]^c$ or $W = \emptyset$ or $W = LbP(A)$. By the last theorem from the previous section one of the players $\epsilon \in \{0,1\}$ has a winning strategy in the game G which respects the congruence

$$E = \{(x,y) \in A \times A \mid (A,C)_x \cong (A,C)_y\}.$$

Since it is clear that $E_1^{d,p} \subseteq E$, it follows that the winning strategy respects $E_1^{d,p}$. Hence the conclusion of the theorem follows for this case.

We show by induction on $Card(S)$ that the theorem holds for every S with $Card(S) \geq 1$. We may assume now that $Card(S) \geq 2$. Note that either

$$\bigcap_{s \in S} [C^s] \subseteq W$$

or

$$\bigcap_{s \in S} [C^s] \subseteq W^c.$$

Assuming that the theorem holds for every S' with $Card(S') < Card(S)$, we prove the theorem for the case

$$\bigcap_{s \in S} [C^s] \subseteq W.$$

The alternative case can be proved similarly. Recall that we abbreviate for every $q \in A$, $G_q(0) = (A_q, W_q, 0)$. Consider the set of winning nodes for the player 1 :

$$D = \{q \in A \mid \exists f' : T_1 \cap A_q \to T_2 \times \Sigma[(f' \ wins \ G_q(0)) \wedge (f' \text{ strictly respects } E_1^{d,p} \text{ over } A_q)]\}$$

Two cases arise with respect to the set D.

Case 1. For any left ray α such that $l\alpha \subseteq A$, we have $l\alpha \cap D \neq \emptyset$. By sewing lemma consider a strategy $H' : T_1 \to T_1 \times \Sigma$ which wins all games $G_q(0)$ for the player 1, where $q \in D$, and which strictly respects the congruence $E_1^{d,p}$. It follows that H' wins G for the player 1 in this case. Just choose $q \in l\alpha \cap D$ and note that H' wins $G_q(0)$.

Case 2. Assume that there exists a left ray α such that $l\alpha \cap D = \emptyset$. We seek a winning ortiented strategy for the player 0 which is a refinement of the oriented strategy $avoid_D^0$. Therefore we suppose that $D = \emptyset$. We need to prove that for each $k \in l\alpha$ the player 0 has a winning strategy in the game $G_k(0) = (A_k, W_k, 0)$ which respects the congruence $E_1^{d,p}$. Then by sewing lemma, the player 0 will have a winning strategy in the game G which respects the congruence $E_1^{d,p}$. Thereby the theorem will be proved.

For this purpose fix any $k \in l\alpha$. For every color $s \in S$, put $D^s = A \backslash C^s(0)$. We temporarily fix an $s \in S$. Consider now an arbitrary $q \geq k, q \in D^s$. Consider a game $G_q = (D_q^s, W^s, 0)$, with $D_q^s = A_k \cap D^s$, and the set W^s defined as follows. Consider the set W presented as a union of intersection of the sets $[C_t], t \in S$ and their complements. Consider such an intersection. If it refers to the set $[C_s]$, omit the entire intersection from W. Furthermore, if an intersection refers to $[C_s]^c$, omit this complement from the intersection, but do not omit the intersection unless it refers to $[C_s]$. This is the set W^s referred to in the game above. This winning set is based on the set of colors $S = S \setminus \{s\}$ of strictly lesser cardinality than that of the set S.

It easy to check that $W \cap Path(D^s) = W^s \cap Path(D^s)$.

We wish to use the inductive assumtion in order to show that the player 0 wins the game G_q with a strategy respecting $E_1^{d,p}$. To this end we define a congruence $E_1^{d',p'}$ over D_q^s. Let $p(k) = max\{j \in l\alpha \mid j \leq k\}$. We consider two cases to define the congruence. We will abbreviate the list of sets C with color s omitted by (C, s).

Case A. Suppose k is not in ξ, which is equivalent to $k \neq p(k)$. Then

$$E_1^{d',p'} = \{(x, y) \in D_q^s \times D_q^s \mid q \leq x, y \wedge$$

$$x, y \in Sect_{k(p)} \wedge (D^s_q, (C,s))_x \cong (D^s_q, (C,s))_y$$
$$\wedge LAR(q, d', x) = LAR(q, d', y)\}$$

with $d' = LAR(p(k), d, q) = LAR(k, LAR(p(k), d, k), q)$ and $p' = q$.

Case B. Assume that $k \in \xi$ which is equivalent to $k = p(k)$. We have two sucases.

Subcase B1. Assume that $k \leq q \leq p$. In this case define

$$E_1^{d',p'} = \{(x,y) \in D^s_q \times D^s_q \mid q \leq x, y \wedge$$

$$\exists j x, y \in Sect_j \wedge (D^s_q, (C,s))_x \cong (D^s_q, (C,s))_y$$
$$\wedge LAR(j, d', x) = LAR(j, d', y)\}$$

with $d' = d$ and $p' = p$.

Subcase B2. Assume that q is not in the linear segment $[k, p]$ of ξ. In this case $E_1^{d',p'}$ is defined as in **Case A**.

We check that $E_1^{d',p'} \subseteq E_1^{d,p} \cap D^s_q \times D^s_q$. Let $(x,y) \in E_1^{d,p}; x, y \geq q$. It follows that if $x \in D^s$, then $y \in D^s$. There is an isomorphism $\tau : (A,C)_x \to (A,C)_y$. Since τ is an isomorphism and the sets $C^s_i(0)$, $i \in \omega$, are definable in the language of (A,C), we get that

$$\tau(A_x \cap C^s(0)) = A_y \cap C^s(0).$$

Hence the restriction of τ on the set $A_x \setminus C_s(0)$ is an isomorphism from $(D^s_q, (C,s))_x$ into $(D^s_q, (C,s))_y$. We now have to show that $ELAR_1(x, y, d', p')$ holds in the game G_q.

Cases A or B2. According to the cases listed above we assume that $k \neq p(k)$ or $k \in \xi$ and q is not in the segment $[p(k), p] \subseteq \xi$.

We note that $ELAR_1(x, y, d, p)$ holds in the game G. It follows that

$$LAR(p(k), d, x) = LAR(p(k), d, y),$$

and

$$LAR(p, d, x) = LAR(q, LAR(p(k), d, q), x) = LAR(p(k), d, y).$$

Therefore

$$LAR(q, d', x) = LAR(q, d', y).$$

Case B1. For this case $k \leq q \leq p$ holds. From the definition of $E_1^{d',p'}$, we get that $ELAR_1(x, y, d', p')$ holds.

By induction one of the players has a winning strategy $f_{\epsilon,q}$ which strictly respects the congruence $E_1^{d',p'}(G_q)$. We will show that $\epsilon = 0$. To see this, suppose that $\epsilon = 1$ instead. We get a contradiction by constructing a winning strategy for the player 1 in the game $G_q(0) = (A_q, W_q, 0)$ which strictly respects $E_1^{d,p}$. Put

$$F_q(x) = \begin{cases} f_{\epsilon,q}(x) & \text{if } x \in D_q^s, \\ \emptyset & \text{if } x \in T_0, \\ Suc(x) & \text{if } x \geq q \text{ and } x \notin D_q^s \end{cases}$$

Since $q \in D_q^s$, the set $P_{F_q,q}$ of all elements consistent with F_q from q is a subset of D_q^s. It follows that this set is also consistent with $f_{\epsilon,q}$. Thus $P_{F_q,q} = P_{f_{\epsilon,q}}$ and

$$Path(P_{F_q,q}) \subseteq W_q^s \subset W_q.$$

Therefore F_q wins the game $G_q(0)$. For a contradiction we have to show that F_q strictly respects $E_1^{d,p}$.

Let $x, y \geq q$ and $(x, y) \in E^{d,p}$. We have to consider only two cases.

Case 1. Suppose that $x \in D_q^s$. In this case $y \in D_q^s$ too. As we have shown $(x, y) \in E_1^{d',p'}(G_q)$. Thus for any isomorphism $\tau : (A, C)_x \to (A, C)_y$ using the fact that $f_{\epsilon,q}$ strictly respects the congruence $E_1^{d',p'}(G_q)$, we get

$$\tau(P_{F_q,x}) = \tau(P_{f_{\epsilon,q},x}) = P_{f_{\epsilon,q},y} = P_{F_q,y}.$$

Case 2. Suppose x is not in D_q^s. Then y is not in D_q^s. Hence using the Case 1 and that $F_q(x) = Suc(x)$ we get similarly to the the Case 1 that $\tau(P_{F_q,x}) = P_{F_q,y}$.

We have thus demonstrated that $q \in D$, however D was assumed empty or in other words we have picked q outside of D. This forces us to discard the assumption $\epsilon = 1$ and conclude that $f_{\epsilon,q}$ is a strategy for player $\epsilon = 0$. Furthermore, the strategy F_q defined above wins the game $G_q(0)$ for the player 0 and strictly respects $E_1^{d,p}$. By the sewing lemma an oriented strategy H^s exists which strictly respects $E_1^{d,p}$ and wins $G_q(0)$ for all $q \in D^s$.

We now define a strategy F_k that would win the game $G_k(0)$ and would respect $E_1^{d,p}$ as follows:

$$F_k(x) = \begin{cases} Suc(x) & \text{if } x \in D, \\ H^{s*(x)}(x) & \text{if } x \in D^{s*(x)}, \\ decr_{C^{s*(x)}(0)}(x) \cap avoid_D^0 & \text{if } x \in C^{s*(x)}(0), \end{cases}$$

where

$$s * (x) = Head(LAR(p, d, x)),$$

if the length of the list $LAR(p, d, x)$ is equal to the number n of sets in the list C with $Head$ denoting the function of taking the leftmost item from a list of items, and

$$s * (x) = min\{s \in S \mid s \text{ does not occur in } \ LAR(p, d, x)\},$$

if the length of the list $LAR(p, d, x)$ is strictly less than n.

Now we prove that F_k wins the game $G_k(0) = (A_k, W_k, 0)$. Let $\mu = z_0, z_1, ...$ be a path consistent with F_k with $z_0 = k$.

If for each $s \in S$ the set $\mu \cap C^s$ is infinite, then $\mu \in W_k$ since the latter set contains $\cap\{[C^s], s \in S\}$.

Suppose now the opposite, i.e. that for some $s \in S$, μ does not meet C_s after some $t \in \tau$. Let $S* \subseteq S$ be the set of all such s. By the definition of $s * (x)$, it follows that for some $z \in \mu$, we have

$$z \geq t \forall x \in \mu (x \geq z \Rightarrow s * (x) = s$$

for some fixed $s \in S*$. It follows that for every $x \geq z, x \in \mu$ x is not in C_s, otherwise $F_k(x) = decr_{C^s(0)}(x)$ and the path μ will meet C^s contradicting the property $s \in S*$.

Since k is not in D, it follows that $\mu \cap D = \emptyset$ and $\mu_z \subseteq D^s$. Thus μ is consistent with H^s after z. Therefore $\mu \in W_z$. It follows immediately that F_k wins $G_k(0)$. We have shown that F_k wins the game $G_k(0)$.

We now check that F_k strictly respects $E_1^{d,p}$. Since $Suc(x)$ strictly respects any congruence, H^s strictly respects $E_1^{d,p}$ after k, $avoid_D^0$ strictly respects $E_1^{d,p}$ over the arena A, and $decr_{C^{s*(x)}(0)}(x)$ strictly respects $E_1^{d,p}$ over A, one can get that F_k strictly respects $E_1^{d,p}$. This completes the proof of the theorem. □

The following corollary follows from the statement of the theorem above and the the definition of invariant subsets.

Corollary 5.1 *Let A be an arena. Suppose that the sets C from the premise of the preceding theorem are invariant. Then one of the players has a winning apparent strategy which strictly respects $E^{d,p}$ in the game $(A,W,0)$, where W is as in the theorem.* □

For every node p and any display $d \in Display(S)$, we define a new congruence E over the arena A as follows. First abbreviate $ELAR$ a binary relation generated over arena A by:

$$ELAR(x,y) \Leftrightarrow ELAR_1(x,y) \bigvee \exists jj'(x \in Sect_j \& y \in Sect_{j'} \& j \neq j' \rightarrow LAR(j,d,x) = LAR(j'd,y)).$$

We sometimes write $ELAR(x,y,d,p)$ to display the parameters d,p which gave rise to $ELAR(x,y)$ at hand. Define a congruence by

$$E^{d,p} = \{(x,y) \mid x,y \in A \wedge (A,C)_x \cong (A,C)_y \wedge ELAR(x,y)\}.$$

One can repeat the proof of the uniform determinacy theorem and obtain the next result:

Corollary 5.2 *Consider an arena A, a finite set of colors S, a list C of colored subsets of A, a set of paths $W \in Bool(C)$ and a game $G = (A,W,0)$. Fix a node $p \in A$ and a display $d \in Didsplay(S)$ and a congruence $E^{d,p}$ over the arena A. Then one of the players $\epsilon \in \{0,1\}$ wins the game G and has a winning strategy which strictly respects $E^{d,p}$.* □

The theorem and corolaries above give us more elegant discribtion of winning strategies in terms of isomorphism types.

Definition 5.2 *Let $G = (A,W,0)$ be a game, where $W \in Bool(C)$. For every $x \in A$ consider a model (A_x, C).* **The isomorphism type of the node x** *is the class of all models isomorphic to (A_x, C). The set of all isomorphism types of the nodes from A we denote by $IT(A)$.*

Now we can formulate the following theorem which is a reformulation of the UDT and its corollaries.

Theorem 5.2 1. *Let $G = (A,W,0)$ be a game, where $W \in Bool(C)$. In this game one of the players has a winning strategy which can be regarded as a function from $IT(A) \times Display(S)$ to $IT(A) \times \Sigma$.*

2. Let $G = (A, W, 0)$ be a game, where $W \in Bool(C)$, and let C be a collection of invariant subsets of A. In this game one of the players has a winning strategy which can be regarded as a function from $IT(A) \times Display(S)$ to Σ. $\square$

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