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Abstract. In this note we disprove the following conjecture by M. van Lambalgen: *"We conjecture that at least for* **low** *degrees x, i.e. x with $x' \equiv_T \emptyset'$, even $I(x(n)) \leq n + c$"*. We also exhibit a class of nonrandom sequences with natural complexity-theoretic properties.

Keywords: Random sequence, Chaitin complexity, computably enumerable set, Low Basis Theorem, fixed-point free function, low degree.

Our notation is standard and follows [1], [9]. In particular, $\omega = \{0, 1, \ldots\}$ is the set of natural numbers and $\{W_e\}_{e\in\omega}$ is a standard enumeration of all computably enumerable sets. Let $\{0,1\}^*$ be the set of binary strings. By $\lambda y H(y)$[1] we denote the Chaitin complexity measure defined on $\{0,1\}^*$ and by μ we denote the usual product measure on $\{0,1\}^\omega$. The Turing degree of the Halting Problem is denoted by $\mathbf{0}'$; it is the degree of the Turing jump of the empty set (denoted by $\emptyset'$).

We use lower-case Greek letters for binary strings, and upper-case Latin letters for infinite binary sequences and sets of binary strings. We identify sets of natural numbers with their characteristic functions. For any $X \subseteq \omega$ by $\sigma \subseteq X$ we denote the fact that σ is an initial segment of X. Let $\sigma\{0,1\}^\omega$ be the class of all sequences X for which $\sigma \subseteq X$, and for a set $W \subseteq \{0,1\}^*$ let $W\{0,1\}^\omega$ be the union of all $\sigma\{0,1\}^\omega$, for $\sigma \in W$.

[1] Van Lambalgen has used the old notation $\lambda y I(y)$.

A class $\mathcal{A}$ of sequences from $\{0,1\}^\omega$ is a Σ^0_1-*class* if $\mathcal{A} = W_e\{0,1\}^\omega$ for some $e \in \omega$. A Π^0_1-*class* is the complement of a Σ^0_1-class. A class $\mathcal{A}$ of sequences from $\{0,1\}^\omega$ is of *effective Lebesgue measure zero* if there is a recursive sequence of Σ^0_1-classes $\mathcal{U}_0, \mathcal{U}_1, \ldots$ such that $\mu(\mathcal{U}_i) \le 2^{-i}$ for all i, and $\mathcal{A} \subseteq \bigcap_{i\in\omega} \mathcal{U}_i$.

Assume $n > 0$. Computably enumerable (c. e.) sets are called 1-*c. e.* A set X is $(n+1)$-*c. e.* if $X = X_1 \setminus X_2$ for some c. e. set X_1 and n-c. e. set X_2. A Turing degree is called n-c. e. if it contains an n-c. e. set. In particular, 2-c. e. sets (also known as the d-c. e. sets) are precisely the differences of c. e. sets, i. e. the sets of the form $B \setminus C$ with both B and C being c. e.

An equivalent way to define the notion of n-c.e. sequence is the following (e.g. see [9]). A sequence X is n-c. e. if there exists a recursive function f such that for every k,

$$X(k) = \lim_{s\to\infty} f(s,k), f(0,k) = 0 \text{ and } |\{s : f(s,k) \ne f(s+1,k)\}| \le n,$$

where $|X|$ denotes the cardinality of X.

One can extend this definition as follows: A sequence D is ω-*c. e.* if there exist two recursive functions f and g such that for every k, $X(k) = \lim_{s\to\infty} f(s,k)$, $f(0,k) = 0$ and $|\{s : f(s,k) \ne f(s+1,k)\}| \le g(k)$.

M. van Lambalgen states the following (in [6], p. 141):

Conjecture. *"We conjecture that at least for* **low** *degrees x, i.e. x with $x' \equiv_T \emptyset'$, even $I(x(n)) \le n + c$"*, where $x(n)$ is an initial segment of length n of the infinite sequence x and I is the Chaitin complexity measure.

Chaitin [2] gave the following important complexity-theoretic characterization of random sequences (see [1], p. 135):

A sequence X is random iff $\lim_{n\to\infty}(H(X(n)) - n) = \infty$.

Therefore, van Lambalgen's conjecture implies that every low sequence should be nonrandom. However, the first part of our theorem below proves the existence of a random sequence of low degree, which refutes this conjecture. The second part of the theorem exhibits a class of nonrandom sequences with natural complexity-theoretic properties.

Theorem. 1) *There exists an ω-c. e. random sequence A such that $A' \equiv_T \emptyset'$.*

2) *For $n \in \omega$, every sequence of n-c.e. degree strictly below $0'$ is nonrandom.*

Proof. The proof uses the Low Basis Theorem by Jockusch and Soare [8] which states that *every nonempty Π^0_1-class of infinite binary sequences contains a low sequence, i.e. a sequence A such that $A' \equiv_T \emptyset'$.*

We consider an universal Martin-Löf test, i.e. a recursive sequence of Σ^0_1-classes $\{\mathcal{U}_i\}_{i\in\omega}$ with following properties:

a) $\mathcal{U}_0 \supseteq \mathcal{U}_1 \supseteq \ldots$,

b) For every n, $\mu(\mathcal{U}_n) < 2^{-n}$,

c) Every class of effective Lebesgue measure zero is contained in the intersection $\bigcap_{i\in\omega} \mathcal{U}_i$.

The existence of such a sequence $\{\mathcal{U}_i\}_{i\in\omega}$ was proved by P. Martin-Löf [7]; see also [1].

Now we define the class $\mathcal{P}$ as the set of all $\{0,1\}$-sequences which are not included in some $\mathcal{U}_i$. Clearly, $\mathcal{P}$ is a Π^0_1-class. Martin-Löf's definition says, that *a sequence is random if it does not belong to $\bigcap_{i\in\omega} \mathcal{U}_i$, for some universal Martin-Löf test $\{\mathcal{U}_i\}_{i\in\omega}$.* So our class $\mathcal{P}$ contains only random sequences. By the Low Basis Theorem there exists a sequence $A \in \mathcal{P}$ such that $A' \equiv_T \emptyset'$.

The proof of the Low Basis Theorem actually shows that the sequence A' is an ω-c. e. sequence. Since for all X, X is 1-reducible to X', we conclude that A is an ω-c. e. low sequence. This proves part 1) of the theorem.

To prove part 2) we use the Generalised Completeness Criterion (see [9]) and a result of Kučera from [5]. Recall that a function h is called *fixed-point free* if $W_x \neq W_{h(x)}$, for all $x \in \omega$. The Generalized Completeness criterion states that *for every n-c. e. sequence X, if there is a fixed-point free function f such that $f \leq_T X$ then $X \equiv_T \emptyset'$.* Kučera proved in [5] that *every random sequence computes a fixed-point free function* (i. e. for every random sequence X there exists a fixed-point free function f with $f \leq_T X$). Now consider an arbitrary sequence X of n-c. e. degree strictly below $0'$. Suppose X is random. By Kučera's result, X computes some fixed-point free function. By the Generalised Completeness Criterion we see that X computes the Halting Problem. This is a contradiction, so the theorem is proved.

Remark. In [4] S. Kautz noted that there exists a random degree strictly below $\mathbf{0}'$.

Open Problem. G. J. Chaitin [3] has asked whether the methods in this paper might help to answer the more general question of which Turing degrees contain random sequences and which Turing degrees don't contain any random sequences.

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