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Uniform Orthogonal Group Divisible Designs with Block Size Three

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Abstract

The spectrum of orthogonal group divisible designs with block size three, and u groups each of size g, is studied. Existence is settled with few possible exceptions for each group size g.

1 Definitions and Background

A *pairwise balanced design* (or PBD) is a pair $(X, \mathcal{A})$ such that X is a set of elements called *points*, and $\mathcal{A}$ is a set of subsets (called *blocks*) of X, each of cardinality at least two, such that every pair of points is in a unique block of $\mathcal{A}$. If v is a positive integer and K is a set of positive integers, each of which is greater than or equal to two, then we say that $(X, \mathcal{A})$ is a (v, K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in \mathcal{A}$. We denote by $B(K)$ the set $\{v : \exists (v, K)\text{-PBD}\}$. A set is said to be PBD-*closed* if $B(K) = K$.

A PBD is *resolvable* if its blocks can be partitioned into parallel classes; a *parallel class* is a set of point-disjoint blocks whose union is the set of all points. The notation (v, K)-RPBD is used for a resolvable PBD. When $K = \{k\}$, a (v, K)-PBD is a *balanced incomplete block design*; the notations (v, k)-BIBD and (v, k)-RBIBD are sometimes used in this case.

A *group divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{A})$, which satisfies the three properties: (1) $\mathcal{G}$ is a partition of X into subsets called *groups*; (2) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point; and (3) every pair of points from distinct groups occurs in a unique block. Taking the groups of a GDD as blocks yields a PBD, and taking a parallel class of blocks of a PBD as groups yields a PBD.

The *group type* of a $GDD(X, \mathcal{G}, \mathcal{A})$ is a multiset $\{|G| : G \in \mathcal{G}\}$. We use exponential notation to describe group types: a group type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ denotes u_i occurrences of a group of size g_i for $1 \le i \le s$. Notationally, we permit $g_i = g_j$ when $i \ne j$. When all groups have the same size, the GDD is *uniform*. Groups of size 0 can be added as convenient. As with PBDs, we say that a GDD is a K-GDD if $|A| \in K$ for every $A \in \mathcal{A}$.

Let K be a set of positive integers, and let k be a positive integer. PBD$(v, K \cup k^\star)$ denotes a PBD containing a block of size k. If $k \notin K$, this indicates that there is exactly one block of size k in the PBD. On the other hand, if $k \in K$, then there is at least one block of size k in the PBD.

Theorem 1.1 *(Wilson's Fundamental Construction, WFC) [11]*
Let $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ be a GDD (the master *GDD) with groups $G_1, G_2, \ldots, G_t$. Suppose there exists a function $w : V \to \mathsf{Z}^+ \cup \{0\}$ (a* weight function*) which has the property that for each block $B = \{x_1, x_2, \ldots, x_k\} \in \mathcal{B}$ there exists a K-GDD of type $(w(x_1), w(x_2), \ldots, w(x_k))$ (an* ingredient *GDD). Then there exists a K-GDD of type $(\sum_{x \in G_1} w(x), \sum_{x \in G_2} w(x), \ldots, \sum_{x \in G_t} w(x))$.*

Let $(X, \mathcal{G}, \mathcal{B})$ and $(X, \mathcal{G}, \mathcal{D})$ be two 3-GDDs of type $g_1^{u_1} \cdots g_s^{u_s}$ having the *same* groups. These form *orthogonal 3-GDDs* if two *orthogonality conditions* are met:

1. if $\{x, y, z\} \in \mathcal{B}$ and $\{x, y, w\} \in \mathcal{D}$, then z and w are in different groups;
2. if $\{\{a, b, c\}, \{a, d, e\}\} \subset \mathcal{B}$ and $\{\{x, b, c\}, \{y, d, e\}\} \subset \mathcal{D}$, then $x \neq y$.

A pair of orthogonal 3-GDDs of type $g_1^{u_1} \cdots g_s^{u_s}$ is called an *OGDD of type* $g_1^{u_1} \cdots g_s^{u_s}$. We are concerned here with OGDDs with uniform group type g^u.

OGDDs of type 1^u are equivalent to orthogonal Steiner triple systems, and OGDDs played a central role in completing the existence theorem for orthogonal STSs:

Theorem 1.2 *[8] An orthogonal Steiner triple system of order v (or OGDD of type 1^v) exists if and only if $v \equiv 1, 3 \pmod 6$ and $v \notin \{3, 9\}$.*

In turn, orthogonal STSs were introduced in an effort to attack the existence question for Room squares [13].

The existence question for OGDDs in general, however, has not been previously addressed. OGDDs are of interest in their own right, and can be used to produce Room frames.

In this paper, we examine the existence question for uniform OGDDs; for every group size, we are left with few possible exceptions for the number of groups allowed. The basic necessary conditions for an OGDD of type g^u to exist are

1. $g \cdot (u - 1) \equiv 0 \pmod 2$;
2. $g \cdot u \cdot (u - 1) \equiv 0 \pmod 3$; and
3. $u \geq 4$.

The first two conditions are necessary (and sufficient when $u \neq 2$) for a 3-GDD of type g^u to exist. The third is a simple consequence of the definition of orthogonality.

2 Main Constructions

A few basic constructions are employed throughout. We introduce them here:

Lemma 2.1 *(Filling In Holes) Suppose there is an OGDD of type $g_1^{u_1} \cdots g_s^{u_s}$. Let x be a nonnegative integer, and suppose, for each $1 \leq i \leq s$ there is a OGDD of type $g_{i,1}^{u_{i,1}} \cdots g_{i,s_i}^{u_{i,s_i}} x^1$ with $\sum_{j=1}^{s_i} u_{ij} g_{ij} = g_i$ for $1 \leq i \leq s$. Then there is an OGDD whose type consists of $u_i \cdot u_{i,j}$ groups of size $g_{i,j}$ for $1 \leq i \leq s$ and $1 \leq j \leq s_i$, and one group of size x.*

The proof of Lemma 2.1 is routine and therefore omitted. When $x = 0$, the groups are simply broken up into smaller groups; when $x > 0$, one new group is added "at infinity".

A second main tool that we require is a PBD construction:

Lemma 2.2 *Let $(V, \mathcal{G}, \mathcal{B})$ be a K-GDD of group type $h_1^{u_1} \cdots h_s^{u_s}$. Let g be a positive integer, and suppose that, for every $k \in K$, there is an OGDD of type g^k. Then there is an OGDD of type $(g \cdot h_1)^{u_1} \cdots (g \cdot h_s)^{u_s}$.*

This is a simple variant of WFC (give every point weight g). Since a K-PBD of order v is a K-GDD of type 1^v, we obtain an important corollary:

Corollary 2.3 *If a K-PBD of order v exists, and for every $k \in K$, there is an OGDD of type g^k, then an OGDD of type g^v exists.*

Lemmas 2.1 and 2.2 are most useful in recursions that employ ingredients with "few groups" to make OGDDs with "many groups". We also require recursions that increase the group size.

Stinson and Zhu [14] employed conjugate orthogonal quasigroups (COQs) to provide such an inflation construction. Unfortunately, the existence spectrum for COQs is far from completed. Hence we elect to modify their definition somewhat, essentially by weakening a condition on COQs while strengthening a condition on OGDDs, that makes the inflation succeed. We make this precise next. Let Sym_3 be the symmetric permutation group on three letters, which we write as $\{123, 132, 213, 231, 312, 321\}$; ijk is a shorthand for $\pi(1) = i$, $\pi(2) = j$ and $\pi(3) = k$. A *quasigroup* is a set Q and a binary operation $\oplus$, so that for $x, y \in Q$, each of the three equations $x \oplus y = a$, $x \oplus b = y$ and $c \oplus x = y$ has a unique solution for the single unknown. Two quasigroups $(Q, \oplus)$ and $(Q, \otimes)$ are *orthogonal* if, given $a, b \in Q$, there is a unique pair (x, y) with $x, y \in Q$ that satisfy $x \oplus y = a$ and $x \otimes y = b$.

Let $(Q, \oplus)$ be a quasigroup with $|Q| = g$. Then on $Q \times \{1, 2, 3\}$, form a 3-GDD of type g^3, by including the block $\{x_1, y_2, (x \oplus y)_3\}$ for all $x, y \in Q$ (here and elsewhere, we write x_i as a shorthand for (x, i)). Given any 3-GDD of type g^3, this can be reversed to recover a quasigroup. Now consider a permutation $\pi \in Sym_3$, and permute the second coordinate of the symbols $Q \times \{1, 2, 3\}$ of the 3-GDD by mapping $\pi(i)$ to i for $i = 1, 2, 3$; then recover the quasigroup. It is easy to see that when π is not the identity, the quasigroup so obtained may differ from the original. The quasigroup obtained in this way is the π-*conjugate* of $(Q, \oplus)$.

Two quasigroups are *conjugate orthogonal* if every conjugate of one is orthogonal to every conjugate of the other. This is evidently a strong condition, and so we examine how to relax the condition (in a useful way). Let ψ be a relation from Sym_3 to itself. Then call a pair of (not necessarily distinct) quasigroups Q_1 and Q_2 a ψ-*COQ* if, whenever $\pi, \pi' \in Sym_3$ and $\pi' \in \psi(\pi)$, the π-conjugate of Q_1 is orthogonal to the π'-conjugate of Q_2. By choosing ψ so that all permutations are related, one recovers the definition of conjugate orthogonal quasigroups. Instead choosing ψ so that no permutations are related, an arbitrary pair of quasigroups form a ψ-COQ. But the useful choices lie between these two extremes.

We have described a way to weaken the definition of COQ; now we need a corresponding way to strengthen the definition of OGDD. For every block $\{x, y, z\}$ in both 3-GDDs forming the OGDD, we impose some total order on the elements of the block (any of six possible orders can be chosen independently for each block). Let (x_1, x_2, x_3) be an ordered block, and let $\pi \in Sym_3$; then $\pi((x_1, x_2, x_3)) = (x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$. Now consider any pair x, y of elements. These appear as elements (x_i, x_j) in some (ordered) block of the first 3-GDD, and as elements (y_a, y_b) in some (ordered) block of the second 3-GDD in an OGDD. Let π be the permutation in Sym_3 that maps $(1, 2)$ to (i, j), and let π' be the permutation in Sym_3 that maps $(1, 2)$ to (a, b). If for every choice of elements x and y, we find $\pi' \in \psi(\pi)$, the (ordered) OGDD is a ψ-*OGDD*.

Adding the constraint of a ψ-compatible ordering typically strengthens the requirements for an OGDD to exist. Now we establish that for various choices of ψ, the weakened notion of ψ-COQs and strengthened one of ψ-OGDDs provide the correct match.

Theorem 2.4 *Let ψ be a relation from Sym_3 to itself. Suppose that there is a ψ-OGDD of type $g_1^{u_1} \cdots g_s^{u_s}$. Let k be the order of a ψ-COQ. Then there is an OGDD of type $(k \cdot g_1)^{u_1} \cdots (k \cdot g_s)^{u_s}$.*

Proof: The proof parallels [14] closely. Let $(Q, \odot_1)$ and $(Q, \odot_2)$ be the first and second quasigroups of order k in a ψ-COQ. Let $(V, \mathcal{G}, \mathcal{B}_1)$ and $(V, \mathcal{G}, \mathcal{B}_2)$ be the first and second (ordered) 3-GDDs in a ψ-OGDD of type $g_1^{u_1} \cdots g_s^{u_s}$. We form the OGDD on $\widehat{V} = V \times Q$, with groups $\widehat{\mathcal{G}} = \{G \times Q : G \in \mathcal{G}\}$. Define

$$\mathcal{D}_i = \bigcup_{(x,y,z) \in \mathcal{B}_i} \bigcup_{u,v \in Q} \{\{(x, u), (y, v), (z, (u \odot_i v))\}\}.$$

Evidently $(\widehat{V}, \widehat{\mathcal{G}}, \mathcal{D}_i$ for $i \in \{1, 2\}$ form two 3-GDDs of type $(k \cdot g_1)^{u_1} \cdots (k \cdot g_s)^{u_s}$; our task is to show that they are orthogonal. We check the two conditions in turn. Consider two points (x, u) and (y, v) in different groups of $\widehat{V}$. They appear in a block with a third element (z_i, w_i) in $\mathcal{D}_i$ for $i \in \{1, 2\}$. Now z_1 and z_2 are in different groups since the triples $\{x, y, z_1\}$ and $\{x, y, z_2\}$ arise from the two orthogonal 3-GDDs (the ordering is immaterial for this).

To check the second requirement, suppose to the contrary that, for $i, j \in \{1, 2\}$, one has $\{(x_i, u_i), (y_i, v_i), (z_j, w_j)\} \in \mathcal{D}_j$. Immediately $z_1 \neq z_2$ since $\{x_1, y_1, z_j\} \in \mathcal{B}_j$ for $j \in \{1, 2\}$, but these two 3-GDDs are orthogonal. Moreover, $\{x_1, y_1\} = \{x_2, y_2\}$ by the orthogonality of $\mathcal{B}_j$ for $j \in \{1, 2\}$, so without loss of generality, we take $x = x_1 = x_2$ and $y = y_1 = y_2$. Hence $u_1 \neq u_2$ and $v_1 \neq v_2$. If in the ordered blocks of $\mathcal{B}_j$ ($j \in \{1, 2\}$), we find x in position α and y in position β, define the permutation π_j to be the one with $\pi_j(1) = \alpha$ and $\pi_j(2) = \beta$. Since $\mathcal{B}_j$ for $j \in \{1, 2\}$ form a ψ-OGDD, we have $\pi_2 \in \psi(\pi_1)$. So let $(Q, \oplus)$ be the π_1-conjugate of $(Q, \odot_1)$, and let $(Q, \otimes)$ be the π_2-conjugate of

$(Q, \odot_2)$; the property of the ψ-COQ ensures that $(Q, \oplus)$ and $(Q, \otimes)$ are orthogonal. But by construction, $w_1 = u_1 \oplus v_1 = u_2 \oplus v_2$ and $w_2 = u_1 \otimes v_1 = u_2 \otimes v_2$, a contradiction. /

One case of interest occurs when ψ is the universal relation; here we recover our usual notions of COQ and OGDD, and the theorem is then the inflation technique of Stinson and Zhu [14]. Zhang [17] explored the use of orderings defined by the relation $\psi(abc) = \{bca, cab\}$ for every $abc \in Sym_3$. Unfortunately in this case relatively few ψ-OGDDs are known.

We examine a relation that is less strict than Stinson and Zhu's, but stricter than that Zhang. Partition Sym_3 into three classes: $\{123, 213\}$, $\{132, 312\}$, and $\{231, 321\}$. Then $\pi \in \phi(\pi')$ if and only if π and π' are in different classes.

Our key observation is the following:

Theorem 2.5 *Every OGDD underlies a ϕ-OGDD.*

Proof: We must determine suitable orderings of the two 3-GDDs $(X, \mathcal{G}, \mathcal{B}_1)$ and $(X, \mathcal{G}, \mathcal{B}_2)$ forming the OGDD. To do this, form a graph H whose vertex set is $\mathcal{B}_1 \cup \mathcal{B}_2$. For $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$, place an edge between B_1 and B_2 if $|B_1 \cap B_2| \geq 2$ (since $\mathcal{B}_1$ and $\mathcal{B}_2$ are disjoint, this requires $|B_1 \cap B_2| = 2$). By construction, H is a cubic, bipartite graph; hence its edges can be properly coloured using three colours, say 0, 1, and 2.

Now for $i = 0, 1$ and for $B = \{x_0, x_1, x_2\} \in \mathcal{B}_{i+1}$, suppose that the three edges incident with B in H receive colours as follows: colour $\alpha + i$ to $\{x'_0, x_1, x_2\}$, colour $\beta + i$ to $\{x_0, x'_1, x_2\}$, and colour $\gamma + i$ to $\{x_0, x_1, x'_2\}$. Colour numbers are computed modulo 3. Then order the triple as $(x_\alpha, x_\beta, x_\gamma)$. Now consider a pair $\{x, y\}$ of elements, and find $\{x, y, z_j\}$ in $\mathcal{B}_j$ for $j = 1, 2$. These two blocks are connected by an edge in H, so suppose that that edge was coloured δ. Then $\{x, y\}$ occupy positions $\{\delta - 1, \delta + 1\}$ in the ordered block of the first system, but occupy positions $\{\delta - 2, \delta\}$ in the ordered block of the second system. /

The importance of this theorem is that, while at first blush we appear to have strengthened the restrictions on the OGDD, in practice we have not. But we have indeed weakened the prerequisites on COQs, so we take advantage of this here. First we establish the existence of some ingredient ϕ-COQs:

Lemma 2.6 *There is an idempotent ϕ-COQ of order q whenever $q \geq 7$ and q is a prime power. There are ϕ-COQs of order 4 and 5.*

Proof: See [6] for a finite field construction for $q = 7$ or $q \geq 9$; the two quasigroups in the COQs produced have a common mate, and hence have a common partition into transversals (any one of which ensures idempotence). For $q = 8$, there is an idempotent quasigroup which is orthogonal to each of its conjugates except itself (see [6]); two copies of the quasigroup form an idempotent ϕ-COQ. Finally, for $q = 4$ and 5, a construction of nonidempotent COQs is given in [6]. /

We employ a basic PBD construction to establish a fairly complete existence spectrum.

Lemma 2.7 *Let $(V, \mathcal{G}, \mathcal{B})$ be a K-GDD of type $g_1^{u_1} \cdots g_2^{u_s}$, for which*

1. *for each $k \in K$, there is an idempotent ϕ-COQ of order k;*
2. *for every $1 \leq i \leq s$, there is a ϕ-COQ of order g_i.*

Then there is a ϕ-COQ of order $v = |V|$.

Corollary 2.8 *If m is a positive integer and $m \notin \{2, 3, 6, 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, 30, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48, 51, 52, 54, 55, 60, 62\}$, then there is a ϕ-COQ of order m.*

Proof: If m is an integer not in the list given, then there is a PBD of order m (or GDD of type 1^m) whose block sizes are prime powers at least seven (see [4]); apply Lemma 2.7. /

We eliminate a few more exceptions to obtain the current existence theorem:

Theorem 2.9 *If m is a positive integer and $m \notin \{2, 3, 6, 10, 12, 14, 18, 26, 30, 38, 42\}$, then there is a ϕ-COQ of order m.*

Proof: First employ Corollary 2.8. Now when an OSTS(m) exists, consider the two orthogonal Steiner triple systems $(V, \mathcal{B}_1)$ and $(V, \mathcal{B}_2)$ of which the OSTS consists. Define two Steiner quasigroups $(V, \otimes_i)$ for $i = 1, 2$, by setting $x \otimes_i x = x$ for $x \in V$, and $x \otimes_i y = z$ whenever $\{x, y, z\} \in \mathcal{B}_i$. A Steiner quasigroup is equal to each of its six conjugates; hence if the quasigroups $(V, \otimes_i)$ for $i = 1, 2$ are orthogonal, then they form an idempotent COQ. But orthogonality of the STS implies orthogonality of their Steiner quasigroups; hence by Theorem 1.2, we have idempotent COQs of orders 15, 21, 33, 39, 45, 51, and 55.

Now the direct product of two ϕ-COQs is a ϕ-COQ, so this settles $20 = 4 \cdot 5$, $28 = 4 \cdot 7$, $35 = 5 \cdot 7$, $36 = 4 \cdot 9$, $40 = 5 \cdot 8$, $44 = 4 \cdot 11$, $52 = 4 \cdot 13$, and $60 = 4 \cdot 15$. Similarly, if a ϕ-COQ of order m in which the quasigroups have α common transversals exists, ϕ-COQs of orders a and $a + 1$ both exist, and a ϕ-COQ of order x exists with $0 \leq x \leq \alpha$, then a ϕ-COQ of order $ma + x$ exists by singular direct product; In this way, we obtain $33 = 8 \cdot 4 + 1$ and $48 = 11 \cdot 4 + 4$. Form a {7,8,9}-GDD of type $7^7 5^1$ by deleting two points from a group of TD(8,7); and one of type $8^7 5^1 1^1$ by truncating two groups of a TD(9,8) to 1 and 5 points. Then apply Lemma 2.7 to settle 54 and 62. Finally, it is easy to check that an OGDD of type g^u, upon application of the Steiner quasigroup construction to both 3-GDDs, yields two holey quasigroups (of type g^u) which are conjugate orthogonal. Hence when an OGDD of type g^u exists and a ϕ-COQ of order $g + 1$ exists, there is a ϕ-COQ of order $gu + 1$. Applying this with $g = 3$ and $u \in \{7, 11, 15\}$ (these OGDDs are constructed later) settles 22, 34, and 46. /

Theorem 2.10 *If m is a positive integer and $m \notin \{2, 3, 6, 10, 12, 14, 18, 26, 30, 38, 42\}$, and there is an OGDD of type g^u, then there is an OGDD of type $(m \cdot g)^u$.*

Proof: Apply Theorem 2.4 using $\psi = \phi$, and Theorem 2.9 to provide the ϕ-COQ. Theorem 2.5 orders the given OGDD as a ϕ-OGDD. /

3 Direct Constructions

A computer search was used to directly construct a number of small OGDDs needed to provide a basis for the recursive constructions described in §2. The hill-climbing method used is an extension of that described in [10] to construct orthogonal STSs. Following [10] we restrict the search space by imposing a prescribed automorphism structure on D_1 and D_2, and generate a set of *tactical configurations* for the GDDs to be constructed. Each tactical configuration prescribes which orbits need to be represented in each block so as to cover all differences within orbits (i.e. *pure* differences) and between orbits (i.e. *mixed* differences). We then construct a random *base* design D_1 by hill-climbing, and then attempt to construct a *mate* design D_2, again by hill-climbing, but subject to the constraint of orthogonality with D_1. Tactical configurations for D_1 and D_2 are selected in advance from the set already generated. As is the case with many types of triple systems and related structures hill-climbing was very successful in constructing many families of OGDDs, including some quite large ones.

For details of the hill-climbing algorithm the reader is referred to [10]. The algorithm for OGDDs is the same as that for orthogonal STSs, apart from the provision of a group structure and the implementation of the extra orthogonality constraint (1). In the following subsections we describe the automorphism and group structures that were used to construct the basic uniform OGDDs required in §4.

3.1 m-Cyclic Designs

A GDD $D = (X, \mathcal{G}, \mathcal{A})$ of type g^u where $gu = mt$ is *m-cyclic* if X can be represented as $Z_t \times I_m$ where $I_m = \{0, 1, \ldots, m-1\}$, and D has an automorphism of the form $\alpha = (0_0, 1_0, ..., (t-1)_0))(0_1, 1_1, ..., (t-1)_1)...(0_{m-1}, 1_{m-1}, ..., (t-1)_{m-1})$, where the element (x, i) of $Z_t \times I_m$ is written as x_i. Note that α must preserve both $\mathcal{A}$ and $\mathcal{G}$. A necessary condition for the existence of D is that $g^2 u(u-1) \equiv 0 \pmod{6m}$. D can be represented by a set of $g^2 u(u-1)/6m$ base blocks, with the remaining blocks obtained by applying the automorphism $\alpha^i, i = 1, 2, ..., t-1$ to the base blocks.

We describe now a possible group structure for an m-cyclic GDD of type g^u. Take an integer s where $1 \leq s \leq u$, $u \equiv 0 \pmod{s}$, $t \equiv 0 \pmod{s}$, and $g \equiv 0 \pmod{h}$ where $h = t/s$. Let $a = u/s$ and

$b = g/h$. Define group $G_{x,i} = \{(x+y)_{i+j} \mid y = 0, s, ..., (h-1)s, j = 0, 1, ..., b-1\}$ where $0 \le x < s$ and $i \equiv 0 \pmod{b}, 0 \le i < m$. Then the set of groups $\mathcal{G} = \{G_{x,i} \mid x = 0, 1, ..., s-1, i = 0, b, ..., (a-1)b\}$ is invariant under the action of α. Given an m-cyclic GDD of type g^u, knowledge of s enables us to uniquely construct $\mathcal{G}$. We therefore refer to s as the *group structure type* of the GDD.

For example, Table 1 gives the group structure for a 3-cyclic GDD of type 5^9 and group structure type 3. Each row of elements in the table defines a group. Table 2 gives the base blocks of the OGDD.

0_0	3_0	6_0	9_0	12_0
1_0	4_0	7_0	10_0	13_0
2_0	5_0	8_0	11_0	14_0
0_1	3_1	6_1	9_1	12_1
1_1	4_1	7_1	10_1	13_1
2_1	5_1	8_1	11_1	14_1
0_2	3_2	6_2	9_2	12_2
1_2	4_2	7_2	10_2	13_2
2_2	5_2	8_2	11_2	14_2

Table 1: Group structure for 3-cyclic GDD of type 5^9.

			Base blocks of D_1			
$0_0 14_0 4_0$	$0_0 7_0 14_1$	$0_0 13_0 2_1$	$0_0 8_1 0_1$	$0_0 6_1 14_2$	$0_0 5_1 1_2$	$0_0 13_1 8_2$
$0_0 11_1 3_2$	$0_0 1_1 7_2$	$0_0 3_1 4_2$	$0_0 10_1 0_2$	$0_0 12_1 9_2$	$0_0 9_1 11_2$	$0_0 10_2 12_2$
$0_0 13_2 5_2$	$0_0 6_2 2_2$	$0_1 14_1 4_1$	$0_1 2_1 0_2$	$0_1 4_2 3_2$	$0_1 14_2 9_2$	
			Base blocks of D_2			
$0_0 4_0 1_1$	$0_0 5_0 0_1$	$0_0 2_0 9_1$	$0_0 8_0 11_1$	$0_0 14_0 0_2$	$0_0 6_1 4_1$	$0_0 13_1 12_2$
$0_0 14_1 5_2$	$0_0 2_1 14_2$	$0_0 5_1 9_2$	$0_0 8_1 8_2$	$0_0 11_2 10_2$	$0_0 6_2 13_2$	$0_0 4_2 2_2$
$0_0 3_2 7_2$	$0_1 4_1 11_2$	$0_1 8_1 9_2$	$0_1 5_1 10_2$	$0_1 1_1 3_2$	$0_1 13_2 8_2$	

Table 2: Base blocks of 3-cyclic OGDD of type 5^9.

Most of the designs constructed as bases for the recursive constructions needed in §4 are m-cyclic OGDDs. Table 3 lists the parameters and group structure types of these designs.

For some OGDDs (particularly those of small order), a more effective strategy is to use hill-climbing to construct D_1 and then to construct D_2 by backtracking. This was used in [8] to construct the OGDDs of types 2^u for $u \in \{7, 9, 10, 12, 13, 15, 16\}$, $4^9 2^1$, 4^{10}, 6^8, and 6^9 which are required in §4. In addition Table 4 contains parameters of some other required OGDDs which were constructed by this method.

3.2 2-rotational and near relative difference set designs

We define a GDD $D = (X, \mathcal{G}, \mathcal{A})$ of type 2^u where $2u = 2(t+1)$ to be *2-rotational* if X can be represented as $\{\infty_0, \infty_1\} \cup (Z_t \times Z_2)$ and D has an automorphism of the form $\alpha = (\infty_0)(\infty_1)(0_0, 1_0, ..., (t-1)_0))(0_1, 1_1, ..., (t-1)_1)$. A 2-rotational structure was used to construct OGDDs of type 2^{6r}. If the group structure $\mathcal{G} = \{\{\infty_0, \infty_1\}\} \cup \{\{i_0, i_1\} \mid i = 0, 1, ..., (t-1)\}$ is used, then each GDD must cover, in $4r$ base blocks, a total of $3r-1$ pure differences and $6r-4$ mixed differences, as well as pairing off a representative from each orbit with ∞_0 and ∞_1. A possible tactical configuration achieving this is shown in Table 2. In the table $r_1 = \lfloor r/2 \rfloor$ and $r_2 = \lceil r/2 \rceil$.

Using this structure 2-rotational OGDDs of type g^u were constructed for $u = 18, 36, 42, 48, 54, 66, 102$ and 114.

In [8] near relative difference sets were used to construct a number of small OGDDs. Using a combination of hill-climbing for D_1 and backtracking for D_2 OGDDs of types 2^{24}, 2^{27} and 2^{39} were constructed using this structure. They are displayed in Appendix A4.

g	2	2	2	2	2	2	3	3	3	3	3	3
u	19	21	25	28	40	52	7	9	13	15	19	21
m	2	3	1	1	1	1	1	3	1	1	1	1
s	19	7	25	28	40	52	7	9	13	15	19	21

g	3	3	3	3	3	3	3	3	3	3	3	3
u	23	25	27	29	33	37	39	47	53	59	67	83
m	1	1	1	1	1	1	1	1	1	1	1	1
s	23	25	27	29	33	37	39	47	53	59	67	83

g	3	4	4	4	4	4	4	4	4	4	4	4
u	87	6	9	12	16	18	22	24	28	34	40	52
m	1	3	3	3	1	3	1	3	1	1	1	1
s	87	2	3	4	16	6	22	8	28	34	40	52

g	4	4	5	6	6	6	6	6	6	6	6	6
u	58	94	9	5	7	11	12	13	16	17	19	20
m	1	1	3	2	2	2	1	1	1	1	2	1
s	58	94	3	5	7	11	12	13	16	17	19	20

g	6	6	6	6	6	6	6	7	9	9	9	9
u	23	24	27	32	39	44	52	9	5	9	11	17
m	2	1	2	1	2	1	1	3	1	1	1	1
s	23	24	27	32	39	44	52	3	5	9	11	17

g	9	9	12	12	12	12	12	12	12	12	18	21
u	23	83	5	10	12	16	18	20	24	32	11	5
m	1	1	1	1	1	1	1	1	1	1	2	1
s	23	83	5	10	12	16	18	20	24	32	11	5

Table 3: OGDDs constructed by hill-climbing.

g	15	18	24	27	33
u	5	5	6	5	5
m	1	1	1	1	1
s	5	5	6	5	5

Table 4: OGDDs constructed by hill-climbing and backtracking.

4 Group Size 0 (mod 6)

Existence is attacked by a combination of direct constructions, recursive constructions that break up groups, and recursive constructions that inflate groups. Solutions with small groups are inflated to make solutions with large groups, which in turn are filled in to make more solutions with small groups. In order to organize the presentation so that all required ingredients are in place before a result is presented, one requires much repetition. Hence we present the arguments organized by group size, trusting the reader to verify that ingredients needed are indeed produced in the paper.

In this section, we treat cases in which the group size g satisfies $g \equiv 0 \pmod 6$.

4.1 $g = 36$

Lemma 4.1 *There is an OGDD of type* 36^u *for* $5 \leq u \leq 9$.

Proof: Apply Theorem 2.10 with $m = 4$, $g = 9$, and $u \in \{5, 7, 9\}$ (from Lemma 5.2) to handle 5, 7, and 9. There is a 5-GDD of type 4^6 [16]; apply Lemma 2.2 with $g = 9$. There is a 7-GDD of type 6^8 [1]; apply Lemma 2.2 with $g = 6$. /

Corollary 4.2 *There is an OGDD of type* 36^u *for* $u = 21, 25, 26, 30, 31$ *and for all* $u \geq 35$.

Block type	No. base blocks
$\{\infty_0, x_0, y_1\}$	1
$\{\infty_1, x_0, y_1\}$	1
$\{0_0, x_0, y_0\}$	r_1
$\{0_0, x_0, y_1\}$	$3r_2 - 1$
$\{0_0, x_1, y_1\}$	$3r_1 - 1$
$\{0_1, x_1, y_1\}$	r_2

Table 5: Tactical configuration for 2-rotational OGDDs.

Proof: The stated values are all in $B(\{5, 6, 7, 8, 9\})$; apply Corollary 2.3. /

Now we clean up the remaining cases. For all remaining odd values of u, apply Theorem 2.10 with $m = 4$ and $g = 9$ (from Lemma 5.2). For $u \in \{10, 12, 16, 18, 22, 24, 28, 34\}$, apply Theorem 2.10 with $m = 9$ and $g = 4$ (using Lemma 6.3). Bennett and Yin give $\{5, 9, 13\}$-GDDs of type 4^u for $u \in \{14, 20, 32\}$; apply Theorem 2.3 with $g = 9$. In summary, we can state:

Lemma 4.3 *An OGDD of type* 36^u *exists for all* $u \geq 5$.

4.2 $g = 12$

Lemma 4.4 *There is an OGDD of type* 12^u *whenever* $u \geq 5$ *is odd.*

Proof: If $u \geq 7$, apply Theorem 2.10 with $m = 4$ and $g = 3$ (from Lemma 5.3). For $u = 5$, a direct construction is given in §3. /

Lemma 4.5 *There is an OGDD of type* 12^u *for* $u = 6, 8, 26, 30$ *and all* $u \geq 36$.

Proof: For $u = 6$, a direct construction is given in §3. For $u = 8$, there is a 7-GDD of type 6^8; apply Lemma 2.2 with $g = 2$. The remaining numbers specified are in $B(\{5, 6, 7, 8, 9\})$ [4]; apply Corollary 2.3. /

Deleting a point from a TD(13,13) gives a 13-GDD of type 12^{14}; apply Lemma 2.2 with $g = 1$ to settle $u = 14$. Deleting a point from a (169,7,1)-design [1] gives a 7-GDD of type 6^{28}; apply Lemma 2.2 with $g = 2$ to settle $u = 28$ (using Lemma 6.2). Extending three parallel classes in a resolvable 9-GDD of type 3^{33} gives a {9,10}-GDD of type 3^{34}; apply Lemma 2.2 with weight 4 (using Lemma 6.3) to treat $u = 34$. Solutions for u =10, 12, 16, 18, 20, 22, 24, and 32 are given in §3.

In summary, we state

Lemma 4.6 *There is an OGDD of type* 12^u *for all* $u \geq 5$.

4.3 $g = 18$

Lemma 4.7 *There is an OGDD of type* 18^u *for all* $u \geq 5$ *except possibly when* $u \in \{6, 22, 23, 34\}$.

Proof: A direct construction when $u \in \{5, 11\}$ is given in §3. For $u \in \{7, 9, 10\}$, apply Lemma 2.10 with $m = 9$ and $g = 2$. For $u = 8$, apply Lemma 2.2 with $g = 3$ to a 7-GDD of type 6^8. Now by Lemma 2.3, if $u \in B(\{5, 7, 8, 9\})$, then there is an OGDD of type 18^u. This handles u =21, 25, 35, 36, 37, 40, 41, 45, 47, 48, 49, 50, 53, 54, 55, 56, 57, 58, 59, all numbers $61 \leq u \leq 93$, and all numbers $u \geq 244$ [12]. We first treat the remaining cases for $u \leq 60$, then clean up the range $94 \leq u \leq 243$.

For $u \in \{12, 13, 15, 16, 18, 19, 24, 27, 28, 30, 31, 33, 39, 42, 43, 52, 60\}$, use Lemma 2.10 with $m = 9$ and $g = 2$ to produce the needed OGDDs. For $u = 29$, apply Lemma 2.1 using 72^7 and 18^5; and using

180^5 and 18^{11} to handle $u = 51$. Now when a TD(10,m) exists, truncating a group to 9 points leaves a $\{9, 10, m\}$-GDD of type 9^{m+1}; since OGDDs of type 2^9 and 2^{10} exist, if an OGDD of type 2^m also exists then an OGDD of type 18^{m+1} exists by Lemma 2.2. Apply with $m+1 \in \{14, 17, 20, 26, 32, 38, 44\}$. Adding a point to the groups of TD(5,9) gives a $\{5, 10\}$-PBD on 46 points; apply Corollary 2.3 with $g = 18$.

Now we finish off the interval $94 \leq u \leq 243$. Truncating three groups of a TD(10,m) to a, b, and c points gives a $\{7, 8, 9, 10, m, a, b, c\}$-PBD on $7m + a + b + c$ points. Then Corollary 2.2 can be applied if OGDDs of type 18^u for $u = m, a, b, c$ all exist. In a similar way, adding an infinite point to the groups establishes that a $\{7, 8, 9, 10, m+1, a+1, b+1, c+1\}$-PBD on $7m + a + b + c + 1$ points exists, and Corollary 2.2 can be applied if OGDDs of type 18^u for $u = m+1, a+1, b+1, c+1$ all exist. The determination of particular values of m, a, b and c for every $94 \leq u \leq 243$ is routine and therefore omitted. /

4.4 $g = 6$

Lemma 4.8 *There is an OGDD of type* 6^u *for* $u = 5$*, and all* $u \geq 7$ *except possibly when* $u \in \{10, 14, 18, 22, 26, 30, 38, 42, 46\}$.

Proof: Direct constructions when $u \in \{5, 7\}$ are in §3, and for $u = 9$ in [8]. When $u = 8$, apply Lemma 2.2 with $g = 1$ to a 7-GDD of type 6^8. Now by Corollary 2.3, if $u \in B(\{5, 7, 8, 9\})$, then there is an OGDD of type 6^u. By [12], this handles u =21, 25, 35, 36, 40, 41, 45, 47, 48, 49, 50, 54, 56, 57, 58, 59, all numbers $61 \leq u \leq 93$, and all $u \geq 244$. First we treat smaller values.

There are 7-GDDs of type 3^{15} and 6^{28}; apply Lemma 2.2. There is a resolvable 9-GDD of type 3^{33} and hence a $\{9, 10\}$-GDD of type 3^{34}; apply Lemma 2.2 to both. Now apply Lemma 2.1 using OGDDs of types 6^5 and $24^{\frac{u-1}{4}}$ to handle $u \in \{29, 53\}$, types 6^7 and $36^{\frac{u-1}{6}}$ when $u \in \{31, 37, 43, 55\}$, types 6^8 and 42^7 when $u = 50$, types 6^{11} and 60^5 when $u = 51$, and types 6^{12} and 72^5 for $u = 60$. The values

$$u \in \{11, 12, 13, 16, 17, 19, 20, 23, 24, 27, 32, 39, 44, 52\}$$

are treated in §3.

To handle the range $94 \leq u \leq 243$, we truncate two groups in a TD(9,m), and possibly extend the groups, to get a $\{7, 8, 9, m, a, b\}$-PBD on $7m + a + b$ points or a $\{7, 8, 9, m+1, a+1, b+1\}$-PBD on $7m + a + b + 1$ points. Then Corollary 2.3 can be applied when OGDDs of type 6^u exist for $u \in \{m, a, b\}$ to get $7m + a + b$, and for $u \in \{m+1, a+1, b+1\}$ to get $7m + a + b + 1$. Some easy arithmetic (taking m =11, 13, 16, 17, 19, 23, 25, 27, and 31) yields solutions except when $u = 118$, but $118 \in B(\{5, 7, 8, 9\})$. /

4.5 Wrapping up

First we treat the case when $g = 24$:

Lemma 4.9 *There is an OGDD of type* 24^u *for all* $u \geq 5$.

Proof: Apply Lemma 2.10 with $m = 4$, using Lemma 4.8 to produce OGDDs of type 24^u for u =5, 8, 12, 16, 20, 24, 28, 32, and 34. Apply Lemma 2.10 with $m = 8$ and OGDDs of type 3^u for all odd $u \geq 7$. An OGDDs of type 24^6 is given in §3. Now $B(\{5, 6, 7, 8, 9\})$ contains 26, 30, and all $u \geq 36$ [4]; apply Corollary 2.3. For $u = 10$, remove a block from a TD(10,13) to get a $\{9, 10\}$-GDD of type 12^{10} and apply Lemma 2.2 with $g = 2$. For $u = 14$, delete a point from TD(13,13) to get a 13-GDD of type 12^{14} and apply Lemma 2.2 with $g = 2$. There is a $\{5,13\}$-GDD of type 4^{18} and a $\{5,9\}$-GDD of type 4^{22}; apply Lemma 2.2 with $g = 6$. /

Next we treat the case when $g = 72$:

Lemma 4.10 *There is an OGDD of type* 72^u *for all* $u \geq 5$.

Proof: Apply Lemma 2.10 with $m = 4$ to handle all values of u except when $u \in \{6, 22, 23, 34\}$ (using Lemma 4.7). Apply Lemma 2.10 with $m = 8$ to handle u =23 (using Lemma 5.2). There is a $\{5,6\}$-GDD of type 6^6 (remove a block and a group from TD(7,7)); apply Lemma 2.2 with $g = 12$. By [5], there is a $\{5,9\}$-PBD of type 4^{22} and of type 4^{34}; apply Lemma 2.2 with $g = 18$. ⁄

Our summary result when $g \equiv 0 \pmod 6$ is

Theorem 4.11 *There is an OGDD of type $(6m)^u$ whenever $m \geq 1$ and $u \geq 5$, except possibly when*

1. $m = 1$ *and* $u \in \{6, 10, 14, 18, 22, 26, 30, 38, 42, 46\}$*;*
2. $m = 3$ *and* $u \in \{6, 22, 23, 34\}$*;*
3. $m = 9$ *and* $u \in \{6, 14, 22, 26\}$*;*
4. $m \equiv 1, 5 \pmod 6$*,* $m \geq 5$*, and* $u \in \{6, 14, 22, 26, 38\}$*; and*
5. $m \equiv 3 \pmod 6$*,* $m \geq 15$ *and* $u \in \{6, 22\}$*.*

Proof: Suppose $m = 2s$ is even. If $s \notin \{2, 3, 6, 10, 12, 14, 18, 26, 30, 38, 42\}$, then apply Lemma 2.10 with weight s to an OGDD of type 12^u from Lemma 4.6. If $s \in \{2, 10, 14, 18, 26, 38\}$, apply Lemma 2.10 with weight $s/2$ to an OGDD of type 24^u from Lemma 4.9. If $s = 3$, the result is Lemma 4.7. If $s \in \{6, 30, 42\}$, apply Lemma 2.10 with weight $s/6$ to an OGDD of type 72^u from Lemma 4.10. This completes the cases when m is even.

Now we handle cases when m is odd. Lemma 4.8 handles $m = 1$ and Lemma 4.7 handles $m = 3$. Now when $m \equiv 1, 5 \pmod 6$, Lemma 2.10 handles all values of u except those missing for $m = 1$.

When $u \in \{10, 18, 30, 42, 46\}$ and $m > 1$, apply Lemma 2.10 with weight $3m$ using Lemma 6.2.

Finally, when $m \equiv 3 \pmod 6$ write $m = 3t$. When $t \geq 5$, apply Lemma 2.10 using weight t using Lemma 4.7 to handle all values of u except for $u \in \{6, 22, 34\}$. Then apply Lemma 2.10 with weight $3t$ when $u = 34$.

The case when $g = 54$ remains. Applying Lemma 2.10 with weight 27 and Lemma 6.2, and with weight 9 and Lemma 4.8, leaves the values $u \in \{6, 14, 22, 26, 38\}$. Truncate a group of TD(27,37) to produce a $\{27,28\}$-GDD of type 27^{38} and apply Lemma 2.2 with weight 2 to treat $u = 38$. The remaining cases are $u \in \{6, 14, 22, 26\}$. ⁄

5 Group Size 3 (mod 6)

Lemma 5.1 *Let γ be odd. If there is an OGDD of type $(3\gamma)^u$ for*

$$u \in \{7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37, 39, 47, 53, 59, 67, 83, 87\},$$

then an OGDD of type $(3\gamma)^u$ exists for all odd $u \geq 7$.

Proof: If $v \in B(\{5, 7, 8, 9\})$, then there is an OGDD of type 3^{2v-1} using Lemma 2.2 with weight 6, deleting a point from the $\{5,7,8,9\}$-PBD of order v to form groups. This handles u =41, 49, 69, 71, 73, 79, 81, 89, 93, 95, 97, 99, all values $105 \leq u \leq 185$ except 119, and all values $u \geq 487$.

Apply Lemma 2.1 using $(3\gamma)^7$ and $(18\gamma)^{\frac{u-1}{6}}$ to obtain

$$u \in \{43, 49, 55, 61, 73, 79, 91, 97, 103\};$$

using $(3\gamma)^7$ and $(21\gamma)^{\frac{u}{7}}$ to obtain $u \in \{35, 63, 77\}$; using $(3\gamma)^9$ and $(24\gamma)^{\frac{u-1}{8}}$ to obtain $u \in \{57, 65\}$; using $(3\gamma)^9$ and $(27\gamma)^{\frac{u}{9}}$ to obtain $u = 45$; using $(3\gamma)^{11}$ and $(30\gamma)^{\frac{u-1}{10}}$ to obtain $u \in \{51, 101\}$; using $(3\gamma)^{13}$ and $(42\gamma)^7$ to obtain $u = 85$; using $(3\gamma)^{15}$ and $(45\gamma)^5$ to obtain $u = 75$, and using $(3\gamma)^{17}$ and $(51\gamma)^7$ to obtain $u = 119$. Now we have a solution for all odd u satisfying $7 \leq u \leq 185$.

For $189 \leq u \leq 485$, one can always write $\frac{u-1}{2}$ in the form $7m + a + b$ where $m \in \{13, 16, 19, 25, 31\}$, and $0 \leq a, b \leq m$, $\{a, b\} \cap \{1, 2\} = \emptyset$. Then truncate two groups of a TD(9,m) to a and b points and apply Lemma 2.2 with weight 6γ to get an OGDD of type $(6\gamma m)^7(6\gamma a)^1(6\gamma b)^1$; then apply Lemma 2.1. For $u = 187$, form a $\{7, 8, 13\}$-PBD of type $7^{12}9^1$ by taking $a = b = 1$ and $m = 13$ in the truncation. ⁄

Now we clean up group size 9:

Lemma 5.2 *There is an OGDD of type 9^u for all odd $u \geq 5$.*

Proof: Apply Lemma 5.1 first. Now apply Lemma 2.10 with $m = 9$ using Theorem 1.2 to handle all remaining cases for u satisfying $u \equiv 0, 1 \pmod 3$, except for $u = 9$. An OGDD of type 9^5 is given in §3. Use this with Lemma 2.1 and OGDDs of types $36^{\frac{u-1}{4}}$ to obtain $u \in \{29, 53\}$. Apply Lemma 2.1 using 9^5 and 45^{19} to handle $u = 95$. Apply Lemma 2.2 with weight 9 to a $\{5, 7, 9\}$-PBD of order 47 [3] to handle $u = 47$. Truncate a group of TD(10,27) to 18 points and apply Lemma 2.2 with weight 2 to get an OGDD of type $54^9 36^1$; fill holes with nine infinite points to handle $u = 59$. The remaining cases, when $u \in \{9, 11, 17, 23, 83\}$, are given in §3. /

Lemma 5.3 *There is an OGDD of type $(3\gamma)^u$ whenever $u \geq 7$, and $u \equiv \gamma \equiv 1 \pmod 2$.*

Proof: If $\gamma \geq 5$, apply Lemma 2.10 with weight γ to the solution for 3^u. If $\gamma = 3$, apply Lemma 5.2. So assume henceforth that $\gamma = 1$. Apply Lemma 5.1 to get a finite list of cases to treat. Then apply Lemma 2.1 using 3^7 and 18^5 to obtain $u = 31$. Solutions for $u \in \{11, 17\}$ appear in [8]. The cases when $u \in \{7, 9, 13, 15, 19, 21, 23, 25, 27, 29, 31, 33, 37, 39, 47, 53, 59, 67, 83, 87\}$ are all in §3. /

There remains one case for which our solution is incomplete, namely OGDDs of type $(3\gamma)^5$. When $\gamma = 1$, an exhaustive backtrack similar to that undertaken in [9], demonstrates that *no solution exists.* In §3, solutions are given when $\gamma \in \{3, 5, 7, 9, 11\}$, however. We return to this point later.

6 Group Size $2, 4 \pmod 6$

6.1 $g = 2$

A general construction to be applied repeatedly is of much use:

Lemma 6.1 *Let $x \in \{0, 1\}$. Suppose that a TD(9,m) exists, that $0 \leq a, b \leq m$, and that OGDDs of type 2^{3m+x}, 2^{3a+x} and 2^{3b+x} all exist. Then there is an OGDD of type $2^{27m+3a+3b+x}$.*

Proof: Truncate two groups of a TD(9,m) to a and b points. Apply Lemma 2.2 with weight 6 to get an OGDD of type $(6m)^7(6a)^1(6b)^1$, and fill its holes (with $2x$ infinite points) using the stated ingredients. /

For later use, we record the inadmissible values for m, a, and b. When $x = 0$, none can be in $\{1, 2, 17, 23, 29, 31, 41\}$; when $x = 1$, none can be in $\{1, 7, 11, 19, 31, 47\}$. These lists arise from the definite and possible exceptions from Lemma 6.2.

Lemma 6.2 *There is an OGDD of type 2^u whenever $u \equiv 0, 1 \pmod 3$, $u \geq 7$, except possibly when $u \in \{22, 34, 51, 58, 69, 87, 93, 94, 123, 142\}$.*

Proof: Direct constructions are given in §3 when

$$u \in \{18, 19, 21, 24, 25, 27, 28, 33, 36, 39, 40, 42, 48, 52, 54, 66, 102, 114\},$$

and in [8] when $u \in \{7, 9, 10, 12, 13, 15, 16\}$. Applying Lemma 2.1 using 2^7 and $12^{\frac{u-1}{6}}$ handles all $u \equiv 1 \pmod 6$ when $u \geq 31$. Applying Lemma 2.1 using 2^7 and $14^{\frac{u}{7}}$ handles $u \in \{70, 84, 105, 112\}$; using 2^9 and $16^{\frac{u-1}{8}}$ handles $u = 57$; using 2^9 and $18^{\frac{u}{9}}$ handles $u \in \{45, 63, 72, 135\}$; using 2^{10} and $18^{\frac{u-1}{9}}$ handles $u \in \{46, 64\}$; using 2^{12} and $22^{\frac{u-1}{11}}$ handles $u = 78$; using 2^{12} and $24^{\frac{u}{12}}$ handles $u \in \{60, 96, 132\}$; using 2^{15} and $28^{\frac{u-1}{14}}$ handles $u \in \{99, 141\}$; and using 2^{15} and $30^{\frac{u}{15}}$ handles $u = 75$.

There is a {7,9}-GDD of type $16^8 10^1$ (by lineflip; see [11]) to which Lemma 2.3 can be applied, giving $u = 138$.

Now when a TD(10,m) exists, by truncating a group to a points, and adding 0 or 1 infinite points, we obtain

1. if OGDDs of type 2^m and 2^a exist, then an OGDD of type 2^{9m+a} exists; and

2. if OGDDs of type 2^{m+1} and 2^{a+1} exist, then an OGDD of type 2^{9m+a+1} exists.

In a similar way, if a TD$(9+x,m)$ exists, truncating x groups to one point each so that a block of size $9+x$ remains produces an OGDD of type 2^{9m+x} given OGDDs of type 2^m and 2^{9+x}. Some consequences are tabulated next; applications with a spike are underlined:

m	Numbers of Groups Obtained
9	81, 82, 88, 90
11	100, 106, 108, 111
13	117, <u>118</u>, <u>120</u>, 124, 126, 129, 130
16	144, <u>147</u>, <u>148</u>, <u>150</u>, 153, 154, 156, 159, 160
17	154, 160, 162, 165, 166, 168, 171
19	<u>172</u>, <u>174</u>, <u>175</u>, <u>177</u>, 178, 180, 183, 184, 186, 189, 190

At this point, we have treated all values $u \le 190$. The remainder of the proof treats the cases $u \equiv 0 \pmod 3$ and $u \equiv 4 \pmod 6$ separately. First we treat $u \equiv 4 \pmod 6$ (so that, in each application of Lemma 6.1, $x = 1$). Apply Lemma 6.1 with $m = 8, 9$ to handle $196 \le u \le 244$. Write $u = 9 \cdot 27 + a$ for $u \in \{250, 256, 262, 268\}$ (and proceed as above). Apply Lemma 6.1 with $m = 13, 16, 17$ to handle $274 \le u \le 460$. Write $u = 9 \cdot 49 + a$ for $u \in \{466, 472, 478\}$. Apply Lemma 6.1 with $m = 23, 29, 37, 47, 53$ to handle $484 \le u \le 1432$. Now for $u \ge 1438$, find the largest integer m for which $21m + 10 \le u$ and $\gcd(u, 210) = 1$. Then a TD(10,m) exists, and there are suitable selections of a and b to apply Lemma 6.1.

The case when $u \equiv 0 \pmod 3$ has a similar flavour, but an easier closure can be obtained. When $\frac{u}{3} + 1 \in B(\{7, 8, 9\})$, one can remove a point from the PBD to obtain a GDD with block sizes 7, 8, and 9, and group sizes 6, 7, and 8; applying Lemma 2.2 and filling in holes gives an OGDD of type 2^u. By [12], this handles (among others) $u \in \{333, 336, 339\}$ and all $u \ge 1029$. Now apply Lemma 6.1 with $m = 8, 9, 11$ to obtain $192 \le u \le 297$; with $m = 16, 19$ to handle $336 \le u \le 513$; and with $m = 25, 27, 32, 37, 47$ to handle $534 \le u \le 1026$. It remains only to clean up a few specific cases. Writing $u = 12 \cdot 25 + a$ for $a \in \{0, 3, 6, 9, 12, 15, 18, 21, 24\}$, we handle 2^u as follows. If $a \le 6$, form a TD($12+a$,25) and truncate a groups to leave a spike. If $a \ge 9$, truncate a group of TD(13,25) to a points. In either case, treat the result as a PBD and apply Corollary 2.3. For $u = 327$, start with a TD(10,17); use weight 4 on points in nine groups and weights 0, 2, and 4 on the points of the last group (with OGDDs of types 4^9 and 4^{10} from Lemma 6.3, and $4^9 2^1$ from [8]) to produce an OGDD of type $68^9 42^1$; filling its groups using 2^{34} and 2^{21} handles $u = 327$. For $u = 330$, employ Lemma 2.1 with OGDDs of type 2^{66} and 132^5. For $516 \le u \le 531$, write $u = 12 \cdot 43 + a$ as before. This completes all of the cases. /

6.2 $g = 4$

Lemma 6.3 *An OGDD of type 4^u exists whenever $u \equiv 0, 1 \pmod 3$ and $u \ge 6$.*

Proof: Whenever u is odd and $u \ne 9$, apply Lemma 2.10 with weight 4 and Theorem 1.2. In §3, OGDDs of type 4^u are given for

$$u \in \{6, 9, 12, 16, 18, 22, 24, 28, 34, 40, 52, 58, 94\};$$

in [8], a solution for 4^{10} is given. Apply Lemma 2.1 with 4^6 and $24^{\frac{u}{6}}$ to obtain all $u \equiv 0 \pmod 6$ with $u \ge 30$, with 4^{10} and $36^{\frac{u-1}{9}}$ to obtain $u \in \{46, 64, 82, 100\}$, and with 4^{10} and $40^{\frac{u}{10}}$ to obtain $u \in \{70, 100, 280, 370\}$.

Now we finish up the case when $u \equiv 4 \pmod 6$. In each such case that remains, $\frac{u+2}{3} \in B(\{5, 6, 7, 8\})$. From such a PBD, delete a point to get a $\{5, 6, 7, 8\}$-PBD with group sizes 4, 5, 6, and 7; use Lemma 2.2 with weight 12, and fill the holes using 4 infinite points forming the final group. /

6.3 Wrapping up

We now summarize the consequences when the group size g satisfies $g \equiv 2, 4 \pmod 6$:

Lemma 6.4 *Let $g = 2\gamma$ and $\gamma \equiv 1, 2 \pmod 3$. Then an OGDD of type g^u exists whenever $u \equiv 0, 1 \pmod 3$, $u \geq 6$, except when $g = 2$ and $u = 6$, and possibly when*

1. *$u \in \{6, 22, 34, 58, 94, 142\}$, and γ is odd or $\gamma \in \{4, 20, 28, 52, 76\}$;*
2. *$u \in \{51, 69, 87, 93, 123\}$ and $\gamma \in \{1, 5, 7, 13, 19\}$.*

Proof: When u is odd, Theorem 1.2 gives an OGDD of type 1^u except when $u = 9$. Give weight g when $g \notin \{2, 10, 14, 26, 38\}$. For $u = 9$, give weight γ to an OGDD of type 2^9 if γ is odd, or weight $\gamma/2$ to an OGDD of type 4^9 if γ is even.

When $u \notin \{6, 22, 34, 58, 94, 142\}$, and u is even, proceed as for $u = 9$. Otherwise, when $\gamma \notin \{4, 20, 28, 52, 76\}$ and γ is even, give weight $\gamma/2$ to an OGDD of type 4^u. /

7 Group Size $1, 5 \pmod 6$

In this case, simple inflation from OGDDs of type 1^u (from Theorem 1.2) establishes that when $g \equiv 1, 5 \pmod 6$ and $u \equiv 1, 3 \pmod 6$, $u \geq 7$, there is an OGDD of type g^u except possibly when $u = 9$. The lack of an OGDD of type 1^9 seems to be a singularity, however, as OGDDs of types 5^9 and 7^9 are given in §3.

8 Orthogonal Modified GDDs

A *K-modified group divisible design* (K-MGDD) of type (u, v) is a set V of $u \cdot v$ points, a partition $\mathcal{G}$ (*first groups*) into v sets of size u, a partition $\mathcal{H}$ (*second groups*) of V into u sets of size v, and a set $\mathcal{B}$ of *blocks* whose sizes appear in K. Every first group meets every second group in exactly one point. Every pair of points lies in exactly one block if and only if it does not appear in a first or second group. This notion generalizes uniform GDDs by introducing a second partition into groups.

The notion of orthogonality for 3-GDDs extends naturally to 3-MGDDs, by requiring in the first orthogonality condition that if $\{x, y, z\} \in \mathcal{B}$ and $\{x, y, w\} \in \mathcal{D}$, then z and w are in different first groups *and* in different second groups. Two orthogonal 3-MGDDs of type (u, v) form an OMGDD of type (u, v). The relevance of OMGDDs is that one of type (u, v) may exist despite the nonexistence of OGDDs of type u^v and v^u. It is easily verified that when an OMGDD of type (u, v) and an OSTS(v) both exist, then an OGDD of type u^v exists.

The hill-climbing algorithm described in §3 can be modified to construct OMGDDs. We can again restrict the search by prescribing an automorphism structure for D_1 and D_2. Specification of a suitable group structure becomes more difficult as the automorphism must preserve both first and second groups. We describe now a possible group structure for an m-cyclic MGDD of type (g, u) of group structure type s. We have $gu = mt$, $u \equiv 0 \pmod s$, $t \equiv 0 \pmod s$, and $g \equiv 0 \pmod h$ where $h = t/s$. A further necessary condition is $gcd(s, h) = 1$. Let $a = u/s$ and $b = g/h$. Define first group $G_{x,i} = \{((y - x) \pmod t)_{i+j} \mid y = 0, s, ..., (h-1)s, j = 0, 1, ..., b-1\}$ where $x \equiv 0 \pmod h$, $i \equiv 0 \pmod b$, and $0 \leq x, i < m$, and second group $H_{y,j} = \{((y-x) \pmod t)_{i+j} \mid x = 0, h, ..., (s-1)h, i = 0, b, ..., (a-1)b\}$ where $y \equiv 0 \pmod s$, $0 < j < b$, and $0 \leq y < m$. Then the set of first groups $\mathcal{G} = \{G_{x,i} \mid x = 0, h, ..., (s-1)h, i = 0, b, ..., (a-1)b\}$ and the set of second groups $\mathcal{H} = \{H_{y,j} \mid y = 0, s, ..., (h-1)s, j = 0, 1, ..., (b-1)\}$ are invariant under the action of α.

The method described in §3 to generate m-cyclic OGDDs can be modified to generate m-cyclic OMGDDs. Numerous 1-cyclic examples have been found. Over Z_{28}, the systems generated by {0,1,6}, {0,2,11}, {0,3,13} and by {0,1,11}, {0,2,5}, {0,6,19} form an OMGDD of type $(4, 7)$.

An important use of OMGDDs follows:

Lemma 8.1 *Suppose there is a K-GDD of order v and type $g_1 \cdots g_t$ for which an OMGDD of type (k, u) exists for every $k \in K$, and an OGDD of type g_i^u exists for $1 \leq u \leq t$. Then there is an OGDD of type v^u.*

Proof: Form the K-GDD on V with groups $G_1, \ldots, G_t$. The OGDD formed has point set $V \times \{1, \ldots, u\}$, with groups $\{V \times \{i\} : 1 \leq i \leq u\}$. For every block B in the K-GDD, place an OMGDD of type $(|B|, u)$, aligning the first groups on $\{x \times \{1, \ldots, u\} : x \in B\}$ and the second groups on $\{B \times \{i\} : 1 \leq i \leq u\}$. For every group G_i, place an OGDD of type $|G_i|^u$, aligning its groups on $\{G_i \times \{i\} : 1 \leq i \leq u\}$. Verification that the result is an OGDD is straightforward. /

Corollary 8.2 *There is a constant g_0 such that if $g > g_0$ and $g \equiv 0 \pmod 3$, an OGDD of type g^5 exists.*

Proof: Let $K = \{13, 16, 19, 22\}$. The closure $B(K)$ contains all sufficiently large integers congruent to 1 modulo 3, so choose g_0 so that $g_0 + 1$ is the largest such integer not in $B(K)$. Then form a K-PBD on $g + 1$ points, and delete one point to form a K-GDD with group sizes in {12,15,18,21}. OGDDs of types 12^5, 15^5, 18^5, and 21^5 have all been given in §3, and OMGDDs of types (5,13), (5,16), (5,19), and (5,22) are given in Appendix A5. Apply Theorem 8.1. /

For every group size g, we have shown that there is a finite number of values of u so that g^u meets basic necessary conditions, but an OGDD of type g^u is not known to exist. However, our results fall short of leaving a finite total number of exceptions. When u =6, 14, 22, 26, 34, 38, 58, 94, 142, Lemmas 4.11 and 6.4 leave an infinite number of values of g for which the OGDD is unknown; in the same way, when $u = 9$ and $g \equiv 1, 5 \pmod 6$, infinitely many values remain open. Orthogonal MGDDs can be used, as in Corollary 8.2, to reduce the number of cases for g, often to a finite number. However, we omit the details here.

9 Concluding Remarks

The main question that remains open is whether there is any value of g for which an OGDD of type g^4 exists. On the basis of the nonexistence when $g = 2$ and $g = 4$, one might be tempted to conjecture that the answer is negative. Our computational techniques did not succeed in finding any example with four groups; with no example in hand, our recursive techniques are not able to produce further examples. Before one advances the conjecture that no such OGDDs exist, the OMGDD of type (4,7) given in §8 merits attention. Indeed, we have also produced OMGDDs of type $(4, g)$ for $g \in \{11, 13, 15, 17, 19, 21\}$ as well, and hence it appears that the lack of OGDDs of type g^4 has much more to do with the difficulty of finding them, than with any suggestion that none exists.

Of most interest is the remarkable continued success of hill-climbing in finding triple systems with additional properties. The success in producing a wide variety of uniform OGDDs underlies the results here.

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Web Access

An electronic version of this paper, in postscript format, is available on the World Wide Web at http://www.cs.auckland.ac.nz/CDMTCS/docs/pubs.html. Copies of the Appendices are also available in ascii format.

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Appendix

A1. 1-cyclic OGDDs

Each design of type g^u has an automorphism $\alpha = (0, 1, ..., gu - 1)$ and the group structure type u as defined in §3. Only the base blocks of the OGDDs are listed.

2^{25}

0,47,17	0,14,26	0,29,19	0,35,48	0,22,18	0,7,23	0,11,6	0,41,49

0,39,40	0,31,34	0,44,12	0,20,5	0,4,41	0,8,22	0,21,23	0,17,24

2^{28}

0,13,42	0,35,3	0,31,50	0,20,8	0,15,54	0,30,7	0,16,34	0,5,52
0,10,55							

0,54,40	0,27,46	0,6,55	0,48,15	0,35,24	0,39,26	0,36,31	0,18,52
0,3,12							

2^{40}

0,23,4	0,78,49	0,74,79	0,35,28	0,69,37	0,10,77	0,21,36	0,46,26
0,56,18	0,14,64	0,9,17	0,53,12	0,25,58			

0,3,21	0,12,67	0,61,23	0,7,34	0,28,58	0,47,32	0,51,60	0,54,5
0,63,74	0,37,78	0,4,14	0,16,24	0,45,44			

2^{52}

0,84,22	0,92,11	0,58,102	0,43,18	0,51,3	0,69,68	0,26,97	0,77,94
0,55,95	0,70,38	0,30,80	0,76,91	0,75,83	0,67,4	0,99,14	0,88,57
0,65,59							

0,66,72	0,91,51	0,17,85	0,12,34	0,95,28	0,100,44	0,2,31	0,93,16
0,90,7	0,84,25	0,78,23	0,101,58	0,57,33	0,69,74	0,86,96	0,1,63
0,50,89							

3^{7}

0,2,20	0,9,17	0,6,11

0,2,18	0,15,4	0,8,9

3^{13}

0,37,34	0,31,6	0,30,23	0,21,20	0,29,17	0,11,15

0,27,29	0,3,35	0,15,9	0,16,11	0,22,8	0,1,21

3^{15}

0,26,5	0,36,22	0,1,3	0,11,18	0,20,12	0,17,13	0,35,6

0,25,38	0,8,17	0,31,27	0,43,33	0,34,40	0,19,16	0,21,44

3^{19}

0,13,47	0,17,12	0,20,22	0,46,39	0,53,54	0,33,48	0,8,29	0,32,6
0,14,30							

0,51,34	0,35,7	0,9,54	0,25,55	0,52,13	0,10,14	0,46,31	0,36,56
0,33,41							

3^{21}

0,27,47	0,29,37	0,9,12	0,33,46	0,39,49	0,23,25	0,44,62	0,35,31
0,41,48	0,11,6						

0,10,49	0,60,62	0,29,33	0,51,19	0,58,13	0,35,41	0,23,15	0,17,43
0,36,25	0,9,56						

3^{23}

0,1,25	0,53,51	0,50,60	0,7,54	0,39,66	0,36,5	0,41,29	0,13,17
0,48,37	0,14,63	0,43,35					

0,55,57	0,21,61	0,64,10	0,9,51	0,65,7	0,52,39	0,66,44	0,20,36
0,41,35	0,45,19	0,1,32					

3^{25}

0,27,41	0,74,70	0,23,56	0,59,65	0,12,36	0,64,72	0,54,28	0,46,55
0,43,30	0,37,44	0,2,17	0,53,35				

0,20,67	0,73,4	0,49,59	0,12,66	0,30,19	0,40,41	0,32,61	0,13,51
0,17,48	0,15,22	0,18,23	0,33,72				

3^{27}

0,6,61	0,49,52	0,9,42	0,62,73	0,80,36	0,56,22	0,24,41	0,50,65
0,60,2	0,67,71	0,12,7	0,38,68	0,46,18			

0,22,30	0,11,60	0,77,72	0,16,34	0,80,2	0,39,25	0,7,71	0,37,75
0,61,46	0,45,68	0,62,29	0,26,50	0,41,53			

3^{29}

0,48,56	0,20,47	0,66,2	0,16,81	0,49,25	0,17,86	0,34,78	0,72,82
0,33,59	0,14,50	0,75,30	0,4,84	0,11,52	0,74,55		

0,83,61	0,71,25	0,85,55	0,66,59	0,39,74	0,53,11	0,6,20	0,33,23
0,8,78	0,63,68	0,84,40	0,27,12	0,69,31	0,86,36		

$\mathbf{3^{33}}$

0,28,87	0,23,90	0,80,39	0,72,89	0,69,1	0,64,49	0,11,16	0,14,36
0,91,97	0,4,29	0,42,24	0,37,92	0,26,78	0,3,46	0,13,61	0,34,54

0,58,7	0,18,13	0,95,10	0,35,37	0,1,43	0,53,21	0,47,17	0,28,73
0,8,24	0,29,20	0,19,87	0,34,23	0,39,36	0,93,55	0,25,84	0,22,72

$\mathbf{3^{37}}$

0,77,13	0,108,89	0,86,109	0,15,7	0,48,28	0,90,81	0,58,72	0,85,61
0,46,10	0,11,67	0,68,80	0,29,35	0,71,33	0,95,94	0,27,59	0,62,5
0,45,4	0,18,60						

0,14,77	0,36,26	0,93,95	0,54,5	0,84,13	0,70,94	0,45,105	0,39,4
0,52,99	0,69,31	0,103,15	0,81,25	0,58,67	0,33,83	0,110,19	0,32,11
0,89,82	0,68,3						

$\mathbf{3^{39}}$

0,33,95	0,101,71	0,103,37	0,6,67	0,81,98	0,18,70	0,91,59	0,31,29
0,48,73	0,53,113	0,13,107	0,74,89	0,83,45	0,8,1	0,90,49	0,106,97
0,24,21	0,42,54	0,82,77					

0,28,27	0,33,105	0,73,115	0,17,85	0,40,26	0,29,20	0,57,95	0,3,64
0,37,41	0,5,111	0,94,102	0,67,54	0,35,87	0,59,69	0,25,46	0,19,66
0,34,16	0,31,55	0,74,81					

$\mathbf{3^{47}}$

0,55,135	0,70,99	0,16,75	0,74,126	0,7,37	0,17,73	0,128,116	0,41,43
0,117,96	0,64,87	0,63,3	0,106,57	0,132,110	0,1,114	0,95,115	0,103,34
0,122,39	0,123,131	0,5,93	0,62,11	0,105,14	0,40,44	0,76,108	

0,10,103	0,88,75	0,57,80	0,132,60	0,73,116	0,89,91	0,46,62	0,37,56
0,97,115	0,74,45	0,59,117	0,30,51	0,102,114	0,108,100	0,124,130	0,42,105
0,126,55	0,32,1	0,119,54	0,40,5	0,107,121	0,92,64	0,3,7	

$\mathbf{3^{53}}$

0,132,43	0,56,123	0,87,52	0,148,68	0,93,34	0,1,62	0,128,32	0,24,133
0,112,13	0,46,55	0,54,21	0,81,145	0,149,120	0,151,114	0,118,121	0,58,141
0,154,25	0,119,77	0,90,6	0,51,28	0,19,12	0,65,16	0,88,15	0,74,17
0,157,20	0,48,4						

0,140,33	0,123,99	0,104,119	0,44,5	0,75,11	0,73,118	0,27,109	0,116,25
0,97,145	0,130,3	0,89,18	0,17,7	0,129,133	0,42,108	0,76,98	0,121,67
0,63,69	0,125,79	0,37,2	0,47,31	0,65,150	0,58,136	0,57,56	0,110,138
0,139,147	0,87,100						

$\mathbf{3^{59}}$

0,2,168	0,145,62	0,109,93	0,130,144	0,7,146	0,107,30	0,95,96	0,67,52
0,124,13	0,50,151	0,79,21	0,158,55	0,57,121	0,165,85	0,91,114	0,141,116
0,129,75	0,28,73	0,60,106	0,5,34	0,69,4	0,169,134	0,150,167	0,42,39
0,105,6	0,41,128	0,44,20	0,126,89	0,159,137			

0,101,45	0,89,147	0,143,112	0,52,155	0,124,114	0,6,84	0,142,116	0,154,49
0,150,67	0,169,40	0,20,86	0,141,77	0,69,71	0,17,139	0,149,131	0,3,85
0,133,96	0,126,138	0,57,172	0,102,32	0,13,60	0,136,152	0,50,21	0,80,73
0,166,87	0,9,33	0,1,135	0,54,68	0,158,162			

$\mathbf{3^{67}}$

0,43,62	0,75,98	0,141,8	0,130,34	0,151,107	0,28,194	0,95,122	0,53,142
0,51,52	0,186,190	0,78,188	0,195,140	0,169,124	0,196,199	0,153,10	0,39,18
0,40,57	0,100,70	0,31,135	0,56,176	0,164,76	0,36,69	0,152,16	0,24,87
0,115,74	0,47,129	0,155,175	0,9,117	0,102,187	0,22,64	0,147,29	0,12,121
0,128,38							

0,192,1	0,27,91	0,71,11	0,185,143	0,146,95	0,120,189	0,148,66	0,96,7
0,59,131	0,198,168	0,166,127	0,193,169	0,29,85	0,86,101	0,19,113	0,36,175
0,108,154	0,147,79	0,123,180	0,111,136	0,28,153	0,158,163	0,4,49	0,2,151
0,126,104	0,167,83	0,14,87	0,77,157	0,184,164	0,18,31	0,6,98	0,63,40
0,61,160							

$\mathbf{3^{83}}$

0,227,106	0,174,93	0,202,183	0,220,161	0,48,68	0,195,196	0,8,79	0,167,3
0,247,136	0,150,52	0,132,118	0,58,41	0,115,46	0,90,140	0,114,177	0,38,7
0,78,36	0,70,146	0,124,112	0,206,231	0,139,176	0,86,147	0,148,32	0,60,4
0,28,157	0,108,21	0,184,219	0,155,194	0,215,6	0,105,9	0,205,51	0,97,239
0,175,91	0,26,130	0,169,182	0,226,100	0,64,15	0,192,225	0,233,244	0,187,27
0,204,127							

0,29,167	0,162,164	0,234,98	0,47,195	0,201,236	0,231,37	0,193,38	0,95,159
0,163,196	0,245,70	0,88,67	0,205,31	0,176,200	0,165,72	0,7,230	0,126,142
0,118,96	0,58,192	0,222,188	0,63,168	0,62,183	0,169,77	0,160,50	0,248,145
0,229,224	0,217,189	0,102,173	0,241,129	0,65,208	0,97,108	0,109,239	0,30,180
0,198,210	0,207,91	0,46,243	0,122,181	0,117,17	0,23,14	0,43,40	0,171,36
0,125,170							

3^{87}

0,170,131	0,103,232	0,54,80	0,237,254	0,116,242	0,178,2	0,163,239	0,188,51
0,189,119	0,60,212	0,258,179	0,50,95	0,112,12	0,206,183	0,156,31	0,10,203
0,30,217	0,90,241	0,250,157	0,219,25	0,33,256	0,97,199	0,226,21	0,122,140
0,46,255	0,175,94	0,69,77	0,15,1	0,63,128	0,43,59	0,229,106	0,172,71
0,96,162	0,177,224	0,61,234	0,150,146	0,57,48	0,127,13	0,92,120	0,34,220
0,221,113	0,197,53	0,225,118					

0,229,136	0,103,203	0,85,56	0,118,49	0,124,16	0,131,65	0,145,152	0,94,21
0,253,170	0,120,6	0,162,86	0,12,127	0,198,43	0,210,246	0,117,53	0,88,238
0,258,233	0,13,39	0,160,207	0,123,112	0,84,256	0,104,60	0,251,57	0,224,179
0,202,2	0,169,48	0,154,79	0,33,260	0,46,19	0,139,163	0,257,189	0,95,14
0,30,50	0,181,223	0,199,159	0,74,164	0,71,206	0,226,70	0,17,113	0,119,41
0,31,9	0,128,110	0,132,184					

4^{16}

0,36,56	0,9,15	0,30,3	0,54,14	0,33,7	0,21,63	0,59,46	0,45,62
0,53,41	0,29,4						

0,63,40	0,28,2	0,7,34	0,51,56	0,42,46	0,25,45	0,52,61	0,50,15
0,17,11	0,31,10						

4^{22}

0,10,87	0,34,50	0,25,18	0,4,17	0,48,46	0,14,5	0,32,58	0,73,53
0,31,19	0,49,28	0,47,55	0,45,51	0,27,3	0,52,23		

0,24,39	0,54,9	0,4,20	0,60,7	0,13,69	0,29,47	0,33,3	0,51,77
0,52,14	0,65,8	0,86,76	0,61,40	0,87,82	0,17,42		

4^{28}

0,97,80	0,30,75	0,11,27	0,2,31	0,90,93	0,43,76	0,87,46	0,106,92
0,49,58	0,53,74	0,108,1	0,7,51	0,35,47	0,88,40	0,8,102	0,89,39
0,13,55	0,34,60						

0,44,83	0,98,101	0,27,61	0,49,62	0,21,97	0,9,8	0,48,102	0,72,46
0,22,18	0,87,92	0,19,74	0,12,79	0,77,71	0,32,30	0,89,42	0,59,75
0,69,17	0,31,7						

4^{34}

0,46,13	0,57,115	0,135,97	0,64,22	0,67,117	0,110,75	0,134,51	0,10,119
0,4,28	0,74,99	0,73,87	0,118,107	0,77,32	0,55,12	0,100,95	0,3,130
0,70,30	0,80,60	0,44,52	0,113,65	0,121,105	0,82,89		

0,71,60	0,93,36	0,133,42	0,19,35	0,130,84	0,89,104	0,111,103	0,66,48
0,73,96	0,77,64	0,122,53	0,127,98	0,81,82	0,50,112	0,26,30	0,51,131
0,39,37	0,44,105	0,87,115	0,41,119	0,7,116	0,22,12		

4^{40}

0,146,138	0,79,94	0,92,45	0,103,107	0,32,52	0,21,65	0,82,43	0,17,93
0,30,46	0,28,38	0,58,55	0,12,49	0,2,26	0,119,42	0,25,112	0,97,7
0,62,6	0,100,11	0,59,50	0,127,96	0,142,91	0,131,126	0,27,88	0,159,85
0,125,19	0,23,36						

0,105,8	0,75,129	0,26,79	0,101,49	0,154,65	0,30,94	0,91,50	0,1,45
0,32,5	0,7,100	0,139,122	0,127,142	0,123,24	0,34,9	0,13,150	0,14,92
0,22,20	0,72,43	0,148,102	0,73,16	0,121,125	0,149,36	0,86,3	0,109,19
0,132,48	0,104,62						

4^{52}

0,172,157	0,18,6	0,120,91	0,114,170	0,171,96	0,82,166	0,204,203	0,105,73
0,45,35	0,186,59	0,69,184	0,199,153	0,61,195	0,142,125	0,160,116	0,54,97
0,11,143	0,158,188	0,206,39	0,159,80	0,150,86	0,121,14	0,140,148	0,136,47
0,182,27	0,201,102	0,168,31	0,141,63	0,183,70	0,34,180	0,205,16	0,77,187
0,108,85	0,151,33						

0,108,191	0,79,34	0,32,109	0,29,27	0,133,73	0,48,154	0,4,90	0,97,202
0,56,197	0,161,169	0,183,53	0,46,87	0,23,82	0,68,159	0,203,177	0,153,138
0,72,110	0,199,1	0,94,64	0,36,14	0,42,137	0,132,89	0,61,37	0,19,85
0,92,112	0,13,205	0,127,12	0,139,65	0,107,187	0,124,168	0,201,151	0,173,28
0,146,88	0,190,33						

4^{58}

0,195,68	0,10,225	0,141,176	0,59,154	0,229,73	0,216,101	0,33,82	0,112,62
0,198,172	0,226,69	0,118,47	0,65,42	0,110,89	0,98,151	0,133,119	0,166,158
0,55,147	0,46,84	0,139,87	0,181,39	0,18,54	0,217,129	0,169,72	0,61,57
0,77,230	0,100,204	0,108,25	0,1,32	0,48,70	0,64,44	0,24,5	0,126,138
0,109,80	0,40,205	0,86,45	0,136,125	0,30,219	0,121,223		

0,72,203	0,140,76	0,161,44	0,105,48	0,135,139	0,16,41	0,173,1	0,192,155
0,217,107	0,23,17	0,219,49	0,222,74	0,70,224	0,19,104	0,212,176	0,45,165
0,91,65	0,181,186	0,171,149	0,166,53	0,2,204	0,63,201	0,118,32	0,18,68
0,163,151	0,132,80	0,208,3	0,14,96	0,55,98	0,54,197	0,225,130	0,108,87
0,47,38	0,142,39	0,111,153	0,109,34	0,126,159	0,133,221		

$\mathbf{4^{94}}$

0,166,208	0,49,85	0,209,80	0,186,351	0,118,366	0,310,31	0,333,117	0,175,176
0,122,357	0,133,326	0,132,232	0,271,30	0,319,332	0,274,260	0,230,304	0,3,331
0,348,281	0,273,338	0,290,207	0,120,108	0,148,367	0,287,69	0,350,229	0,272,212
0,182,39	0,292,234	0,262,187	0,336,320	0,73,330	0,113,191	0,59,18	0,181,179
0,223,284	0,64,342	0,349,125	0,295,91	0,82,99	0,306,329	0,138,231	0,368,289
0,33,236	0,323,308	0,184,88	0,261,62	0,178,71	0,149,156	0,154,264	0,76,246
0,344,5	0,150,11	0,127,123	0,242,21	0,54,250	0,171,151	0,77,101	0,313,202
0,217,239	0,213,52	0,270,325	0,90,252	0,29,370	0,245,109		

0,323,287	0,13,75	0,236,361	0,325,48	0,170,128	0,198,168	0,245,59	0,61,196
0,160,255	0,184,70	0,189,85	0,292,26	0,10,44	0,209,267	0,181,2	0,138,202
0,303,5	0,18,288	0,177,278	0,29,22	0,185,28	0,63,294	0,365,345	0,355,308
0,309,352	0,46,222	0,367,65	0,163,310	0,227,144	0,357,124	0,39,203	0,159,87
0,118,210	0,71,152	0,326,240	0,234,107	0,77,93	0,120,237	0,137,32	0,165,316
0,54,247	0,368,17	0,246,223	0,194,349	0,324,3	0,220,300	0,286,285	0,41,6
0,279,265	0,230,108	0,274,141	0,214,263	0,264,38	0,171,250	0,57,12	0,4,343
0,218,103	0,253,134	0,228,132	0,161,336	0,307,207	0,260,56		

$\mathbf{6^{12}}$

0,53,67	0,8,17	0,66,35	0,1,27	0,52,22	0,57,68	0,49,2	0,7,39
0,28,38	0,21,3	0,16,29					

0,29,18	0,1,7	0,5,39	0,53,45	0,56,9	0,23,10	0,37,69	0,4,50
0,20,41	0,15,70	0,58,30					

$\mathbf{6^{13}}$

0,32,12	0,18,34	0,54,63	0,23,42	0,64,53	0,77,3	0,48,10	0,6,57
0,2,31	0,70,37	0,5,22	0,28,71				

0,35,67	0,31,53	0,72,30	0,27,23	0,18,33	0,37,77	0,17,19	0,54,34
0,8,29	0,68,75	0,62,50	0,73,64				

$\mathbf{6^{16}}$

0,44,7	0,35,62	0,3,82	0,68,91	0,88,86	0,54,9	0,19,39	0,66,72
0,83,84	0,74,43	0,49,85	0,18,56	0,50,75	0,26,41	0,67,4	

0,21,67	0,15,57	0,23,17	0,59,94	0,27,65	0,84,7	0,13,91	0,70,10
0,14,63	0,56,1	0,72,44	0,51,43	0,9,20	0,66,62	0,74,3	

$\mathbf{6^{17}}$

0,53,81	0,62,61	0,92,83	0,79,91	0,2,77	0,29,67	0,87,37	0,47,4
0,60,3	0,96,33	0,16,70	0,46,22	0,71,66	0,14,72	0,18,94	0,95,82

0,7,86	0,29,81	0,32,77	0,58,22	0,3,46	0,82,83	0,98,65	0,78,96
0,97,87	0,42,53	0,2,76	0,12,39	0,35,48	0,94,64	0,41,55	0,9,40

$\mathbf{6^{20}}$

0,1,50	0,84,58	0,45,4	0,101,8	0,96,86	0,7,12	0,2,90	0,105,28
0,14,3	0,22,47	0,74,35	0,18,56	0,66,29	0,69,63	0,104,87	0,42,97
0,48,61	0,99,31	0,76,67					

0,112,38	0,59,65	0,75,42	0,81,34	0,1,94	0,13,104	0,102,72	0,43,2
0,31,54	0,19,76	0,113,117	0,69,84	0,98,115	0,85,95	0,96,28	0,99,88
0,9,67	0,37,108	0,106,56					

$\mathbf{6^{24}}$

0,124,36	0,29,91	0,43,22	0,55,49	0,27,59	0,2,35	0,15,40	0,130,125
0,17,116	0,113,47	0,1,76	0,51,141	0,11,37	0,81,23	0,30,18	0,74,61
0,52,94	0,9,137	0,77,4	0,38,98	0,103,64	0,134,34	0,79,87	

0,80,74	0,81,128	0,9,53	0,103,68	0,99,11	0,84,112	0,2,92	0,50,139
0,82,46	0,134,65	0,140,115	0,141,31	0,101,102	0,66,51	0,125,118	0,77,114
0,106,57	0,127,104	0,126,105	0,58,117	0,33,20	0,73,12	0,14,22	

$\mathbf{6^{32}}$

0,114,62	0,152,79	0,8,169	0,46,41	0,129,190	0,107,175	0,12,149	0,123,19
0,143,159	0,174,138	0,50,165	0,99,147	0,81,168	0,74,44	0,185,53	0,20,141
0,135,122	0,25,3	0,26,134	0,183,101	0,136,97	0,186,28	0,155,75	0,72,83
0,127,92	0,106,182	0,154,21	0,90,188	0,125,126	0,42,145	0,14,177	

0,95,102	0,65,5	0,172,183	0,26,23	0,131,117	0,52,108	0,57,98	0,114,147
0,87,150	0,112,58	0,113,111	0,19,36	0,122,55	0,119,163	0,153,116	0,50,46
0,74,92	0,49,21	0,85,51	0,124,89	0,10,25	0,12,161	0,152,48	0,47,186
0,162,91	0,72,13	0,93,27	0,77,69	0,38,22	0,24,130	0,109,191	

6^{44}

0,231,248	0,146,124	0,244,191	0,173,255	0,230,80	0,141,37	0,154,239	0,24,43
0,185,86	0,109,206	0,172,67	0,21,14	0,27,78	0,50,122	0,13,218	0,156,194
0,139,203	0,177,195	0,131,41	0,130,196	0,127,233	0,65,216	0,81,201	0,241,54
0,164,170	0,121,95	0,71,56	0,35,171	0,215,207	0,232,103	0,52,148	0,147,102
0,101,39	0,107,111	0,166,47	0,263,11	0,224,254	0,188,152	0,74,149	0,42,180
0,262,3	0,181,209	0,175,235					

0,127,71	0,60,117	0,38,227	0,123,135	0,78,20	0,254,101	0,86,214	0,66,187
0,18,91	0,17,102	0,22,92	0,212,42	0,33,171	0,248,80	0,3,115	0,230,47
0,257,55	0,223,219	0,233,51	0,23,25	0,74,199	0,188,21	0,46,89	0,161,108
0,196,63	0,11,180	0,259,140	0,160,130	0,142,83	0,1,158	0,61,26	0,165,14
0,67,215	0,245,29	0,120,6	0,164,236	0,228,237	0,249,210	0,109,40	0,185,98
0,154,90	0,159,146	0,32,24					

6^{52}

0,41,8	0,205,226	0,150,76	0,152,70	0,217,112	0,246,121	0,295,232	0,77,117
0,43,81	0,72,140	0,124,15	0,134,163	0,185,102	0,84,278	0,79,247	0,49,47
0,116,206	0,201,290	0,252,248	0,42,103	0,264,213	0,28,182	0,23,62	0,259,268
0,164,219	0,36,132	0,305,285	0,129,214	0,237,253	0,12,294	0,25,122	0,131,202
0,192,197	0,101,258	0,234,161	0,6,199	0,218,173	0,177,3	0,46,35	0,142,141
0,254,56	0,146,133	0,159,92	0,293,165	0,136,262	0,281,26	0,137,225	0,37,204
0,189,221	0,302,14	0,69,169					

0,275,34	0,133,8	0,17,19	0,121,136	0,296,146	0,306,94	0,139,245	0,198,86
0,164,233	0,129,238	0,151,214	0,27,60	0,265,219	0,76,280	0,189,48	0,207,143
0,155,244	0,130,256	0,261,239	0,1,273	0,132,163	0,257,165	0,175,53	0,30,222
0,101,286	0,215,62	0,197,210	0,242,7	0,303,78	0,307,284	0,174,287	0,253,181
0,283,88	0,3,61	0,57,209	0,184,269	0,229,41	0,228,263	0,231,38	0,14,10
0,118,262	0,154,142	0,196,276	0,135,217	0,292,201	0,110,66	0,54,267	0,205,65
0,167,178	0,96,21	0,294,24					

9^{5}

0,28,27	0,2,16	0,34,37	0,23,4	0,32,39	0,9,21
0,41,7	0,24,2	0,16,28	0,27,19	0,31,32	0,39,36

9^{9}

0,14,65	0,48,52	0,41,15	0,6,25	0,21,53	0,8,79	0,47,50	0,37,76
0,35,13	0,11,80	0,20,43	0,57,64				

0,15,20	0,52,31	0,47,12	0,73,41	0,53,51	0,80,42	0,56,78	0,74,19
0,48,65	0,10,6	0,57,70	0,23,37				

9^{11}

0,32,59	0,52,62	0,12,57	0,38,84	0,97,24	0,50,9	0,65,83	0,48,78
0,56,64	0,1,5	0,80,63	0,13,92	0,31,28	0,70,93	0,85,25	

0,90,49	0,54,7	0,14,95	0,17,60	0,94,64	0,79,76	0,46,25	0,72,13
0,68,84	0,89,65	0,57,28	0,97,91	0,63,1	0,12,38	0,80,48	

9^{17}

0,146,25	0,145,134	0,24,107	0,44,16	0,91,33	0,93,148	0,143,105	0,45,23
0,141,66	0,114,116	0,1,150	0,92,122	0,111,124	0,41,110	0,15,67	0,57,97
0,139,90	0,99,81	0,118,47	0,126,76	0,36,100	0,79,144	0,6,133	0,73,94

0,111,8	0,135,49	0,93,109	0,127,149	0,10,106	0,21,9	0,133,146	0,94,95
0,88,120	0,73,71	0,54,35	0,89,112	0,45,107	0,150,37	0,148,105	0,61,31
0,126,29	0,142,52	0,77,24	0,128,14	0,6,84	0,79,117	0,83,28	0,138,66

9^{23}

0,152,25	0,3,125	0,11,119	0,142,136	0,194,151	0,64,180	0,111,183	0,29,198
0,158,117	0,165,150	0,149,110	0,37,146	0,77,172	0,193,89	0,36,160	0,51,17
0,189,177	0,186,188	0,163,4	0,179,113	0,137,8	0,134,75	0,87,54	0,1,32
0,76,144	0,79,202	0,197,157	0,181,67	0,102,86	0,107,45	0,74,22	0,154,7
0,106,187							

0,28,68	0,152,125	0,153,197	0,58,128	0,13,182	0,158,15	0,162,89	0,18,14
0,85,119	0,116,9	0,63,30	0,114,150	0,111,17	0,160,51	0,157,199	0,186,170
0,43,124	0,24,108	0,129,110	0,72,1	0,52,147	0,196,75	0,172,106	0,61,39
0,7,5	0,26,201	0,20,140	0,151,48	0,3,77	0,53,41	0,145,65	0,29,105
0,59,176							

9^{83}

0,565,185	0,141,253	0,693,284	0,187,544	0,701,433	0,570,615	0,425,461	0,74,152
0,605,706	0,154,555	0,688,176	0,449,309	0,72,424	0,279,364	0,199,349	0,587,414
0,183,122	0,277,66	0,432,421	0,363,371	0,522,478	0,348,458	0,538,131	0,70,137
0,374,138	0,613,56	0,612,633	0,76,245	0,181,32	0,734,52	0,189,151	0,311,444
0,499,728	0,720,361	0,608,302	0,300,713	0,705,721	0,723,429	0,394,379	0,307,186
0,312,239	0,387,589	0,579,93	0,519,417	0,542,617	0,408,525	0,513,254	0,89,251
0,735,651	0,215,313	0,103,341	0,86,47	0,694,742	0,299,381	0,272,250	0,123,745
0,487,351	0,392,539	0,550,128	0,63,540	0,403,258	0,641,443	0,191,650	0,193,521
0,79,659	0,439,411	0,506,733	0,195,118	0,221,618	0,165,569	0,7,469	0,746,230
0,174,370	0,51,543	0,442,743	0,471,69	0,484,690	0,466,220	0,427,108	0,707,240
0,514,737	0,704,200	0,282,553	0,423,689	0,62,710	0,389,219	0,679,144	0,50,267
0,472,20	0,631,537	0,331,218	0,457,454	0,510,362	0,87,104	0,342,437	0,29,60
0,55,430	0,683,115	0,329,741	0,244,91	0,604,627	0,175,224	0,450,559	0,426,164
0,100,531	0,576,81	0,213,369	0,157,337	0,291,533	0,347,184	0,491,473	0,642,667
0,474,738	0,717,460	0,621,588	0,623,327	0,628,482	0,35,655	0,161,232	0,172,636
0,592,391	0,292,657	0,107,500					

0,169,687	0,120,746	0,473,207	0,447,424	0,68,2	0,700,137	0,283,264	0,372,133
0,152,108	0,514,632	0,382,227	0,316,678	0,335,470	0,21,587	0,573,674	0,292,131
0,493,436	0,453,555	0,91,469	0,630,285	0,236,496	0,602,707	0,386,551	0,197,428
0,306,64	0,100,466	0,497,430	0,444,173	0,577,704	0,226,98	0,712,305	0,286,737
0,408,400	0,648,636	0,33,223	0,77,61	0,97,722	0,457,36	0,399,307	0,32,189
0,199,203	0,545,263	0,124,689	0,585,591	0,212,164	0,280,624	0,394,211	0,318,380
0,738,20	0,18,528	0,149,474	0,387,438	0,288,299	0,179,549	0,489,665	0,134,667
0,486,217	0,611,255	0,450,666	0,425,618	0,349,140	0,725,298	0,525,230	0,668,89
0,732,537	0,717,84	0,688,113	0,671,27	0,684,596	0,432,653	0,409,519	0,330,337
0,742,599	0,532,480	0,652,177	0,87,600	0,657,158	0,472,416	0,241,631	0,144,456
0,313,34	0,204,235	0,682,245	0,730,605	0,556,702	0,572,220	0,86,576	0,188,529
0,744,93	0,542,423	0,676,38	0,705,247	0,374,413	0,53,693	0,343,268	0,588,503
0,411,673	0,314,355	0,515,194	0,376,677	0,389,539	0,342,420	0,130,180	0,561,14
0,256,13	0,379,225	0,471,138	0,482,28	0,692,304	0,238,701	0,163,24	0,488,287
0,350,104	0,187,224	0,354,594	0,302,253	0,495,112	0,615,721	0,167,606	0,534,206
0,384,569	0,329,675	0,126,477					

12^{5}

0,9,33	0,1,49	0,7,39	0,38,34	0,13,44	0,57,43	0,19,37	0,54,52

0,16,43	0,21,19	0,8,36	0,31,34	0,4,18	0,49,12	0,13,51	0,54,53

12^{10}

0,73,36	0,59,67	0,26,102	0,33,91	0,52,28	0,115,2	0,57,88	0,74,117
0,81,119	0,15,9	0,4,103	0,107,95	0,75,34	0,16,109	0,106,55	0,19,85
0,72,49	0,56,78						

0,67,36	0,1,58	0,28,71	0,45,24	0,88,12	0,86,105	0,26,8	0,46,17
0,104,81	0,113,2	0,37,116	0,115,64	0,25,98	0,48,59	0,87,114	0,65,78
0,52,106	0,85,3						

12^{12}

0,71,93	0,111,134	0,59,79	0,13,87	0,35,138	0,38,133	0,64,63	0,21,16
0,104,129	0,88,42	0,27,8	0,14,90	0,75,114	0,29,142	0,26,58	0,55,37
0,53,62	0,45,97	0,78,28	0,140,3	0,17,61	0,101,34		

0,119,21	0,126,16	0,50,28	0,129,64	0,137,134	0,33,75	0,29,20	0,86,92
0,99,68	0,38,143	0,91,17	0,19,56	0,109,82	0,61,142	0,130,71	0,51,95
0,13,5	0,104,26	0,11,41	0,87,32	0,43,47	0,77,54		

12^{16}

0,133,113	0,120,99	0,63,153	0,91,4	0,62,15	0,82,7	0,11,114	0,111,13
0,33,61	0,31,69	0,95,163	0,8,18	0,92,169	0,148,139	0,26,158	0,5,42
0,126,140	0,85,1	0,73,46	0,134,83	0,137,2	0,19,41	0,3,125	0,167,143
0,157,30	0,54,71	0,45,88	0,40,76	0,12,86	0,56,6		

0,2,94	0,172,13	0,76,139	0,91,14	0,182,1	0,17,104	0,106,152	0,22,158
0,148,29	0,81,120	0,45,69	0,93,78	0,3,26	0,42,12	0,58,143	0,164,117
0,90,38	0,57,118	0,37,68	0,51,110	0,25,151	0,21,83	0,174,36	0,19,84
0,79,70	0,5,60	0,186,89	0,71,4	0,8,35	0,43,50		

12^{18}

0,19,4	0,77,152	0,150,17	0,35,22	0,23,33	0,140,132	0,73,138	0,214,112
0,94,45	0,196,157	0,62,130	0,113,26	0,43,185	0,116,211	0,6,47	0,111,207
0,50,71	0,125,155	0,12,172	0,37,48	0,176,147	0,51,209	0,119,182	0,88,215
0,38,14	0,82,174	0,85,101	0,135,80	0,60,28	0,67,213	0,191,93	0,109,52
0,189,110	0,46,99						

0,142,67	0,172,17	0,32,133	0,21,167	0,95,84	0,96,77	0,193,14	0,100,43
0,92,12	0,48,64	0,50,118	0,114,109	0,123,161	0,177,65	0,15,81	0,145,2
0,164,215	0,69,163	0,106,130	0,156,3	0,76,31	0,85,78	0,20,210	0,187,157
0,35,189	0,117,28	0,119,129	0,58,105	0,46,137	0,208,33	0,194,34	0,88,191
0,4,13	0,82,42						

12^{20}

0,67,148	0,117,4	0,177,169	0,199,161	0,75,185	0,133,139	0,2,216	0,208,94
0,7,109	0,149,223	0,53,188	0,66,112	0,164,39	0,210,12	0,134,193	0,231,142
0,99,62	0,121,43	0,82,132	0,18,230	0,88,54	0,77,227	0,145,156	0,179,36
0,87,86	0,57,72	0,118,182	0,69,44	0,207,96	0,3,48	0,35,103	0,213,184
0,218,51	0,221,235	0,209,124	0,70,217	0,49,224	0,219,136		

0,174,239	0,57,155	0,2,63	0,198,35	0,129,178	0,31,39	0,171,139	0,37,92
0,134,210	0,102,214	0,234,227	0,189,212	0,79,104	0,116,143	0,78,59	0,86,196
0,226,89	0,235,225	0,17,182	0,167,122	0,95,166	0,123,53	0,222,115	0,158,206
0,87,99	0,94,207	0,237,81	0,204,150	0,109,202	0,50,29	0,96,152	0,105,121
0,114,67	0,43,52	0,46,22	0,91,199	0,229,72	0,4,68		

12^{24}

0,230,165	0,169,42	0,75,266	0,167,11	0,228,140	0,259,15	0,193,280	0,150,221
0,252,43	0,171,125	0,74,77	0,64,268	0,32,239	0,190,281	0,107,90	0,86,196
0,100,170	0,241,135	0,114,145	0,14,286	0,131,103	0,39,186	0,234,51	0,232,195
0,284,85	0,226,267	0,128,34	0,66,253	0,10,73	0,111,283	0,80,255	0,133,158
0,231,82	0,137,13	0,270,243	0,269,59	0,30,142	0,122,248	0,61,52	0,205,152
0,282,109	0,233,104	0,134,108	0,12,50	0,265,189	0,219,287		

0,237,143	0,242,280	0,88,221	0,175,109	0,128,266	0,54,136	0,73,197	0,277,63
0,196,199	0,251,146	0,167,273	0,284,154	0,174,13	0,153,53	0,44,58	0,27,32
0,102,185	0,119,217	0,157,126	0,262,81	0,60,287	0,180,110	0,10,270	0,80,47
0,151,139	0,68,271	0,29,78	0,2,245	0,223,229	0,125,56	0,112,211	0,236,141
0,84,170	0,159,140	0,171,272	0,36,249	0,265,41	0,122,209	0,55,20	0,248,116
0,279,25	0,50,57	0,42,226	0,195,165	0,212,115	0,21,111		

12^{32}

0,77,63	0,243,73	0,145,20	0,366,181	0,184,22	0,12,354	0,89,139	0,175,220
0,114,67	0,2,355	0,322,132	0,373,286	0,191,163	0,24,318	0,41,171	0,99,216
0,155,43	0,57,159	0,319,53	0,296,150	0,54,16	0,92,82	0,301,306	0,167,237
0,115,363	0,375,197	0,340,233	0,79,137	0,201,121	0,119,116	0,134,40	0,204,347
0,177,211	0,95,235	0,230,336	0,253,144	0,283,358	0,376,219	0,271,35	0,122,232
0,120,105	0,158,332	0,86,281	0,208,6	0,273,135	0,293,312	0,179,196	0,126,133
0,371,335	0,69,1	0,223,284	0,71,257	0,218,76	0,4,361	0,351,51	0,39,299
0,359,56	0,215,338	0,129,55	0,324,93	0,104,276	0,287,59		

0,30,22	0,381,152	0,88,154	0,221,363	0,245,252	0,97,24	0,117,39	0,120,225
0,274,123	0,332,59	0,100,127	0,171,65	0,113,199	0,244,203	0,303,94	0,161,340
0,321,197	0,379,231	0,218,198	0,17,383	0,165,36	0,253,170	0,107,291	0,91,263
0,243,16	0,372,194	0,270,312	0,251,349	0,334,145	0,275,324	0,84,164	0,167,9
0,305,55	0,307,138	0,309,266	0,82,122	0,369,23	0,327,235	0,282,269	0,28,268
0,241,4	0,6,323	0,202,168	0,353,119	0,222,11	0,191,90	0,248,47	0,336,310
0,25,351	0,108,71	0,339,254	0,104,281	0,62,76	0,355,374	0,330,156	0,297,89
0,180,112	0,314,126	0,51,146	0,328,259	0,285,338	0,135,382		

15^{5}

0,1,3	0,4,11	0,6,14	0,9,32	0,12,34	0,13,39	0,16,44	0,17,38
0,18,42	0,19,48						

0,1,59	0,2,6	0,3,24	0,7,33	0,8,61	0,9,43	0,11,38	0,12,56
0,13,36	0,18,46						

18^{5}

0,1,3	0,4,11	0,6,14	0,9,21	0,13,41	0,16,47	0,17,53	0,18,51
0,19,46	0,22,48	0,23,61	0,24,56				

0,1,29	0,2,73	0,3,14	0,4,53	0,6,78	0,7,63	0,8,52	0,9,68
0,13,39	0,16,48	0,21,57	0,23,47				

21^{5}

0,42,78	0,7,19	0,6,67	0,74,33	0,51,2	0,94,23	0,18,26	0,13,4
0,28,52	0,32,29	0,43,57	0,68,84	0,58,104	0,83,66		

0,6,49	0,44,96	0,78,101	0,41,38	0,47,68	0,8,89	0,36,22	0,103,92
0,28,46	0,86,7	0,66,12	0,17,48	0,34,63	0,104,72		

24^6

0,1,142	0,4,62	0,5,40	0,7,87	0,8,28	0,9,103	0,10,61	0,11,45
0,13,59	0,14,29	0,16,33	0,19,75	0,21,47	0,22,101	0,23,91	0,25,74
0,27,100	0,31,112	0,37,89	0,38,105				

0,1,3	0,4,9	0,7,17	0,8,21	0,11,55	0,14,43	0,15,85	0,16,73
0,19,124	0,22,98	0,23,104	0,25,58	0,26,91	0,27,107	0,28,103	0,31,93
0,32,67	0,34,83	0,38,88	0,45,97				

27^5

0,1,97	0,2,18	0,3,52	0,4,21	0,6,13	0,8,71	0,9,112	0,11,88
0,12,41	0,14,42	0,19,76	0,22,101	0,24,67	0,26,74	0,27,89	0,31,82
0,33,69	0,37,81						

0,1,3	0,4,13	0,6,52	0,7,111	0,8,34	0,11,29	0,12,48	0,14,86
0,16,77	0,17,59	0,19,88	0,21,53	0,22,73	0,23,56	0,27,91	0,28,67
0,37,78	0,38,81						

33^5

0,1,52	0,2,98	0,3,97	0,4,78	0,6,27	0,7,128	0,8,36	0,9,31
0,11,84	0,12,151	0,13,46	0,16,48	0,17,41	0,18,79	0,19,107	0,23,66
0,29,112	0,34,76	0,38,102	0,39,93	0,47,103	0,49,106		

0,1,3	0,4,22	0,6,19	0,7,24	0,8,57	0,9,53	0,11,82	0,12,63
0,14,42	0,16,144	0,23,107	0,26,73	0,27,113	0,29,96	0,31,77	0,32,106
0,33,111	0,34,72	0,36,97	0,39,101	0,41,89	0,43,109		

A2. m-cyclic OGDDs $(m > 1)$

Each m-cyclic GDD of type g^u, where $gu = mt$, and group structure type s, has an automorphism $\alpha = (0_0, 1_0, ..., (t-1)_0)...(0_m, 1_m, ..., (t-1)_m)$. Only the base blocks of each GDD are listed.

2^{19}, $m = 2, s = 19$

$0_03_012_0$	$0_017_06_0$	$0_01_05_0$	$0_012_11_1$	$0_010_19_1$	$0_014_117_1$	$0_08_115_1$
$0_013_13_1$	$0_04_16_1$	$0_07_111_1$	$0_018_15_1$	$0_02_116_1$		

$0_04_09_0$	$0_012_018_0$	$0_011_017_1$	$0_02_015_1$	$0_03_02_1$	$0_08_116_1$	$0_09_112_1$
$0_03_14_1$	$0_014_17_1$	$0_011_11_1$	$0_010_15_1$	$0_113_115_1$		

2^{21}, $m = 3, s = 7$

$0_012_03_0$	$0_06_05_1$	$0_04_02_1$	$0_01_01_1$	$0_09_18_1$	$0_010_16_1$	$0_04_16_2$
$0_07_110_2$	$0_011_112_2$	$0_03_19_2$	$0_03_22_2$	$0_07_21_2$	$0_013_211_2$	$0_04_28_2$
$0_00_25_2$	$0_18_112_2$	$0_13_111_2$	$0_12_17_2$	$0_19_19_2$	$0_110_213_2$	

$0_04_05_0$	$0_012_09_1$	$0_011_03_1$	$0_08_012_2$	$0_00_12_1$	$0_05_13_2$	$0_013_16_2$
$0_04_10_2$	$0_08_17_2$	$0_010_111_2$	$0_01_11_2$	$0_07_113_2$	$0_012_12_2$	$0_08_210_2$
$0_09_25_2$	$0_110_111_1$	$0_18_13_2$	$0_19_111_2$	$0_18_25_2$	$0_213_25_2$	

3^9, $m = 3, s = 9$

$0_02_08_0$	$0_04_07_1$	$0_06_11_1$	$0_08_16_2$	$0_05_12_2$	$0_02_14_2$	$0_04_15_2$
$0_03_27_2$	$0_08_21_2$	$0_16_18_1$	$0_15_28_2$	$0_14_23_2$		

$0_05_08_1$	$0_02_07_1$	$0_06_05_2$	$0_08_03_2$	$0_04_11_1$	$0_02_16_1$	$0_01_27_2$
$0_06_22_2$	$0_12_17_2$	$0_18_13_2$	$0_11_22_2$	$0_16_28_2$		

4^6, $m = 3, s = 2$

$0_05_05_1$	$0_07_02_1$	$0_07_12_2$	$0_01_17_2$	$0_04_10_2$	$0_06_13_2$	$0_04_21_2$
$0_05_26_2$	$0_15_17_2$	$0_11_11_2$				

$0_03_07_1$	$0_07_06_2$	$0_06_11_1$	$0_00_12_2$	$0_03_14_2$	$0_05_11_2$	$0_02_15_2$
$0_03_20_2$	$0_11_16_2$	$0_10_27_2$				

4^9, $m = 3, s = 3$

$0_011_07_0$	$0_02_05_1$	$0_011_11_1$	$0_010_19_1$	$0_06_10_2$	$0_02_11_2$	$0_07_111_2$
$0_04_15_2$	$0_00_18_2$	$0_08_110_2$	$0_09_22_2$	$0_06_24_2$	$0_03_27_2$	$0_18_13_2$
$0_17_10_2$	$0_110_29_2$					

$0_07_04_1$	$0_02_07_1$	$0_08_011_2$	$0_01_09_2$	$0_02_13_1$	$0_06_111_1$	$0_01_14_2$
$0_010_16_2$	$0_00_17_2$	$0_08_15_2$	$0_00_21_2$	$0_02_210_2$	$0_18_11_2$	$0_12_10_2$
$0_14_22_2$	$0_111_26_2$					

4^{12}, $m = 3, s = 4$

$0_03_014_0$	$0_01_010_0$	$0_00_115_1$	$0_04_12_1$	$0_06_11_1$	$0_08_15_1$	$0_09_12_2$
$0_07_17_2$	$0_014_112_2$	$0_011_113_2$	$0_03_114_2$	$0_013_19_2$	$0_012_14_2$	$0_010_10_2$
$0_011_210_2$	$0_06_21_2$	$0_015_28_2$	$0_03_25_2$	$0_17_11_2$	$0_16_13_2$	$0_115_25_2$
$0_17_24_2$						

$0_015_08_1$	$0_014_012_1$	$0_06_013_1$	$0_09_03_1$	$0_013_03_2$	$0_05_00_2$	$0_02_15_1$
$0_06_115_1$	$0_00_114_2$	$0_011_115_2$	$0_01_112_2$	$0_04_113_2$	$0_07_21_2$	$0_05_22_2$
$0_08_210_2$	$0_09_24_2$	$0_11_13_2$	$0_114_15_2$	$0_111_110_2$	$0_110_10_2$	$0_112_213_2$
$0_11_28_2$						

$\mathbf{4^{18}}, m = 3, s = 6$

$0_0 1_0 10_0$	$0_0 7_0 2_0$	$0_0 8_0 11_0$	$0_0 20_0 18_1$	$0_0 9_1 0_1$	$0_0 20_1 19_1$	$0_0 8_1 6_1$
$0_0 12_1 7_1$	$0_0 17_1 13_1$	$0_0 16_1 3_1$	$0_0 5_1 10_2$	$0_0 23_1 6_2$	$0_0 4_1 0_2$	$0_0 2_1 1_2$
$0_0 15_1 17_2$	$0_0 1_1 16_2$	$0_0 10_1 19_2$	$0_0 14_1 2_2$	$0_0 11_1 3_2$	$0_0 21_1 7_2$	$0_0 5_2 14_2$
$0_0 18_2 4_2$	$0_0 15_2 20_2$	$0_0 23_2 12_2$	$0_0 11_2 8_2$	$0_0 22_2 21_2$	$0_0 9_2 13_2$	$0_1 21_1 1_2$
$0_1 16_1 0_2$	$0_1 17_1 11_2$	$0_1 10_1 13_2$	$0_1 6_2 22_2$	$0_1 21_2 14_2$	$0_1 17_2 19_2$	

$0_0 22_0 8_0$	$0_0 4_0 7_0$	$0_0 13_0 15_1$	$0_0 19_0 9_1$	$0_0 23_0 21_1$	$0_0 15_0 4_1$	$0_0 23_1 1_1$
$0_0 17_1 8_1$	$0_0 10_1 3_2$	$0_0 19_1 1_2$	$0_0 3_1 10_2$	$0_0 12_1 15_2$	$0_0 0_1 9_2$	$0_0 7_1 18_2$
$0_0 20_1 14_2$	$0_0 5_1 2_2$	$0_0 16_1 17_2$	$0_0 6_1 19_2$	$0_0 18_1 22_2$	$0_0 11_1 7_2$	$0_0 0_2 5_2$
$0_0 8_2 16_2$	$0_0 20_2 6_2$	$0_0 21_2 23_2$	$0_0 11_2 4_2$	$0_0 13_2 12_2$	$0_1 8_1 1_1$	$0_1 21_1 11_1$
$0_1 20_1 22_2$	$0_1 19_1 19_2$	$0_1 10_2 14_2$	$0_1 23_2 8_2$	$0_1 15_2 12_2$	$0_1 5_2 16_2$	

$\mathbf{4^{24}}, m = 3, s = 8$

$0_0 25_0 26_0$	$0_0 4_0 9_0$	$0_0 12_0 30_0$	$0_0 13_0 3_0$	$0_0 17_0 15_1$	$0_0 11_0 4_1$	$0_0 7_1 27_1$
$0_0 19_1 26_1$	$0_0 10_1 5_1$	$0_0 6_1 17_1$	$0_0 13_1 0_1$	$0_0 24_1 14_1$	$0_0 2_1 8_1$	$0_0 20_1 23_1$
$0_0 31_1 2_2$	$0_0 18_1 29_2$	$0_0 29_1 24_2$	$0_0 28_1 8_2$	$0_0 12_1 17_2$	$0_0 11_1 15_2$	$0_0 21_1 28_2$
$0_0 1_1 30_2$	$0_0 3_1 27_2$	$0_0 9_1 19_2$	$0_0 22_1 11_2$	$0_0 16_1 0_2$	$0_0 14_2 20_2$	$0_0 26_2 4_2$
$0_0 13_2 6_2$	$0_0 5_2 9_2$	$0_0 7_2 25_2$	$0_0 31_2 1_2$	$0_0 16_2 3_2$	$0_0 22_2 10_2$	$0_0 23_2 18_2$
$0_0 12_2 21_2$	$0_1 30_1 31_2$	$0_1 17_1 15_2$	$0_1 9_1 2_2$	$0_1 4_1 22_2$	$0_1 14_1 14_2$	$0_1 1_1 9_2$
$0_1 20_2 19_2$	$0_1 6_2 17_2$	$0_1 28_2 13_2$	$0_1 23_2 26_2$			

$0_0 25_0 21_0$	$0_0 30_0 10_0$	$0_0 31_0 26_0$	$0_0 29_0 15_0$	$0_0 9_0 9_1$	$0_0 19_0 7_1$	$0_0 26_1 15_1$
$0_0 23_1 10_1$	$0_0 31_1 29_1$	$0_0 8_1 28_1$	$0_0 6_1 2_1$	$0_0 5_1 30_1$	$0_0 17_1 18_1$	$0_0 21_1 23_2$
$0_0 3_1 12_2$	$0_0 25_1 25_2$	$0_0 11_1 9_2$	$0_0 24_1 11_2$	$0_0 12_1 3_2$	$0_0 13_1 27_2$	$0_0 16_1 15_2$
$0_0 22_1 14_2$	$0_0 1_1 21_2$	$0_0 19_1 4_2$	$0_0 4_1 20_2$	$0_0 14_1 26_2$	$0_0 27_1 8_2$	$0_0 7_2 13_2$
$0_0 6_2 2_2$	$0_0 10_2 28_2$	$0_0 30_2 17_2$	$0_0 16_2 5_2$	$0_0 1_2 18_2$	$0_0 22_2 19_2$	$0_0 31_2 0_2$
$0_0 29_2 24_2$	$0_1 17_1 26_1$	$0_1 18_1 29_2$	$0_1 5_1 10_2$	$0_1 10_1 18_2$	$0_1 3_1 7_2$	$0_1 25_2 27_2$
$0_1 15_2 22_2$	$0_1 26_2 3_2$	$0_1 6_2 28_2$	$0_1 21_2 1_2$			

$\mathbf{5^{9}}, m = 3, s = 3$

$0_0 14_0 4_0$	$0_0 7_0 14_1$	$0_0 13_0 2_1$	$0_0 8_1 0_1$	$0_0 6_1 14_2$	$0_0 5_1 1_2$	$0_0 13_1 8_2$
$0_0 11_1 3_2$	$0_0 1_1 7_2$	$0_0 3_1 4_2$	$0_0 10_1 0_2$	$0_0 12_1 9_2$	$0_0 9_1 11_2$	$0_0 10_2 12_2$
$0_0 13_2 5_2$	$0_0 6_2 2_2$	$0_1 14_1 4_1$	$0_1 2_1 0_2$	$0_1 4_2 3_2$	$0_1 14_2 9_2$	

$0_0 4_0 1_1$	$0_0 5_0 0_1$	$0_0 2_0 9_1$	$0_0 8_0 11_1$	$0_0 14_0 0_2$	$0_0 6_1 4_1$	$0_0 13_1 12_2$
$0_0 14_1 5_2$	$0_0 2_1 14_2$	$0_0 5_1 9_2$	$0_0 8_1 8_2$	$0_0 11_2 10_2$	$0_0 6_2 13_2$	$0_0 4_2 2_2$
$0_0 3_2 7_2$	$0_1 4_1 11_2$	$0_1 8_1 9_2$	$0_1 5_1 10_2$	$0_1 1_1 3_2$	$0_1 13_2 8_2$	

$\mathbf{6^{5}}, m = 2, s = 5$

$0_0 6_0 13_0$	$0_0 3_0 14_0$	$0_0 7_1 6_1$	$0_0 3_1 12_1$	$0_0 8_1 1_1$	$0_0 11_1 14_1$	$0_0 13_1 9_1$
$0_0 4_1 2_1$						

$0_0 7_0 3_0$	$0_0 13_0 7_1$	$0_0 9_0 6_1$	$0_0 1_0 2_1$	$0_0 14_1 8_1$	$0_0 3_1 4_1$	$0_0 13_1 11_1$
$0_1 11_1 8_1$						

$\mathbf{6^{7}}, m = 2, s = 7$

$0_0 18_0 19_0$	$0_0 12_0 4_0$	$0_0 15_0 5_0$	$0_0 1_1 12_1$	$0_0 13_1 15_1$	$0_0 8_1 4_1$	$0_0 18_1 5_1$
$0_0 11_1 6_1$	$0_0 9_1 3_1$	$0_0 10_1 19_1$	$0_0 20_1 2_1$	$0_0 16_1 17_1$		

$0_0 15_0 16_0$	$0_0 18_0 8_0$	$0_0 2_0 19_1$	$0_0 4_0 16_1$	$0_0 9_0 15_1$	$0_0 5_1 8_1$	$0_0 3_1 13_1$
$0_0 1_1 9_1$	$0_0 4_1 10_1$	$0_0 20_1 18_1$	$0_0 11_1 2_1$	$0_1 5_1 4_1$		

$\mathbf{6^{11}}, m = 2, s = 11$

$0_0 13_0 21_0$	$0_0 32_0 28_0$	$0_0 27_0 9_0$	$0_0 31_0 17_0$	$0_0 7_0 10_0$	$0_0 3_1 13_1$	$0_0 9_1 29_1$
$0_0 19_1 23_1$	$0_0 27_1 25_1$	$0_0 31_1 1_1$	$0_0 16_1 15_1$	$0_0 18_1 10_1$	$0_0 28_1 14_1$	$0_0 26_1 17_1$
$0_0 24_1 6_1$	$0_0 2_1 8_1$	$0_0 32_1 20_1$	$0_0 12_1 7_1$	$0_0 21_1 5_1$	$0_0 30_1 4_1$	

$0_0 2_0 16_0$	$0_0 26_0 30_0$	$0_0 1_0 6_0$	$0_0 20_0 14_1$	$0_0 21_0 26_1$	$0_0 25_0 17_1$	$0_0 15_0 21_1$
$0_0 24_0 15_1$	$0_0 23_0 18_1$	$0_0 32_1 2_1$	$0_0 9_1 23_1$	$0_0 8_1 1_1$	$0_0 12_1 16_1$	$0_0 20_1 30_1$
$0_0 4_1 10_1$	$0_0 13_1 29_1$	$0_0 31_1 3_1$	$0_0 19_1 7_1$	$0_1 8_1 32_1$	$0_1 13_1 31_1$	

$\mathbf{6^{19}}, m = 2, s = 19$

$0_0 24_0 51_0$	$0_0 50_0 47_0$	$0_0 21_0 43_0$	$0_0 52_0 13_0$	$0_0 56_0 48_0$	$0_0 28_0 26_0$	$0_0 23_0 11_0$
$0_0 53_0 37_0$	$0_0 32_0 15_0$	$0_0 32_1 10_1$	$0_0 24_1 35_1$	$0_0 12_1 25_1$	$0_0 17_1 7_1$	$0_0 43_1 1_1$
$0_0 29_1 2_1$	$0_0 3_1 9_1$	$0_0 30_1 34_1$	$0_0 39_1 14_1$	$0_0 53_1 48_1$	$0_0 5_1 41_1$	$0_0 42_1 28_1$
$0_0 50_1 47_1$	$0_0 11_1 51_1$	$0_0 45_1 27_1$	$0_0 13_1 44_1$	$0_0 46_1 55_1$	$0_0 20_1 36_1$	$0_0 49_1 56_1$
$0_0 26_1 18_1$	$0_0 4_1 6_1$	$0_0 22_1 23_1$	$0_0 40_1 16_1$	$0_0 37_1 8_1$	$0_0 33_1 21_1$	$0_0 15_1 52_1$
$0_0 31_1 54_1$						

$0_0 33_0 4_0$	$0_0 5_0 27_0$	$0_0 43_0 7_0$	$0_0 32_0 17_0$	$0_0 55_0 49_0$	$0_0 48_0 1_1$	$0_0 1_0 46_1$
$0_0 37_0 2_1$	$0_0 41_0 40_1$	$0_0 45_0 23_1$	$0_0 31_0 36_1$	$0_0 13_0 4_1$	$0_0 34_0 43_1$	$0_0 47_0 37_1$
$0_0 3_0 31_1$	$0_0 39_0 24_1$	$0_0 11_0 8_1$	$0_0 51_1 20_1$	$0_0 14_1 55_1$	$0_0 21_1 26_1$	$0_0 41_1 13_1$
$0_0 17_1 44_1$	$0_0 34_1 25_1$	$0_0 16_1 39_1$	$0_0 6_1 30_1$	$0_0 50_1 32_1$	$0_0 53_1 3_1$	$0_0 12_1 11_1$
$0_0 52_1 7_1$	$0_0 15_1 18_1$	$0_0 27_1 33_1$	$0_0 49_1 29_1$	$0_1 40_1 4_1$	$0_1 32_1 47_1$	$0_1 2_1 13_1$
$0_1 8_1 22_1$						

$\mathbf{6^{23}}, m = 2, s = 23$

$0_0 42_0 9_0$	$0_0 53_0 13_0$	$0_0 24_0 28_0$	$0_0 34_0 15_0$	$0_0 26_0 44_0$	$0_0 31_0 17_0$	$0_0 5_0 2_0$
$0_0 22_0 61_0$	$0_0 10_0 21_0$	$0_0 1_0 63_0$	$0_0 32_0 12_0$	$0_0 9_1 1_1$	$0_0 31_1 58_1$	$0_0 67_1 42_1$
$0_0 28_1 44_1$	$0_0 38_1 16_1$	$0_0 64_1 21_1$	$0_0 51_1 36_1$	$0_0 6_1 66_1$	$0_0 24_1 63_1$	$0_0 30_1 29_1$
$0_0 11_1 62_1$	$0_0 25_1 45_1$	$0_0 65_1 27_1$	$0_0 17_1 4_1$	$0_0 50_1 18_1$	$0_0 34_1 68_1$	$0_0 41_1 8_1$
$0_0 43_1 2_1$	$0_0 5_1 22_1$	$0_0 47_1 37_1$	$0_0 35_1 33_1$	$0_0 54_1 59_1$	$0_0 32_1 13_1$	$0_0 49_1 60_1$
$0_0 40_1 52_1$	$0_0 48_1 19_1$	$0_0 14_1 20_1$	$0_0 39_1 15_1$	$0_0 10_1 3_1$	$0_0 57_1 61_1$	$0_0 12_1 26_1$
$0_0 53_1 56_1$	$0_0 55_1 7_1$					

$0_0 21_0 59_0$	$0_0 19_0 22_0$	$0_0 13_0 25_0$	$0_0 53_0 4_0$	$0_0 6_0 8_0$	$0_0 40_0 68_0$	$0_0 51_0 42_0$
$0_0 7_0 68_1$	$0_0 15_0 2_1$	$0_0 35_0 38_1$	$0_0 37_0 4_1$	$0_0 55_0 17_1$	$0_0 17_0 50_1$	$0_0 30_0 42_1$
$0_0 43_0 15_1$	$0_0 24_0 43_1$	$0_0 58_0 7_1$	$0_0 36_0 25_1$	$0_0 64_0 9_1$	$0_0 44_1 53_1$	$0_0 54_1 10_1$
$0_0 65_1 30_1$	$0_0 37_1 6_1$	$0_0 8_1 34_1$	$0_0 63_1 62_1$	$0_0 51_1 49_1$	$0_0 48_1 60_1$	$0_0 11_1 5_1$
$0_0 39_1 59_1$	$0_0 26_1 22_1$	$0_0 24_1 21_1$	$0_0 66_1 55_1$	$0_0 29_1 45_1$	$0_0 52_1 16_1$	$0_0 28_1 1_1$
$0_0 47_1 32_1$	$0_0 27_1 35_1$	$0_0 64_1 13_1$	$0_0 40_1 57_1$	$0_0 20_1 67_1$	$0_1 19_1 14_1$	$0_1 10_1 40_1$
$0_1 28_1 21_1$	$0_1 32_1 45_1$					

$\mathbf{6^{27}}, m = 2, s = 27$

$0_0 31_0 17_0$	$0_0 58_0 52_0$	$0_0 55_0 65_0$	$0_0 51_0 2_0$	$0_0 68_0 22_0$	$0_0 28_0 62_0$	$0_0 12_0 37_0$
$0_0 8_0 41_0$	$0_0 74_0 63_0$	$0_0 42_0 73_1$	$0_0 72_0 63_1$	$0_0 20_0 56_1$	$0_0 21_0 76_1$	$0_0 43_0 65_1$
$0_0 57_0 11_1$	$0_0 36_0 68_1$	$0_0 80_0 14_1$	$0_0 78_0 17_1$	$0_0 77_0 26_1$	$0_0 66_0 67_1$	$0_0 76_0 42_1$
$0_0 16_1 44_1$	$0_0 7_1 77_1$	$0_0 64_1 33_1$	$0_0 40_1 50_1$	$0_0 21_1 43_1$	$0_0 38_1 61_1$	$0_0 5_1 70_1$
$0_0 10_1 12_1$	$0_0 2_1 45_1$	$0_0 8_1 75_1$	$0_0 29_1 46_1$	$0_0 9_1 6_1$	$0_0 59_1 58_1$	$0_0 37_1 71_1$
$0_0 24_1 80_1$	$0_0 60_1 48_1$	$0_0 53_1 18_1$	$0_0 41_1 4_1$	$0_0 28_1 23_1$	$0_0 66_1 51_1$	$0_0 57_1 25_1$
$0_0 19_1 74_1$	$0_0 79_1 39_1$	$0_0 34_1 52_1$	$0_0 69_1 62_1$	$0_0 3_1 78_1$	$0_0 49_1 13_1$	$0_1 73_1 13_1$
$0_1 20_1 77_1$	$0_1 39_1 30_1$	$0_1 29_1 48_1$				

$0_0 29_0 53_0$	$0_0 61_0 75_0$	$0_0 46_0 8_0$	$0_0 72_0 10_0$	$0_0 13_0 25_0$	$0_0 23_0 78_0$	$0_0 11_0 33_0$
$0_0 60_0 16_0$	$0_0 15_0 49_0$	$0_0 64_0 73_1$	$0_0 63_0 71_1$	$0_0 79_0 77_1$	$0_0 45_0 7_1$	$0_0 76_0 18_1$
$0_0 30_0 21_1$	$0_0 80_0 28_1$	$0_0 31_0 35_1$	$0_0 74_0 39_1$	$0_0 41_0 75_1$	$0_0 4_0 37_1$	$0_0 42_0 24_1$
$0_0 57_1 53_1$	$0_0 36_1 26_1$	$0_0 1_1 52_1$	$0_0 60_1 58_1$	$0_0 20_1 61_1$	$0_0 56_1 44_1$	$0_0 40_1 48_1$
$0_0 32_1 17_1$	$0_0 69_1 11_1$	$0_0 38_1 49_1$	$0_0 68_1 74_1$	$0_0 16_1 41_1$	$0_0 42_1 64_1$	$0_0 3_1 50_1$
$0_0 6_1 45_1$	$0_0 2_1 78_1$	$0_0 51_1 19_1$	$0_0 12_1 65_1$	$0_0 22_1 25_1$	$0_0 66_1 47_1$	$0_0 31_1 5_1$
$0_0 15_1 67_1$	$0_0 76_1 55_1$	$0_0 80_1 62_1$	$0_0 30_1 10_1$	$0_0 70_1 13_1$	$0_0 14_1 59_1$	$0_1 16_1 33_1$
$0_1 35_1 44_1$	$0_1 7_1 38_1$	$0_1 68_1 1_1$				

$\mathbf{6^{39}}, m = 2, s = 39$

$0_0 51_0 49_0$	$0_0 59_0 29_0$	$0_0 91_0 48_0$	$0_0 37_0 92_0$	$0_0 64_0 76_0$	$0_0 23_0 16_0$	$0_0 17_0 31_0$
$0_0 32_0 67_0$	$0_0 111_0 89_0$	$0_0 81_0 114_0$	$0_0 70_0 83_0$	$0_0 19_0 57_0$	$0_0 20_0 44_0$	$0_0 8_0 116_0$
$0_0 18_0 90_0$	$0_0 96_0 56_0$	$0_0 4_0 75_0$	$0_0 65_0 54_0$	$0_0 10_0 15_0$	$0_0 83_1 63_1$	$0_0 105_1 65_1$
$0_0 109_1 99_1$	$0_0 25_1 114_1$	$0_0 28_1 51_1$	$0_0 6_1 86_1$	$0_0 23_1 15_1$	$0_0 103_1 40_1$	$0_0 45_1 60_1$
$0_0 56_1 112_1$	$0_0 89_1 64_1$	$0_0 50_1 5_1$	$0_0 90_1 72_1$	$0_0 73_1 37_1$	$0_0 41_1 88_1$	$0_0 43_1 84_1$
$0_0 16_1 17_1$	$0_0 19_1 87_1$	$0_0 116_1 32_1$	$0_0 33_1 67_1$	$0_0 3_1 47_1$	$0_0 7_1 53_1$	$0_0 100_1 34_1$
$0_0 110_1 96_1$	$0_0 27_1 57_1$	$0_0 29_1 8_1$	$0_0 92_1 13_1$	$0_0 95_1 76_1$	$0_0 74_1 24_1$	$0_0 10_1 42_1$
$0_0 69_1 85_1$	$0_0 54_1 12_1$	$0_0 111_1 68_1$	$0_0 97_1 35_1$	$0_0 21_1 26_1$	$0_0 11_1 2_1$	$0_0 20_1 113_1$
$0_0 22_1 44_1$	$0_0 94_1 101_1$	$0_0 31_1 79_1$	$0_0 61_1 58_1$	$0_0 46_1 52_1$	$0_0 59_1 55_1$	$0_0 18_1 75_1$
$0_0 108_1 77_1$	$0_0 104_1 14_1$	$0_0 91_1 62_1$	$0_0 49_1 102_1$	$0_0 81_1 98_1$	$0_0 4_1 30_1$	$0_0 1_1 66_1$
$0_0 93_1 80_1$	$0_0 48_1 107_1$	$0_0 38_1 36_1$	$0_0 82_1 70_1$	$0_0 115_1 9_1$	$0_0 71_1 106_1$	

$0_0 100_0 13_0$	$0_0 61_0 84_0$	$0_0 114_0 28_0$	$0_0 106_0 97_0$	$0_0 59_0 19_0$	$0_0 44_0 48_0$	$0_0 76_0 111_0$
$0_0 51_0 27_0$	$0_0 53_0 46_0$	$0_0 65_0 79_0$	$0_0 109_0 14_1$	$0_0 50_0 16_1$	$0_0 96_0 92_1$	$0_0 85_0 48_1$
$0_0 70_0 62_1$	$0_0 91_0 101_1$	$0_0 18_0 95_1$	$0_0 102_0 36_1$	$0_0 12_0 99_1$	$0_0 80_0 38_1$	$0_0 101_0 89_1$
$0_0 29_0 52_1$	$0_0 81_0 60_1$	$0_0 72_0 2_1$	$0_0 95_0 93_1$	$0_0 10_0 53_1$	$0_0 63_0 7_1$	$0_0 116_0 55_1$
$0_0 83_0 68_1$	$0_0 5_0 4_1$	$0_0 55_0 104_1$	$0_0 115_0 1_1$	$0_0 92_0 12_1$	$0_0 43_0 58_1$	$0_0 49_0 66_1$
$0_0 60_0 90_1$	$0_0 42_0 106_1$	$0_0 27_1 70_1$	$0_0 26_1 9_1$	$0_0 21_1 91_1$	$0_0 19_1 29_1$	$0_0 50_1 6_1$
$0_0 44_1 86_1$	$0_0 18_1 98_1$	$0_0 11_1 110_1$	$0_0 65_1 35_1$	$0_0 24_1 13_1$	$0_0 42_1 97_1$	$0_0 45_1 31_1$
$0_0 8_1 72_1$	$0_0 88_1 63_1$	$0_0 28_1 32_1$	$0_0 25_1 107_1$	$0_0 76_1 114_1$	$0_0 54_1 57_1$	$0_0 81_1 112_1$
$0_0 108_1 20_1$	$0_0 34_1 85_1$	$0_0 41_1 46_1$	$0_0 5_1 59_1$	$0_0 111_1 84_1$	$0_0 69_1 71_1$	$0_0 74_1 94_1$
$0_0 33_1 79_1$	$0_0 82_1 103_1$	$0_0 73_1 67_1$	$0_0 40_1 100_1$	$0_1 32_1 19_1$	$0_1 83_1 67_1$	$0_1 72_1 95_1$
$0_1 33_1 59_1$	$0_1 65_1 93_1$	$0_1 81_1 69_1$	$0_1 7_1 8_1$	$0_1 76_1 15_1$	$0_1 49_1 40_1$	

$\mathbf{7^{9}}, m = 3, s = 3$

$0_0 7_0 15_1$	$0_0 16_0 1_1$	$0_0 17_0 14_1$	$0_0 2_0 19_1$	$0_0 1_0 0_1$	$0_0 13_0 8_2$	$0_0 11_0 18_2$
$0_0 10_1 2_1$	$0_0 5_1 4_1$	$0_0 3_1 20_2$	$0_0 7_1 6_2$	$0_0 13_1 10_2$	$0_0 12_1 14_2$	$0_0 16_1 9_2$
$0_0 11_1 0_2$	$0_0 9_1 12_2$	$0_0 15_2 2_2$	$0_0 11_2 1_2$	$0_0 17_2 13_2$	$0_0 4_2 5_2$	$0_0 19_2 3_2$
$0_1 4_1 4_2$	$0_1 2_1 15_2$	$0_1 7_1 5_2$	$0_1 5_1 12_2$	$0_1 11_1 1_2$	$0_1 16_2 9_2$	$0_1 6_2 8_2$

$0_0 4_0 5_0$	$0_0 19_0 8_0$	$0_0 14_0 1_1$	$0_0 9_1 7_1$	$0_0 18_1 19_1$	$0_0 15_1 11_1$	$0_0 6_1 13_1$
$0_0 16_1 5_2$	$0_0 3_1 1_2$	$0_0 2_1 4_2$	$0_0 10_1 0_2$	$0_0 17_1 3_2$	$0_0 4_1 19_2$	$0_0 20_1 16_2$
$0_0 14_1 14_2$	$0_0 12_1 7_2$	$0_0 5_1 11_2$	$0_0 0_1 9_2$	$0_0 20_2 6_2$	$0_0 13_2 15_2$	$0_0 12_2 8_2$
$0_0 17_2 18_2$	$0_0 10_2 2_2$	$0_1 5_1 13_2$	$0_1 11_1 14_2$	$0_1 13_1 18_2$	$0_1 1_2 12_2$	$0_1 20_2 4_2$

$\mathbf{18^{11}}, m = 2, s = 11$

$0_0 79_0 1_0$	$0_0 12_0 70_0$	$0_0 59_0 61_0$	$0_0 82_0 8_0$	$0_0 89_0 18_0$	$0_0 90_0 96_0$	$0_0 95_0 80_0$
$0_0 27_0 51_0$	$0_0 31_0 83_0$	$0_0 35_0 65_0$	$0_0 60_0 14_0$	$0_0 62_0 5_0$	$0_0 32_0 45_0$	$0_0 36_0 92_0$
$0_0 50_0 73_0$	$0_0 29_1 30_1$	$0_0 40_1 50_1$	$0_0 65_1 34_1$	$0_0 4_1 53_1$	$0_0 54_1 82_1$	$0_0 36_1 38_1$
$0_0 12_1 20_1$	$0_0 28_1 24_1$	$0_0 78_1 14_1$	$0_0 98_1 41_1$	$0_0 62_1 46_1$	$0_0 9_1 67_1$	$0_0 68_1 95_1$
$0_0 59_1 27_1$	$0_0 91_1 71_1$	$0_0 64_1 49_1$	$0_0 63_1 56_1$	$0_0 1_1 7_1$	$0_0 10_1 39_1$	$0_0 6_1 79_1$
$0_0 96_1 35_1$	$0_0 76_1 58_1$	$0_0 97_1 17_1$	$0_0 70_1 18_1$	$0_0 2_1 80_1$	$0_0 19_1 72_1$	$0_0 16_1 13_1$
$0_0 90_1 21_1$	$0_0 32_1 23_1$	$0_0 8_1 83_1$	$0_0 92_1 5_1$	$0_0 81_1 86_1$	$0_0 94_1 69_1$	$0_0 31_1 87_1$
$0_0 25_1 48_1$	$0_0 89_1 52_1$	$0_0 84_1 45_1$	$0_0 37_1 73_1$	$0_0 51_1 85_1$	$0_0 57_1 3_1$	$0_0 47_1 60_1$
$0_0 74_1 15_1$	$0_0 61_1 75_1$	$0_0 42_1 93_1$	$0_0 26_1 43_1$			

$0_0 98_0 72_0$	$0_0 62_0 59_0$	$0_0 92_0 29_0$	$0_0 12_0 95_0$	$0_0 58_0 71_0$	$0_0 80_0 75_0$	$0_0 79_0 25_0$
$0_0 60_0 52_0$	$0_0 90_0 71_1$	$0_0 21_0 61_1$	$0_0 43_0 1_1$	$0_0 69_0 39_1$	$0_0 23_0 29_1$	$0_0 51_0 59_1$
$0_0 65_0 4_1$	$0_0 81_0 23_1$	$0_0 61_0 45_1$	$0_0 68_0 87_1$	$0_0 14_0 98_1$	$0_0 84_0 17_1$	$0_0 89_0 96_1$
$0_0 57_0 31_1$	$0_0 6_0 27_1$	$0_0 46_0 72_1$	$0_0 35_0 82_1$	$0_0 97_0 68_1$	$0_0 17_0 67_1$	$0_0 50_0 64_1$
$0_0 32_0 74_1$	$0_0 5_1 95_1$	$0_0 75_1 15_1$	$0_0 13_1 18_1$	$0_0 76_1 86_1$	$0_0 62_1 36_1$	$0_0 30_1 97_1$
$0_0 2_1 65_1$	$0_0 92_1 54_1$	$0_0 94_1 46_1$	$0_0 16_1 20_1$	$0_0 91_1 85_1$	$0_0 12_1 28_1$	$0_0 63_1 43_1$
$0_0 89_1 48_1$	$0_0 9_1 51_1$	$0_0 90_1 37_1$	$0_0 56_1 81_1$	$0_0 58_1 34_1$	$0_0 49_1 79_1$	$0_0 25_1 93_1$
$0_0 60_1 52_1$	$0_0 35_1 78_1$	$0_0 10_1 3_1$	$0_0 53_1 24_1$	$0_1 65_1 28_1$	$0_1 15_1 97_1$	$0_1 72_1 54_1$
$0_1 80_1 59_1$	$0_1 85_1 86_1$	$0_1 23_1 35_1$	$0_1 47_1 96_1$			

A3. 2-rotational OGDDs

Each 2-rotational GDD of type 2^u, where $2u = 2(t+1)$, has an automorphism $\alpha = (\infty_0)(\infty_1)(0_0, 1_0, ..., (t-1)_0)(0_1, 1_1, ..., (t-1)_1)$. Only the base blocks of each GDD are listed. The OGDDs have the group structure $\mathcal{G} = \{\{\infty_0, \infty_1\}\} \cup \{\{i_0, i_1\} \mid i = 0, 1, ..., (t-1)\}$.

$\mathbf{2^{18}}$

$0_0 1_0 7_0$	$0_0 15_0 12_0$	$0_0 4_0 7_1$	$0_0 8_0 14_1$	$0_0 5_1 10_1$	$0_0 13_1 16_1$	$0_0 12_1 11_1$
$0_0 9_1 1_1$	$0_0 15_1 8_1$	$0_1 6_1 4_1$	$\infty_0 0_0 4_1$	$\infty_1 0_0 2_1$		

$0_0 8_0 15_0$	$0_0 4_0 3_0$	$0_0 11_0 6_1$	$0_0 12_0 9_1$	$0_0 15_1 10_1$	$0_0 13_1 7_1$	$0_0 5_1 2_1$
$0_0 1_1 3_1$	$0_0 4_1 8_1$	$0_1 9_1 16_1$	$\infty_0 0_0 11_1$	$\infty_1 0_0 16_1$		

$\mathbf{2^{36}}$

$0_0 6_0 30_0$	$0_0 15_0 18_0$	$0_0 26_0 19_0$	$0_0 27_0 20_1$	$0_0 1_0 15_1$	$0_0 14_0 31_1$	$0_0 12_0 25_1$
$0_0 2_0 26_1$	$0_0 10_0 29_1$	$0_0 22_0 23_1$	$0_0 31_0 5_1$	$0_0 18_1 3_1$	$0_0 6_1 33_1$	$0_0 30_1 16_1$
$0_0 32_1 8_1$	$0_0 22_1 4_1$	$0_0 2_1 34_1$	$0_0 7_1 11_1$	$0_0 12_1 21_1$	$0_1 16_1 6_1$	$0_1 30_1 2_1$
$0_1 13_1 1_1$	$\infty_0 0_0 10_1$	$\infty_1 0_0 27_1$				

$0_0 26_0 12_0$	$0_0 13_0 7_0$	$0_0 11_0 1_0$	$0_0 18_0 2_1$	$0_0 32_0 29_1$	$0_0 16_0 3_1$	$0_0 5_0 16_1$
$0_0 33_0 23_1$	$0_0 20_0 5_1$	$0_0 8_0 15_1$	$0_0 4_0 14_1$	$0_0 4_1 12_1$	$0_0 33_1 9_1$	$0_0 24_1 21_1$
$0_0 13_1 31_1$	$0_0 18_1 34_1$	$0_0 28_1 26_1$	$0_0 27_1 1_1$	$0_0 17_1 30_1$	$0_1 4_1 10_1$	$0_1 34_1 14_1$
$0_1 7_1 12_1$	$\infty_0 0_0 8_1$	$\infty_1 0_0 6_1$				

$\mathbf{2^{42}}$

$0_0 14_0 22_0$	$0_0 26_0 28_0$	$0_0 21_0 38_0$	$0_0 34_0 35_1$	$0_0 5_0 37_1$	$0_0 12_0 33_1$	$0_0 10_0 28_1$
$0_0 35_0 3_1$	$0_0 1_0 40_1$	$0_0 30_0 8_1$	$0_0 23_0 12_1$	$0_0 16_0 31_1$	$0_0 4_0 14_1$	$0_0 9_0 13_1$
$0_0 17_1 5_1$	$0_0 24_1 2_1$	$0_0 20_1 36_1$	$0_0 29_1 26_1$	$0_0 7_1 27_1$	$0_0 38_1 34_1$	$0_0 22_1 16_1$
$0_0 25_1 11_1$	$0_1 36_1 8_1$	$0_1 1_1 10_1$	$0_1 2_1 17_1$	$0_1 34_1 23_1$	$\infty_0 0_0 23_1$	$\infty_1 0_0 6_1$

$0_0 15_0 9_0$	$0_0 20_0 25_0$	$0_0 12_0 19_0$	$0_0 40_0 17_1$	$0_0 24_0 25_1$	$0_0 33_0 26_1$	$0_0 4_0 2_1$
$0_0 11_0 32_1$	$0_0 23_0 22_1$	$0_0 39_0 5_1$	$0_0 31_0 9_1$	$0_0 3_0 27_1$	$0_0 27_0 14_1$	$0_0 13_0 3_1$
$0_0 4_1 36_1$	$0_0 15_1 23_1$	$0_0 12_1 10_1$	$0_0 13_1 20_1$	$0_0 16_1 29_1$	$0_0 37_1 6_1$	$0_0 33_1 30_1$
$0_0 38_1 8_1$	$0_1 26_1 25_1$	$0_1 24_1 20_1$	$0_1 6_1 18_1$	$0_1 22_1 36_1$	$\infty_0 0_0 35_1$	$\infty_1 0_0 11_1$

$\mathbf{2^{48}}$

$0_0 25_0 38_0$	$0_0 28_0 10_0$	$0_0 14_0 21_0$	$0_0 35_0 32_0$	$0_0 30_0 37_1$	$0_0 2_0 24_1$	$0_0 1_0 42_1$
$0_0 41_0 9_1$	$0_0 39_0 38_1$	$0_0 24_0 12_1$	$0_0 20_0 21_1$	$0_0 4_0 6_1$	$0_0 5_0 10_1$	$0_0 11_0 8_1$
$0_0 31_0 18_1$	$0_0 40_1 36_1$	$0_0 23_1 17_1$	$0_0 27_1 20_1$	$0_0 11_1 3_1$	$0_0 28_1 45_1$	$0_0 29_1 39_1$
$0_0 26_1 13_1$	$0_0 30_1 14_1$	$0_0 25_1 43_1$	$0_0 4_1 32_1$	$0_0 19_1 31_1$	$0_1 32_1 46_1$	$0_1 20_1 25_1$
$0_1 23_1 26_1$	$0_1 45_1 36_1$	$\infty_0 0_0 16_1$	$\infty_1 0_0 33_1$			

$0_1 12_1 20_1$	$0_1 19_1 42_1$	$0_1 13_1 22_1$	$0_1 4_1 44_1$	$0_0 27_1 13_1$	$0_0 8_1 19_1$	$0_0 23_1 5_1$
$0_0 41_1 11_1$	$0_0 40_1 30_1$	$0_0 15_1 46_1$	$0_0 44_1 18_1$	$0_0 33_1 39_1$	$0_0 25_1 10_1$	$0_0 28_1 29_1$
$0_0 36_1 38_1$	$0_0 33_0 2_1$	$0_0 38_0 12_1$	$0_0 43_0 22_1$	$0_0 46_0 31_1$	$0_0 11_0 14_1$	$0_0 26_0 43_1$
$0_0 37_0 24_1$	$0_0 7_0 42_1$	$0_0 2_0 9_1$	$0_0 39_0 45_1$	$0_0 16_0 20_1$	$0_0 18_0 6_0$	$0_0 24_0 44_0$
$0_0 22_0 5_0$	$0_0 34_0 19_0$	$\infty_0 0_0 37_1$	$\infty_1 0_0 1_1$			

$\mathbf{2^{54}}$						
$0_0 18_0 2_0$	$0_0 29_0 49_0$	$0_0 15_0 12_0$	$0_0 1_0 32_0$	$0_0 23_0 25_1$	$0_0 27_0 34_1$	$0_0 17_0 49_1$
$0_0 40_0 15_1$	$0_0 46_0 11_1$	$0_0 48_0 31_1$	$0_0 8_0 35_1$	$0_0 47_0 16_1$	$0_0 11_0 41_1$	$0_0 28_0 14_1$
$0_0 14_0 4_1$	$0_0 9_0 46_1$	$0_0 34_0 40_1$	$0_0 10_0 1_1$	$0_0 10_1 8_1$	$0_0 38_1 51_1$	$0_0 45_1 13_1$
$0_0 12_1 48_1$	$0_0 24_1 17_1$	$0_0 9_1 23_1$	$0_0 52_1 5_1$	$0_0 26_1 50_1$	$0_0 19_1 20_1$	$0_0 29_1 21_1$
$0_0 3_1 33_1$	$0_1 27_1 12_1$	$0_1 48_1 4_1$	$0_1 18_1 28_1$	$0_1 16_1 50_1$	$0_1 22_1 33_1$	$\infty_0 0_0 42_1$
$\infty_1 0_0 47_1$						
$0_1 22_1 52_1$	$0_1 29_1 33_1$	$0_1 26_1 44_1$	$0_1 46_1 5_1$	$0_0 1_1 41_1$	$0_0 21_1 49_1$	$0_0 47_1 30_1$
$0_0 2_1 23_1$	$0_0 51_1 12_1$	$0_0 48_1 33_1$	$0_0 7_1 5_1$	$0_0 22_1 28_1$	$0_0 42_1 50_1$	$0_0 36_1 46_1$
$0_0 20_1 17_1$	$0_0 52_1 18_1$	$0_0 3_1 14_1$	$0_0 15_1 31_1$	$0_0 50_0 37_1$	$0_0 8_0 24_1$	$0_0 39_0 29_1$
$0_0 52_0 25_1$	$0_0 26_0 11_1$	$0_0 35_0 39_1$	$0_0 37_0 45_1$	$0_0 22_0 32_1$	$0_0 28_0 19_1$	$0_0 24_0 6_1$
$0_0 4_0 13_1$	$0_0 51_0 17_0$	$0_0 38_0 6_0$	$0_0 33_0 42_0$	$0_0 40_0 30_0$	$0_0 7_0 12_0$	$\infty_0 0_0 27_1$
$\infty_1 0_0 34_1$						
$\mathbf{2^{66}}$						
$0_0 1_0 36_0$	$0_0 9_0 5_0$	$0_0 16_0 14_0$	$0_0 6_0 52_0$	$0_0 48_0 40_0$	$0_0 47_0 40_1$	$0_0 26_0 59_1$
$0_0 45_0 5_1$	$0_0 10_0 17_1$	$0_0 15_0 14_1$	$0_0 21_0 19_1$	$0_0 31_0 43_1$	$0_0 41_0 20_1$	$0_0 3_0 35_1$
$0_0 32_0 4_1$	$0_0 7_0 31_1$	$0_0 54_0 49_1$	$0_0 28_0 1_1$	$0_0 42_0 50_1$	$0_0 38_0 9_1$	$0_0 53_0 29_1$
$0_0 43_0 54_1$	$0_0 27_1 10_1$	$0_0 48_1 16_1$	$0_0 61_1 46_1$	$0_0 23_1 47_1$	$0_0 28_1 34_1$	$0_0 42_1 22_1$
$0_0 56_1 57_1$	$0_0 13_1 6_1$	$0_0 26_1 30_1$	$0_0 3_1 55_1$	$0_0 52_1 62_1$	$0_0 2_1 45_1$	$0_0 39_1 53_1$
$0_0 15_1 51_1$	$0_1 54_1 49_1$	$0_1 19_1 56_1$	$0_1 39_1 8_1$	$0_1 44_1 62_1$	$0_1 12_1 42_1$	$0_1 27_1 25_1$
$\infty_0 0_0 18_1$	$\infty_1 0_0 21_1$					
$0_1 63_1 58_1$	$0_1 6_1 28_1$	$0_1 20_1 30_1$	$0_1 52_1 14_1$	$0_1 56_1 41_1$	$0_0 9_1 26_1$	$0_0 22_1 47_1$
$0_0 52_1 64_1$	$0_0 55_1 34_1$	$0_0 30_1 4_1$	$0_0 43_1 35_1$	$0_0 2_1 18_1$	$0_0 40_1 41_1$	$0_0 39_1 36_1$
$0_0 28_1 32_1$	$0_0 21_1 54_1$	$0_0 63_1 29_1$	$0_0 33_1 62_1$	$0_0 24_1 13_1$	$0_0 45_1 27_1$	$0_0 3_1 49_1$
$0_0 8_1 31_1$	$0_0 7_0 60_1$	$0_0 47_0 1_1$	$0_0 56_0 5_1$	$0_0 40_0 17_1$	$0_0 62_0 56_1$	$0_0 63_0 44_1$
$0_0 11_0 23_1$	$0_0 28_0 11_1$	$0_0 15_0 7_1$	$0_0 64_0 50_1$	$0_0 10_0 25_1$	$0_0 24_0 20_1$	$0_0 59_0 10_1$
$0_0 45_0 38_1$	$0_0 8_0 39_0$	$0_0 4_0 27_0$	$0_0 44_0 60_0$	$0_0 33_0 14_0$	$0_0 29_0 17_0$	$0_0 43_0 13_0$
$\infty_0 0_0 37_1$	$\infty_1 0_0 6_1$					
$\mathbf{2^{102}}$						
$0_0 19_0 67_0$	$0_0 95_0 41_0$	$0_0 62_0 84_0$	$0_0 74_0 45_0$	$0_0 91_0 4_0$	$0_0 40_0 66_0$	$0_0 78_0 2_0$
$0_0 3_0 93_0$	$0_0 57_0 58_1$	$0_0 13_0 73_1$	$0_0 38_0 18_1$	$0_0 42_0 79_1$	$0_0 85_0 74_1$	$0_0 32_0 66_1$
$0_0 64_0 19_1$	$0_0 30_0 68_1$	$0_0 12_0 24_1$	$0_0 96_0 64_1$	$0_0 94_0 91_1$	$0_0 83_0 33_1$	$0_0 68_0 11_1$
$0_0 43_0 17_1$	$0_0 36_0 83_1$	$0_0 92_0 61_1$	$0_0 20_0 96_1$	$0_0 73_0 82_1$	$0_0 100_0 7_1$	$0_0 50_0 27_1$
$0_0 55_0 40_1$	$0_0 86_0 84_1$	$0_0 77_0 22_1$	$0_0 80_0 59_1$	$0_0 49_0 65_1$	$0_0 70_0 4_1$	$0_0 77_1 36_1$
$0_0 3_1 55_1$	$0_0 42_1 50_1$	$0_0 29_1 20_1$	$0_0 57_1 28_1$	$0_0 13_1 26_1$	$0_0 2_1 5_1$	$0_0 39_1 62_1$
$0_0 54_1 100_1$	$0_0 94_1 23_1$	$0_0 14_1 32_1$	$0_0 6_1 87_1$	$0_0 97_1 71_1$	$0_0 67_1 95_1$	$0_0 21_1 85_1$
$0_0 49_1 43_1$	$0_0 41_1 92_1$	$0_0 15_1 48_1$	$0_0 10_1 25_1$	$0_0 45_1 93_1$	$0_0 30_1 89_1$	$0_0 63_1 88_1$
$0_0 53_1 31_1$	$0_1 24_1 14_1$	$0_1 85_1 38_1$	$0_1 61_1 27_1$	$0_1 89_1 58_1$	$0_1 66_1 62_1$	$0_1 7_1 5_1$
$0_1 84_1 19_1$	$0_1 11_1 32_1$	$0_1 100_1 56_1$	$\infty_0 0_0 52_1$	$\infty_1 0_0 72_1$		
$0_1 20_1 63_1$	$0_1 45_1 3_1$	$0_1 39_1 46_1$	$0_1 78_1 8_1$	$0_1 17_1 12_1$	$0_1 66_1 30_1$	$0_1 10_1 90_1$
$0_1 50_1 44_1$	$0_0 15_1 6_1$	$0_0 44_1 81_1$	$0_0 28_1 82_1$	$0_0 32_1 54_1$	$0_0 62_1 88_1$	$0_0 9_1 38_1$
$0_0 36_1 37_1$	$0_0 49_1 25_1$	$0_0 26_1 10_1$	$0_0 19_1 93_1$	$0_0 94_1 34_1$	$0_0 2_1 63_1$	$0_0 72_1 3_1$
$0_0 11_1 64_1$	$0_0 87_1 73_1$	$0_0 95_1 76_1$	$0_0 48_1 46_1$	$0_0 69_1 41_1$	$0_0 4_1 29_1$	$0_0 12_1 30_1$
$0_0 21_1 55_1$	$0_0 96_1 47_1$	$0_0 70_1 66_1$	$0_0 17_1 50_1$	$0_0 89_1 1_1$	$0_0 91_1 5_1$	$0_0 39_0 24_1$
$0_0 64_0 99_1$	$0_0 77_0 74_1$	$0_0 99_0 58_1$	$0_0 17_0 33_1$	$0_0 75_0 52_1$	$0_0 45_0 85_1$	$0_0 55_0 7_1$
$0_0 79_0 75_1$	$0_0 5_0 23_1$	$0_0 28_0 79_1$	$0_0 6_0 45_1$	$0_0 51_0 71_1$	$0_0 18_0 83_1$	$0_0 85_0 61_1$
$0_0 100_0 13_1$	$0_0 91_0 57_1$	$0_0 67_0 22_1$	$0_0 15_0 42_1$	$0_0 66_0 8_1$	$0_0 61_0 92_1$	$0_0 20_0 100_1$
$0_0 92_0 59_1$	$0_0 93_0 80_0$	$0_0 31_0 42_0$	$0_0 60_0 98_0$	$0_0 23_0 53_0$	$0_0 29_0 36_0$	$0_0 54_0 97_0$
$0_0 87_0 68_0$	$0_0 57_0 89_0$	$0_0 49_0 74_0$	$\infty_0 0_0 90_1$	$\infty_1 0_0 84_1$		

2^{114}

$0_{0}12_{0}49_{0}$	$0_{0}1_{0}97_{0}$	$0_{0}31_{0}106_{0}$	$0_{0}61_{0}108_{0}$	$0_{0}53_{0}40_{0}$	$0_{0}44_{0}48_{0}$	$0_{0}110_{0}78_{0}$
$0_{0}39_{0}50_{0}$	$0_{0}62_{0}107_{0}$	$0_{0}15_{0}65_{1}$	$0_{0}41_{0}33_{1}$	$0_{0}57_{0}6_{1}$	$0_{0}104_{0}112_{1}$	$0_{0}103_{0}2_{1}$
$0_{0}23_{0}101_{1}$	$0_{0}71_{0}40_{1}$	$0_{0}86_{0}100_{1}$	$0_{0}19_{0}77_{1}$	$0_{0}77_{0}52_{1}$	$0_{0}54_{0}30_{1}$	$0_{0}29_{0}20_{1}$
$0_{0}89_{0}1_{1}$	$0_{0}20_{0}95_{1}$	$0_{0}55_{0}110_{1}$	$0_{0}33_{0}27_{1}$	$0_{0}25_{0}60_{1}$	$0_{0}18_{0}11_{1}$	$0_{0}79_{0}4_{1}$
$0_{0}99_{0}108_{1}$	$0_{0}28_{0}76_{1}$	$0_{0}105_{0}46_{1}$	$0_{0}67_{0}80_{1}$	$0_{0}21_{0}10_{1}$	$0_{0}43_{0}16_{1}$	$0_{0}2_{0}34_{1}$
$0_{0}30_{0}96_{1}$	$0_{0}22_{0}39_{1}$	$0_{0}26_{0}7_{1}$	$0_{0}87_{1}28_{1}$	$0_{0}22_{1}91_{1}$	$0_{0}85_{1}3_{1}$	$0_{0}41_{1}43_{1}$
$0_{0}83_{1}72_{1}$	$0_{0}24_{1}109_{1}$	$0_{0}67_{1}64_{1}$	$0_{0}70_{1}90_{1}$	$0_{0}36_{1}21_{1}$	$0_{0}19_{1}79_{1}$	$0_{0}73_{1}81_{1}$
$0_{0}93_{1}45_{1}$	$0_{0}59_{1}97_{1}$	$0_{0}57_{1}56_{1}$	$0_{0}74_{1}92_{1}$	$0_{0}49_{1}61_{1}$	$0_{0}5_{1}111_{1}$	$0_{0}71_{1}84_{1}$
$0_{0}53_{1}18_{1}$	$0_{0}51_{1}103_{1}$	$0_{0}98_{1}68_{1}$	$0_{0}44_{1}63_{1}$	$0_{0}23_{1}47_{1}$	$0_{0}42_{1}15_{1}$	$0_{0}29_{1}99_{1}$
$0_{0}37_{1}31_{1}$	$0_{1}32_{1}109_{1}$	$0_{1}41_{1}46_{1}$	$0_{1}58_{1}68_{1}$	$0_{1}17_{1}80_{1}$	$0_{1}74_{1}25_{1}$	$0_{1}42_{1}79_{1}$
$0_{1}26_{1}92_{1}$	$0_{1}84_{1}22_{1}$	$0_{1}104_{1}14_{1}$	$0_{1}73_{1}57_{1}$	$\infty_{0}0_{0}26_{1}$	$\infty_{1}0_{0}69_{1}$	
$0_{1}97_{1}62_{1}$	$0_{1}71_{1}67_{1}$	$0_{1}21_{1}28_{1}$	$0_{1}96_{1}22_{1}$	$0_{1}33_{1}53_{1}$	$0_{1}103_{1}54_{1}$	$0_{1}24_{1}76_{1}$
$0_{1}101_{1}110_{1}$	$0_{1}44_{1}29_{1}$	$0_{0}30_{1}88_{1}$	$0_{0}106_{1}76_{1}$	$0_{0}71_{1}79_{1}$	$0_{0}23_{1}95_{1}$	$0_{0}87_{1}93_{1}$
$0_{0}104_{1}64_{1}$	$0_{0}37_{1}11_{1}$	$0_{0}70_{1}4_{1}$	$0_{0}111_{1}34_{1}$	$0_{0}40_{1}54_{1}$	$0_{0}97_{1}9_{1}$	$0_{0}44_{1}101_{1}$
$0_{0}1_{1}51_{1}$	$0_{0}105_{1}103_{1}$	$0_{0}108_{1}6_{1}$	$0_{0}107_{1}89_{1}$	$0_{0}5_{1}43_{1}$	$0_{0}62_{1}67_{1}$	$0_{0}49_{1}81_{1}$
$0_{0}55_{1}24_{1}$	$0_{0}47_{1}2_{1}$	$0_{0}80_{1}10_{1}$	$0_{0}35_{1}100_{1}$	$0_{0}73_{1}96_{1}$	$0_{0}82_{1}48_{1}$	$0_{0}46_{1}59_{1}$
$0_{0}91_{1}72_{1}$	$0_{0}56_{1}83_{1}$	$0_{0}109_{1}110_{1}$	$0_{0}67_{0}66_{1}$	$0_{0}2_{0}77_{1}$	$0_{0}19_{0}22_{1}$	$0_{0}9_{0}42_{1}$
$0_{0}72_{0}53_{1}$	$0_{0}60_{0}85_{1}$	$0_{0}3_{0}15_{1}$	$0_{0}61_{0}13_{1}$	$0_{0}23_{0}39_{1}$	$0_{0}109_{0}41_{1}$	$0_{0}93_{0}18_{1}$
$0_{0}31_{0}57_{1}$	$0_{0}21_{0}29_{1}$	$0_{0}49_{0}20_{1}$	$0_{0}43_{0}32_{1}$	$0_{0}79_{0}27_{1}$	$0_{0}75_{0}60_{1}$	$0_{0}68_{0}7_{1}$
$0_{0}5_{0}68_{1}$	$0_{0}99_{0}78_{1}$	$0_{0}106_{0}21_{1}$	$0_{0}16_{0}90_{1}$	$0_{0}69_{0}86_{1}$	$0_{0}11_{0}69_{1}$	$0_{0}33_{0}19_{1}$
$0_{0}17_{0}31_{1}$	$0_{0}26_{0}56_{0}$	$0_{0}12_{0}39_{0}$	$0_{0}65_{0}6_{0}$	$0_{0}66_{0}84_{0}$	$0_{0}24_{0}105_{0}$	$0_{0}25_{0}10_{0}$
$0_{0}42_{0}55_{0}$	$0_{0}63_{0}28_{0}$	$0_{0}73_{0}51_{0}$	$0_{0}36_{0}37_{0}$	$\infty_{0}0_{0}50_{1}$	$\infty_{1}0_{0}36_{1}$	

A4. Near relative difference set OGDDs

Each near relative difference set GDD of type 2^u, where $2u = 2 + t$, has point set $X = \{\infty_0, \infty_1\} \cup Z_t$ The base blocks are developed under the automorphism $(0, 1, ..., t-1)(\infty_0, \infty_1)$. The OGDDs have the group structure $\mathcal{G} = \{\{\infty_0, \infty_1\}\} \cup \{\{i, i + t/2\} \mid i = 0, 1, ..., (t/2 - 1)\}$.

2^{24}

$0_{0}1_{0}3_{0}$	$0_{0}4_{0}9_{0}$	$0_{0}6_{0}13_{0}$	$0_{0}8_{0}22_{0}$	$0_{0}10_{0}26_{0}$	$0_{0}11_{0}28_{0}$	$0_{0}12_{0}27_{0}$
$\infty_{0}0_{0}21_{0}$						
$0_{0}1_{0}4_{0}$	$0_{0}2_{0}30_{0}$	$0_{0}5_{0}14_{0}$	$0_{0}6_{0}17_{0}$	$0_{0}7_{0}26_{0}$	$0_{0}8_{0}33_{0}$	$0_{0}10_{0}22_{0}$
$\infty_{0}0_{0}15_{0}$						

2^{27}

$0_{0}1_{0}3_{0}$	$0_{0}4_{0}9_{0}$	$0_{0}6_{0}13_{0}$	$0_{0}8_{0}22_{0}$	$0_{0}10_{0}27_{0}$	$0_{0}11_{0}31_{0}$	$0_{0}12_{0}28_{0}$
$0_{0}15_{0}33_{0}$	$\infty_{0}0_{0}23_{0}$					
$0_{0}1_{0}4_{0}$	$0_{0}2_{0}10_{0}$	$0_{0}5_{0}16_{0}$	$0_{0}6_{0}34_{0}$	$0_{0}7_{0}37_{0}$	$0_{0}9_{0}29_{0}$	$0_{0}12_{0}33_{0}$
$0_{0}13_{0}38_{0}$	$\infty_{0}0_{0}17_{0}$					

2^{39}

$0_{0}1_{0}3_{0}$	$0_{0}4_{0}9_{0}$	$0_{0}6_{0}13_{0}$	$0_{0}8_{0}18_{0}$	$0_{0}11_{0}25_{0}$	$0_{0}12_{0}39_{0}$	$0_{0}15_{0}43_{0}$
$0_{0}16_{0}40_{0}$	$0_{0}17_{0}46_{0}$	$0_{0}19_{0}45_{0}$	$0_{0}20_{0}42_{0}$	$0_{0}21_{0}44_{0}$	$\infty_{0}0_{0}35_{0}$	
$0_{0}1_{0}4_{0}$	$0_{0}2_{0}8_{0}$	$0_{0}5_{0}14_{0}$	$0_{0}7_{0}22_{0}$	$0_{0}10_{0}50_{0}$	$0_{0}11_{0}30_{0}$	$0_{0}12_{0}43_{0}$
$0_{0}13_{0}55_{0}$	$0_{0}16_{0}48_{0}$	$0_{0}17_{0}56_{0}$	$0_{0}18_{0}41_{0}$	$0_{0}24_{0}49_{0}$	$\infty_{0}0_{0}29_{0}$	

A5. 1-cyclic OMGDDs

Each design of type (g, u) has an automorphism $\alpha = (0, 1, ..., gu - 1)$ and the group structure type u as defined in §3. Only the base blocks of the OMGDDs are listed.

(5, 13)

0,1,29	0,2,19	0,3,27	0,4,11	0,6,22	0,8,42	0,9,53	0,14,32
0,1,8	0,2,16	0,3,21	0,4,38	0,6,43	0,9,41	0,11,53	0,17,46

(5, 16)

0,1,39	0,2,6	0,3,36	0,7,63	0,8,26	0,9,61	0,11,23	0,13,34
0,14,51	0,22,49						
0,1,68	0,2,39	0,3,22	0,4,21	0,6,29	0,7,56	0,8,54	0,9,27
0,11,44	0,14,42						

(5, 19)

0,1,8	0,2,54	0,3,71	0,4,62	0,6,18	0,9,23	0,11,47	0,13,44
0,16,42	0,17,73	0,21,49	0,29,61				
0,1,9	0,2,6	0,3,67	0,7,21	0,11,63	0,12,61	0,13,72	0,16,53
0,17,41	0,18,62	0,22,48	0,27,56				

(5, 22)

0,1,83	0,2,43	0,3,87	0,4,103	0,6,74	0,8,37	0,9,56	0,12,64
0,13,34	0,14,38	0,16,49	0,17,79	0,18,57	0,19,78		
0,1,102	0,2,34	0,3,61	0,4,41	0,6,103	0,11,38	0,12,68	0,14,31
0,16,87	0,18,51	0,19,47	0,21,64	0,24,53	0,26,62		