Combinatorics on Adjacency Graphs and Incidence Pseudographs

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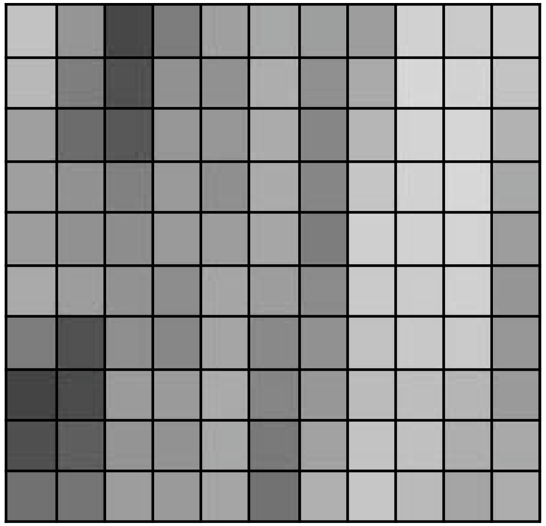
Adjacency graphs:
generalization of adjacencies in
grid cell or grid point model, in this talk: 2D case

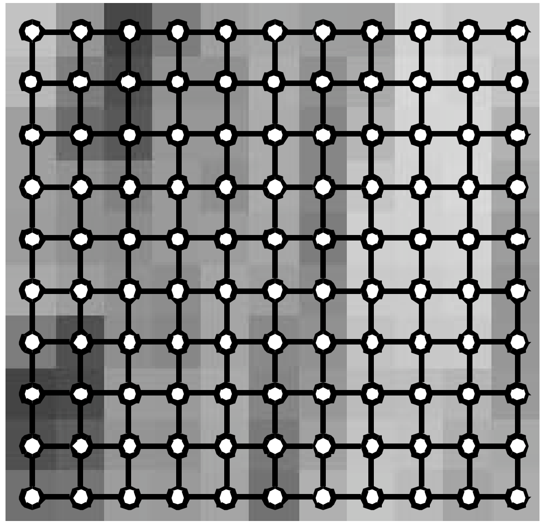
all nodes equal symmetric and irreflexive adjacency relation

Incidence Pseudographs:
generalization of Euclidean complexes (poset topologies)
grid cell incidence model, for nD case, $n \geq 1$

nodes characterized by dimension 0,1,…,n symmetric and reflexive incidence relation



grid cell model



grid point model

2D: 4- and 8-adjacencies, 3D: 6-, 18- and 26-adjacencies

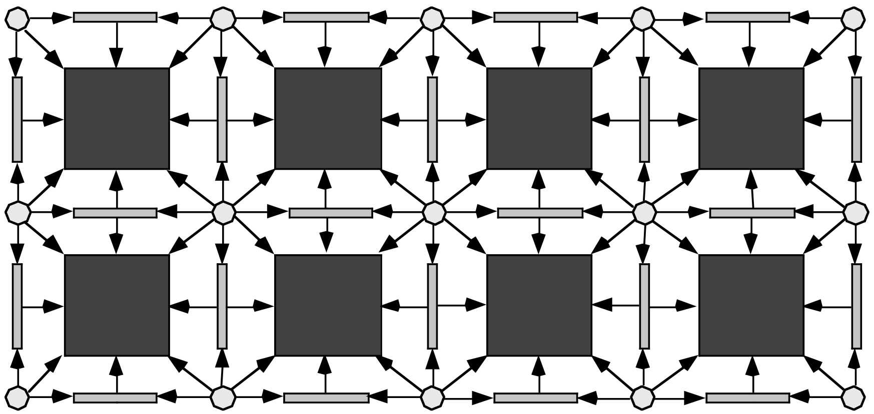
2D: 1- and 0-adjacencies, 3D: 2-, 1-, and 0-adjacencies

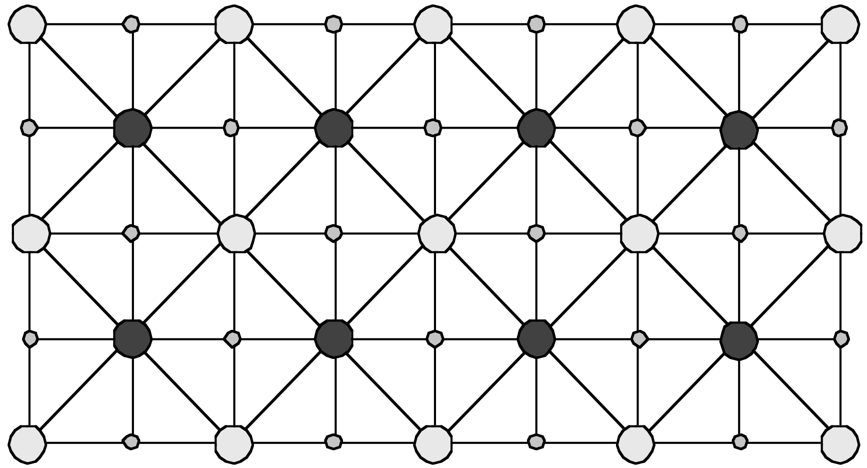
2D nodes: pixel, 3D nodes: voxel or frontier faces

Rosenfeld 1970, …, Artzy/Frieder/Herman 1981, …

Aleksandrov-Hopf 1935 Khalimsky 1986

homeomorphic poset topologies for 2D picture grids





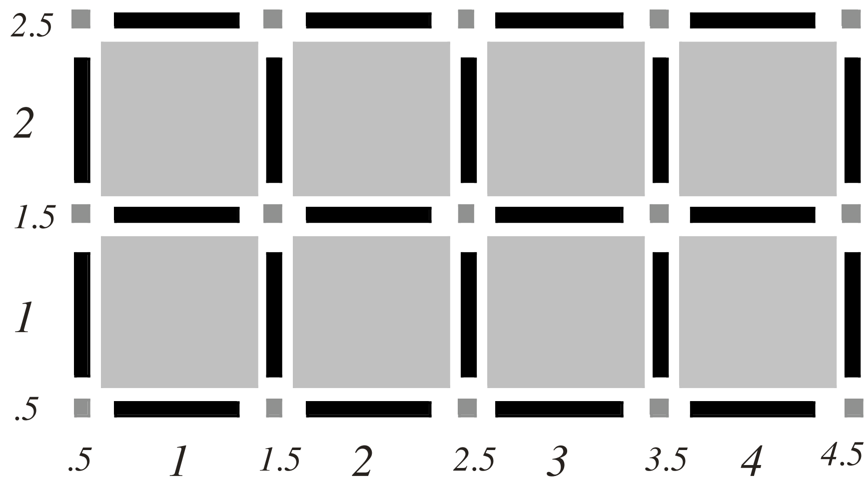
Kovalevsky 1989

m x n picture grid and $(m+1)$ x $(n+1)$ frontier grid

``maximum-label rule"

Voss 1993

incidence relations in nD grid



ORIENTED ADJACENCY GRAPHS [S, A, □]

countable set S, adjacency relation A (irreflexive, symmetric)

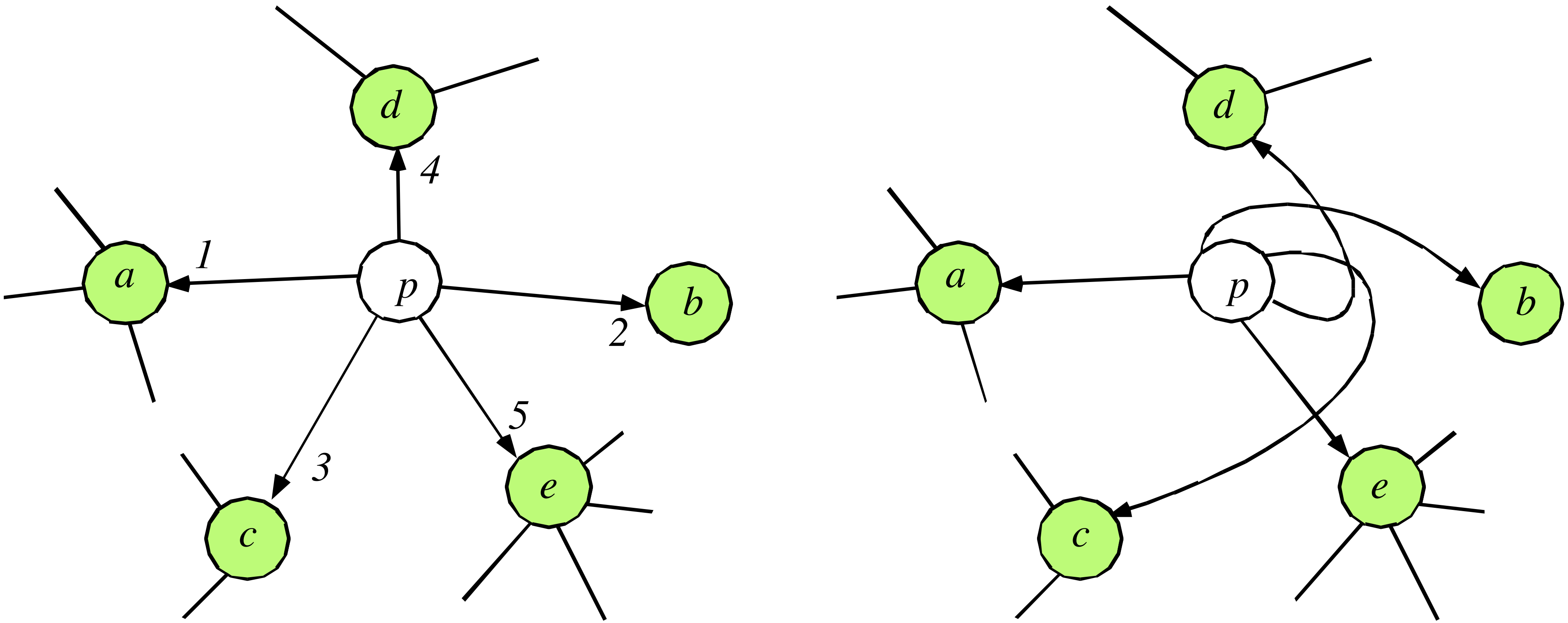
local cyclic orders □

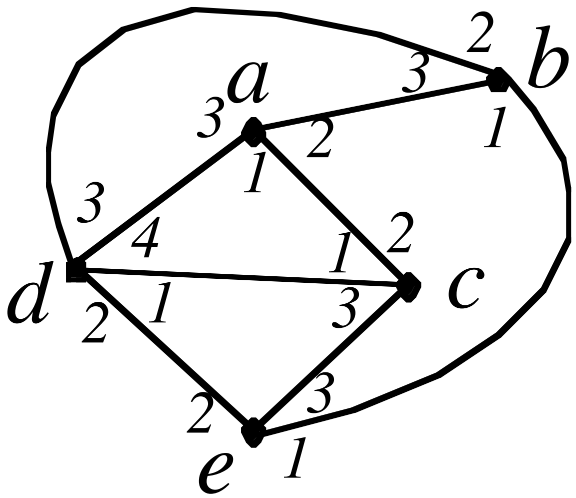
- $A(p)$ is finite for any p in S
- [S, A] is a connected undirected graph (finite or infinite)
- any finite subset M of S possesses at most one infinite complementary component
- any directed edge generates a periodic path with respect to □

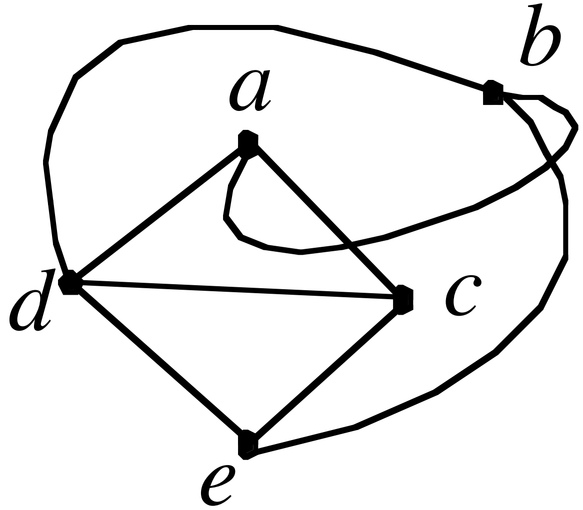
note: a generalization of oriented 2D tilings or 2D combinatorial maps

local circular order □(p) = [a, b, c, d, e]

of all points in the adjacency set $A(p)$





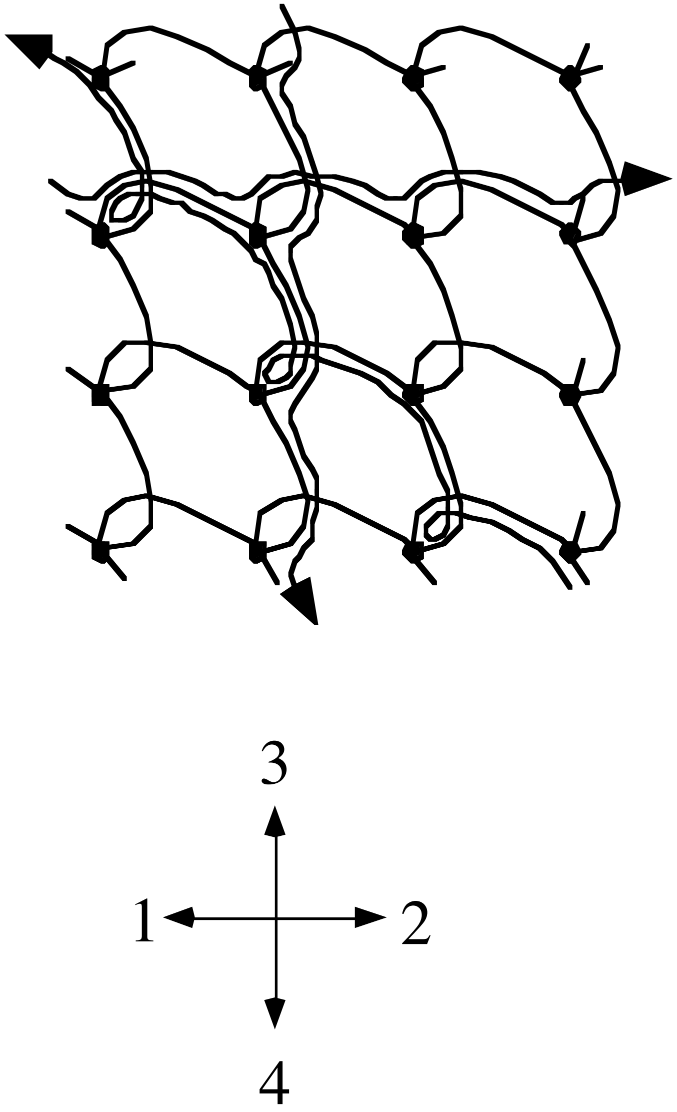


the undirected graph needs not to be planar (as in 2D tilings)
and not to be finite (as in 2D combinatorial maps)

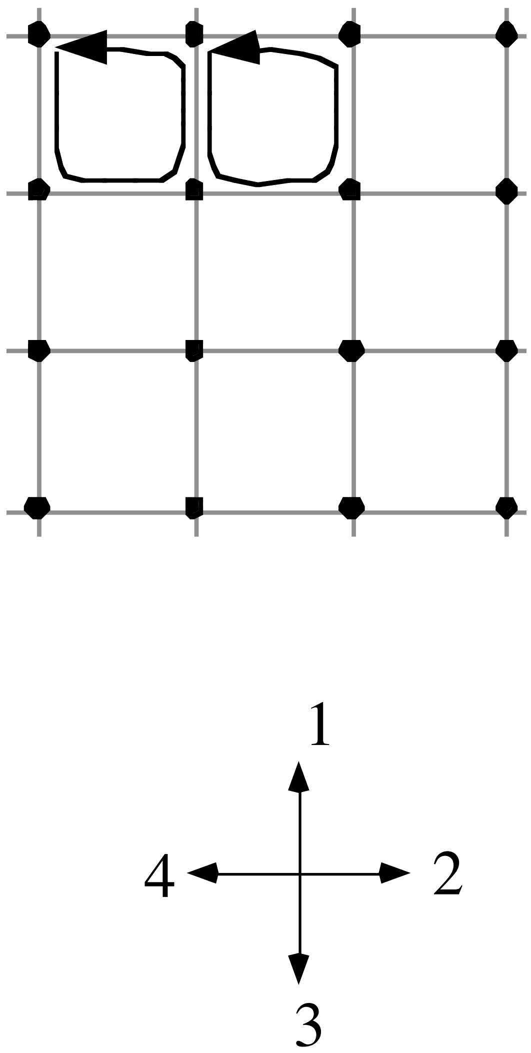
- LEFT: numberings of local circular orders
- RIGHT: drawing convention: clockwise order of outgoing edges

$\square(a) = [c, b, d]$ $\square(b) = [e, d, a]$ $\square(c) = [d, a, e]$
$\square(d) = [c, e, b, a]$ $\square(e) = [b, d, c]$

directed edge (d,a) generates circuit $\square(d,a) = <d, a, c, e, b>$
$\square(a,d) = <a, d, c>$, ...



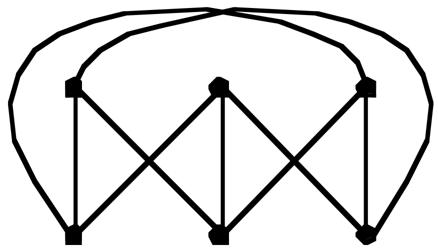
not an oriented adjacency graph (infinite paths)



cycle = generated circuit

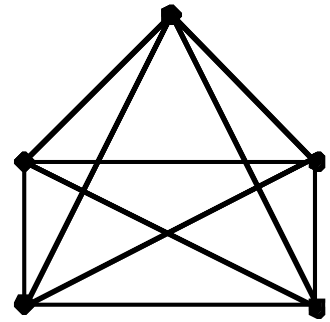
oriented adjacency graph $[S, A, \square]$: $\quad \underset{p \square S}{\square} \square(p) = 2\square_1 \qquad \underset{\square}{\square} \square(\square) = 2\square_1$

$\square_0 \;=\;$ card (S)
$\square_1 \;=\;$ card (A)
$\square(p) =$ card$(A(p))$
$\square(\square) =$ length of cycle $\square$
$\square_2 \;=$ # cycles



$\square_0 = 6$
$\square_1 = 9$
$\square_2 = 3$

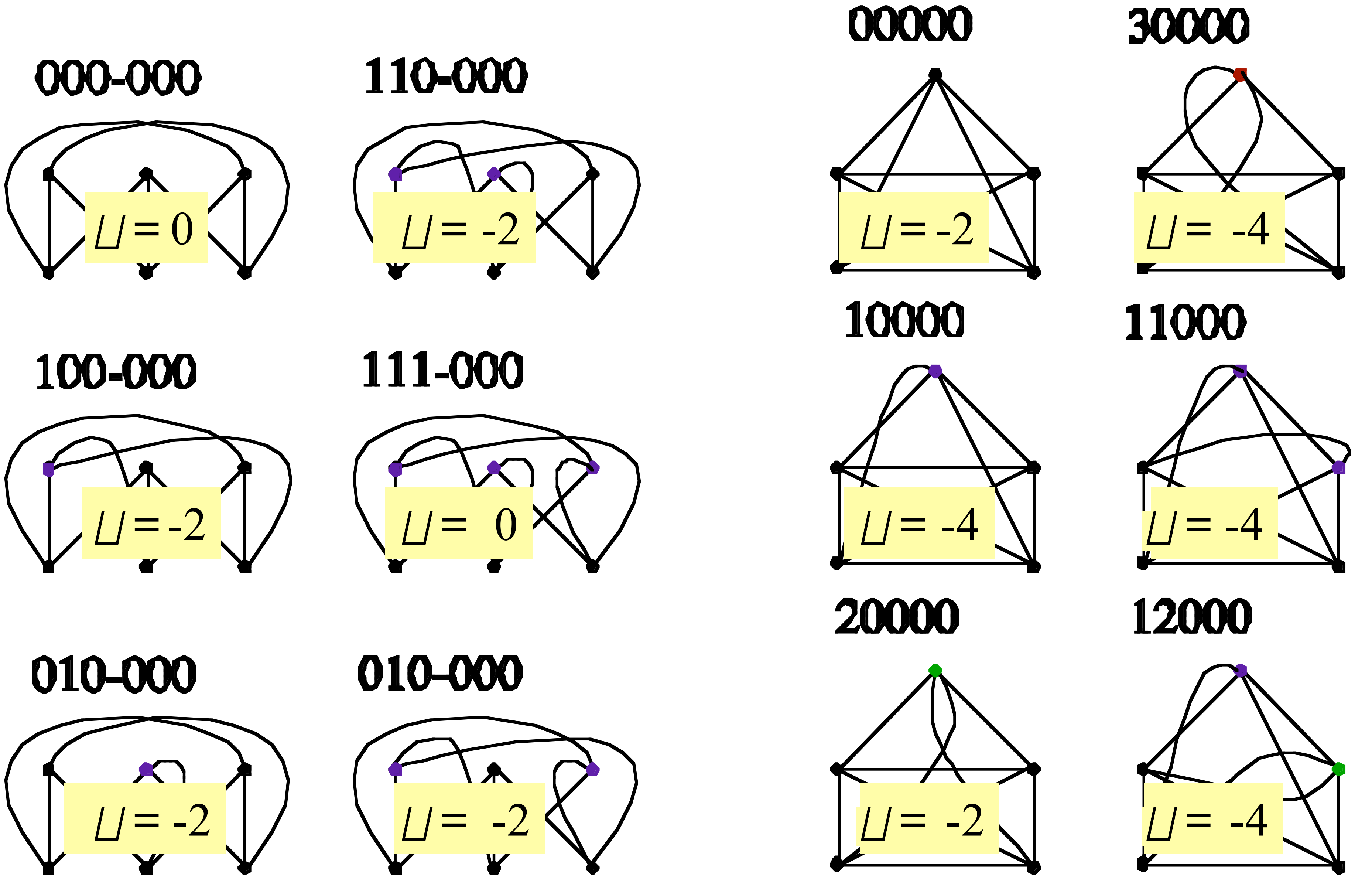
$\square = 0$



$\square_0 = 5$
$\square_1 = 10$
$\square_2 = 3$

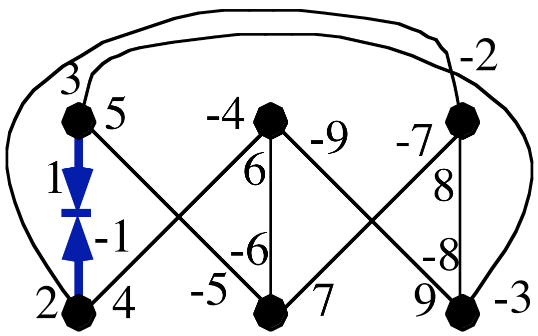
$\square = -2$

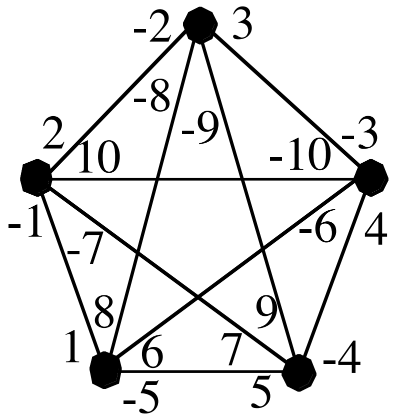
Euler characteristic $\square = \square_0 - \square_1 + \square_2$


000-000
U = 0
110-000
U = -2
00000
U = -2
30000
U = -4
100-000
U = -2
111-000
U = 0
10000
U = -4
11000
U = -4
010-000
U = -2
010-000
U = -2
20000
U = -2
12000
U = -4

combinatorial maps: each directed edge = two **darts**

Heffter 1895, Edmonds 1960, Tutte 1963





anti-clockwise

□ = (1,□1)(2,□2)(3,□3)(4,□4)(5,□5)(6,□6)(7,□7),(8,□8),(9,□9)

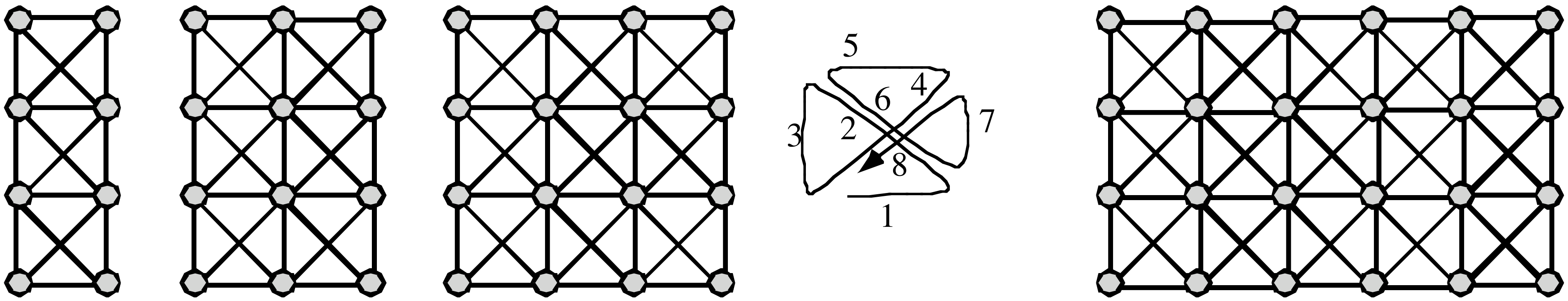
□ = (5,3,1)(-4,6,-9)(-7,8,-2)(-1,2,4)(-6,-5,7)(-8,9,-3)

□ = □ ∘ □ = (1,2,-7,-6,-9,-3)(-1,5,7,8,9,-4)(-2,4,6,-5,3,-8)

clockwise

□ = (2,10,-7,-1)(-2,3,-9,-8)(-3,4,-6,-10)(9,-4,5,7)(8,6,-5,1)

□ = □ ∘ □ = (-2,10,-3,-9,-4,-6,-5,7,-1,8)(1,2,3,4,5)(6,-10,-7,9,-8)



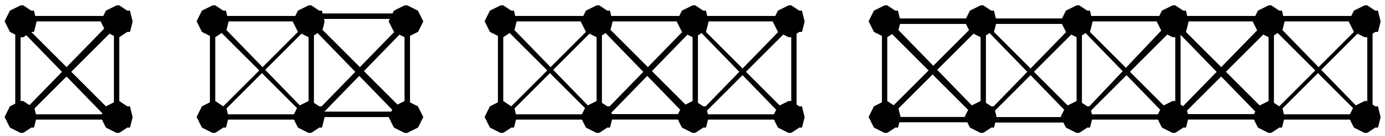
-4 -10 -16 -28

$\chi \leq 2$ for any finite oriented adjacency graph

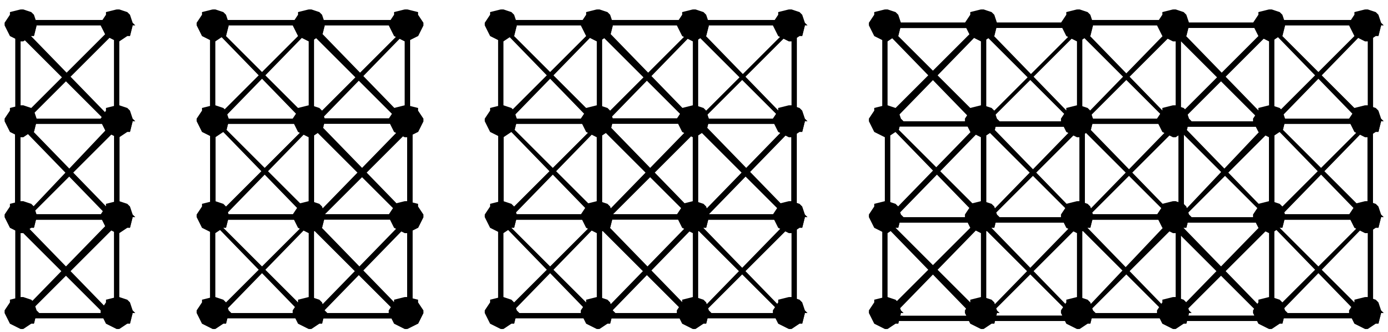
Voss and Klette 1986

finite: **planar** iff $\chi = 2$
infinite: **planar** iff any non-empty finite connected subgraph planar

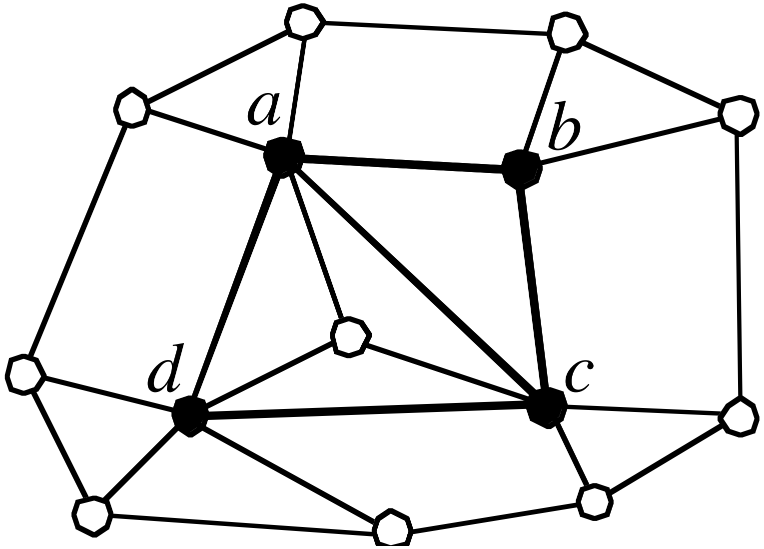
□$_0$,□$_1$,□$_2$=	(4,6,2)	(6,11,3)	(8,16,4)	(12,26,6)	(2*n*,5*n*-4,*n*)
Ц:	0	-2	-4	-8	$\boxed{\mathbf{-2(n-2)}}$



□$_0$,□$_1$,□$_2$=	(8,16,4)	(12,29,7)	(16,42,10)	(24,68,16)	(4*n*,13*n*-10,3*n*-2)
Ц:	-4	-10	-16	-28	$\boxed{\mathbf{-2(3n-4)}}$



$M \square S$ generates **restricted local circular orders** $\square_M(a) = [b,c,d]$



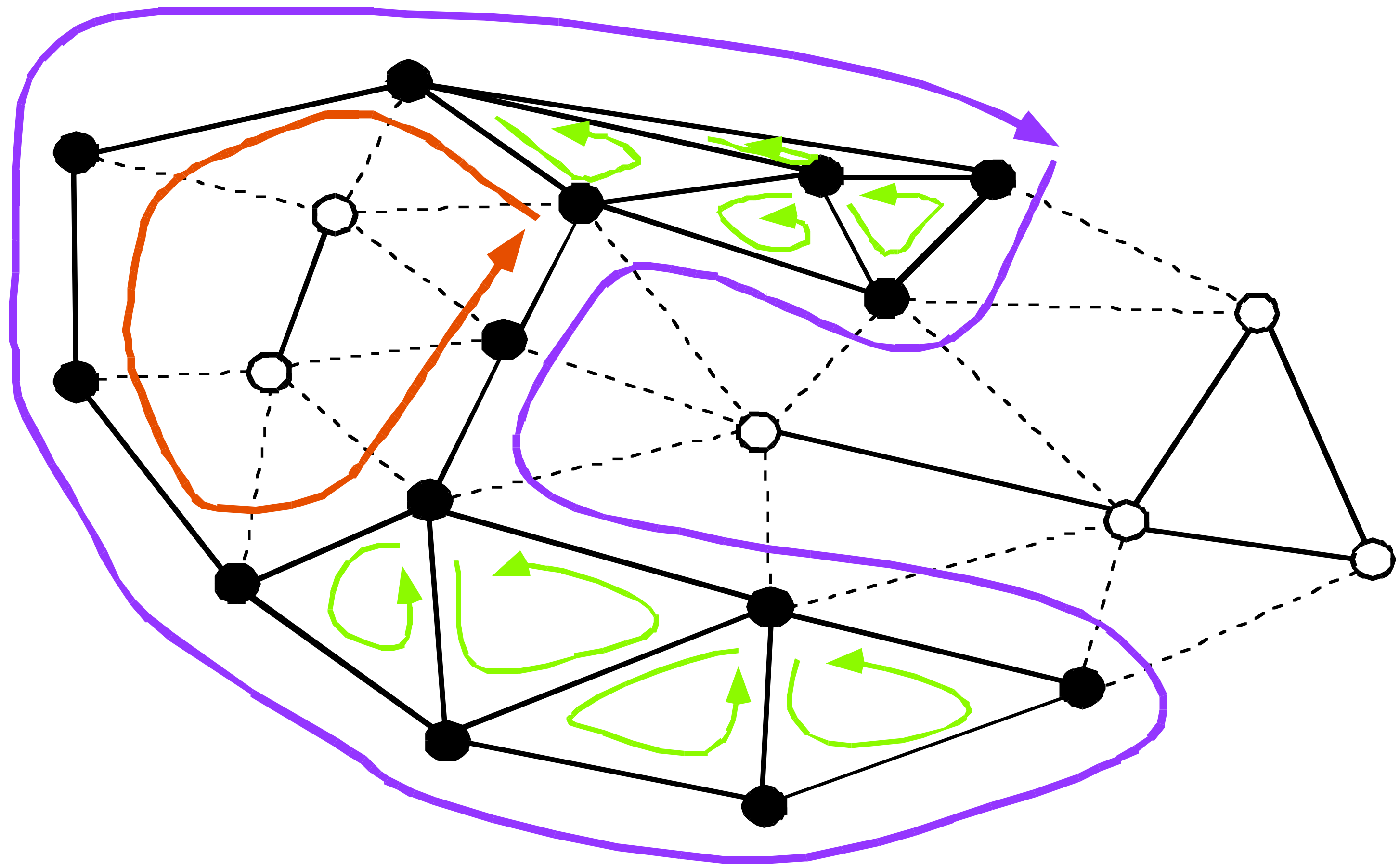
$(a,b),(b,c),(c,d),(d,a)$

$(b,a),(a,c),(c,b)$

$(d,c),(c,a),(a,d)$

$<b,a,c>$ is cycle in $[S, A, \square]$: **atomic cycle**

$<a,b,c,d>$ and $<d,c,a>$ are not cycles in $[S, A, \square]$: **border cycles**

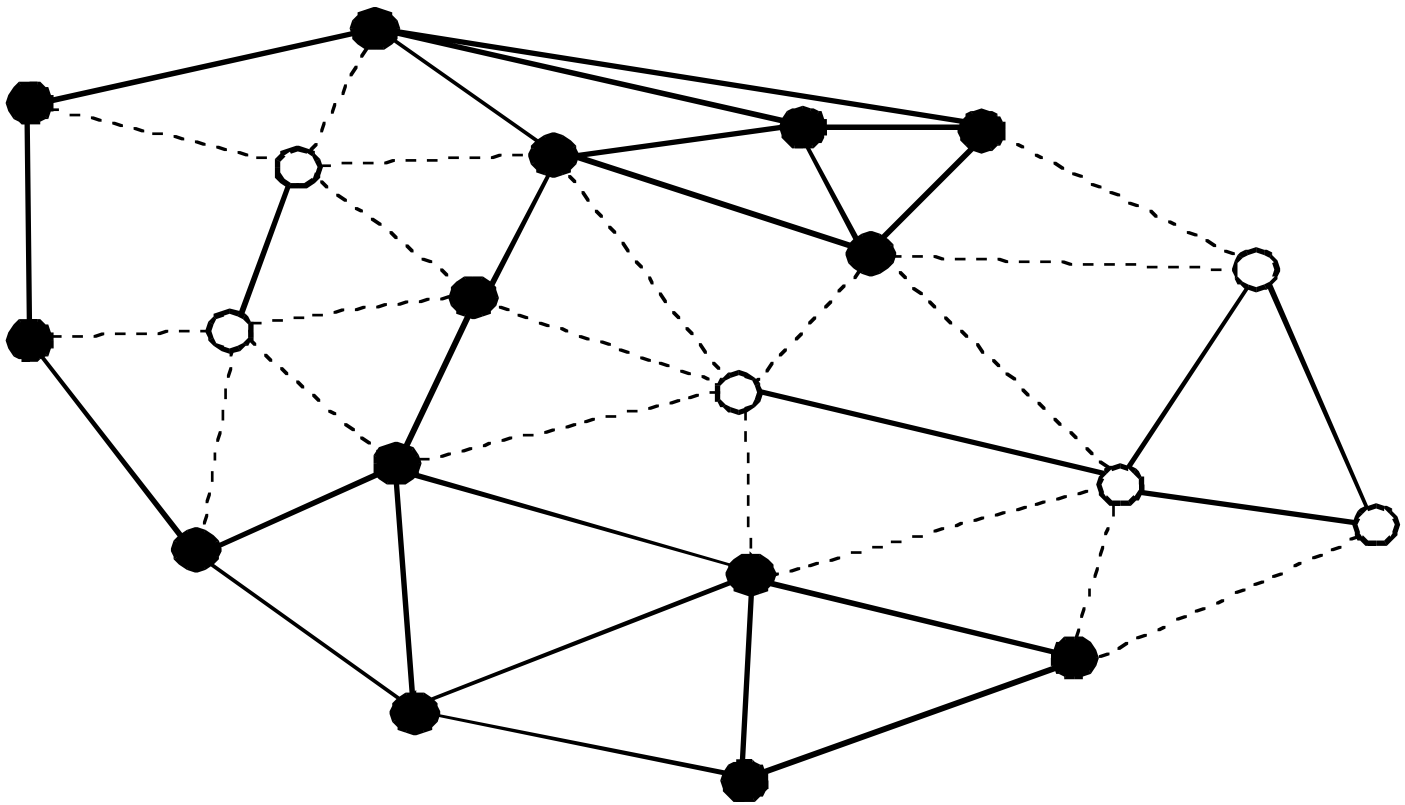


$[S, A, \square_{\square}]$: 8 atomic cycles

2 border cycles

undirected invalid edges **assigned to a border cycle**

[S, A, □]: $\quad \square_0 = 20 \quad \square_1 = 46 \quad \square_2 = 28 \quad U = 2$



[S, A, $\square_{\square}$]: $\quad \square_0 = 14 \quad \square_1 = 22 \quad \square_2 = 10 \quad U = 2$

Note: Euler characteristic of graphs, also counting the ``infinite exterior''

Voss and Klette 1986: *separation theorem*

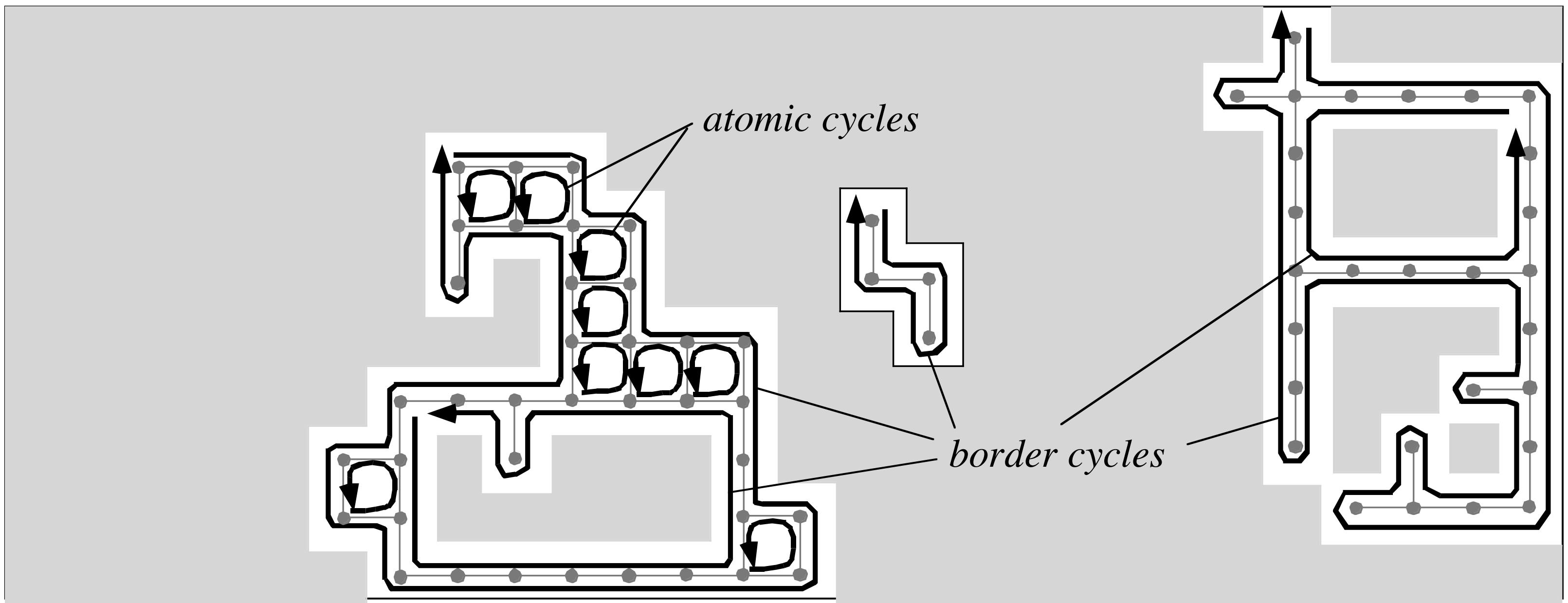
Let [*S*, *A*, □] be a planar oriented adjacency graph.

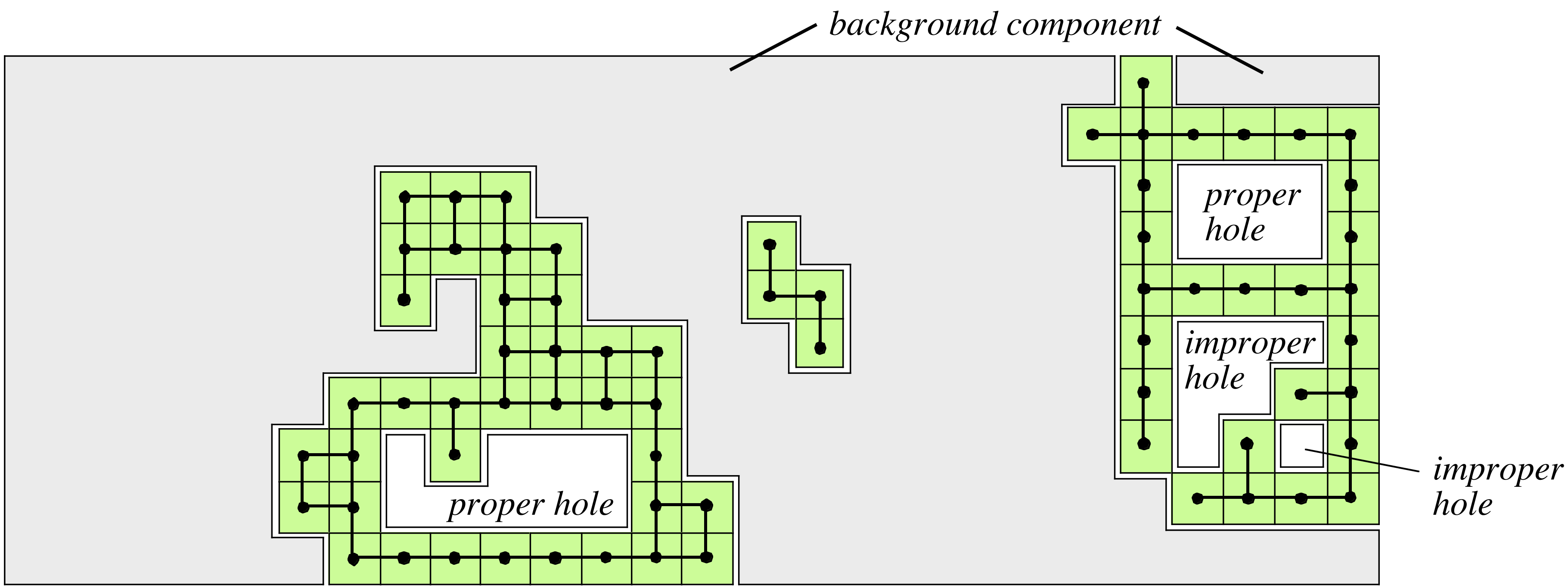
Let *M* be a non-empty finite connected proper subset of *S*.

By deleting all undirected invalid edges assigned to one of the border cycles of *M*, [*S*, *A*, □] splits into **at least** two non-connected substructures.

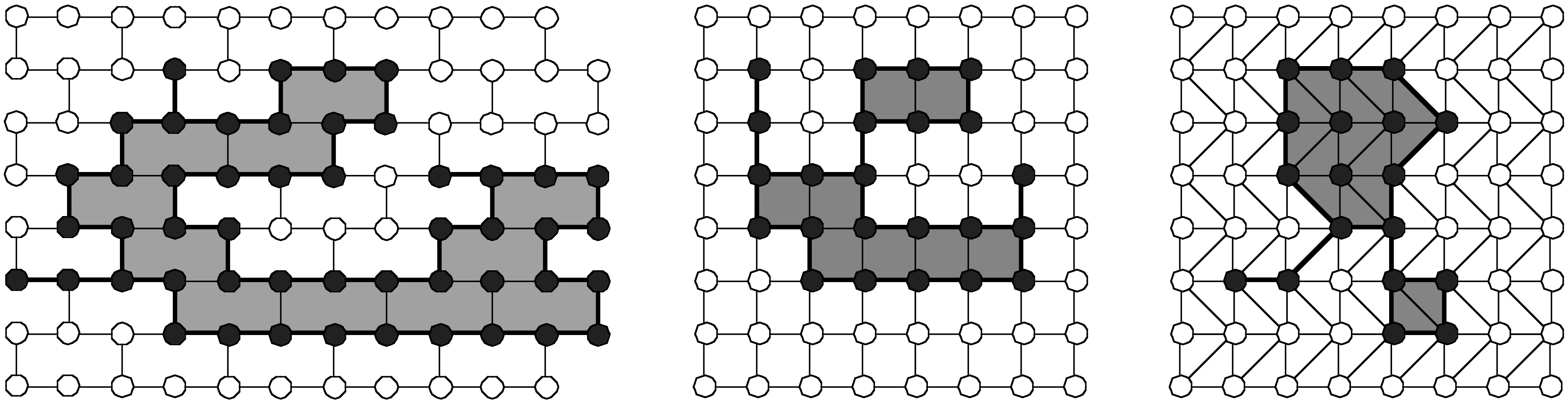
the uniquely defined **outer border cycle** of M separates one (infinite) background component and a finite number of improper holes from M

any **inner border cycle** of M separates a finite number of proper holes from M


atomic cycles
border cycles


background component
proper hole
proper hole
improper hole
improper hole

tiling = planar oriented (finite or infinite) adjacency graph
regular tiling = $\square(p)$ and $\square(\square)$ constants



left : $\square = 3$, $\square = 6$, $\square_0 = 49$, $\square_1 = 59$, $\square_2 = 12$, $l = 52$, $k = 29$, $f = 11$
middle : $\square = 4$, $\square = 4$, $\square_0 = 23$, $\square_1 = 30$, $\square_2 = 9$, $l = 28$, $k = 32$, $f = 8$
right : $\square = 6$, $\square = 3$, $\square_0 = 18$, $\square_1 = 32$, $\square_2 = 16$, $l = 19$, $k = 44$, $f = 15$

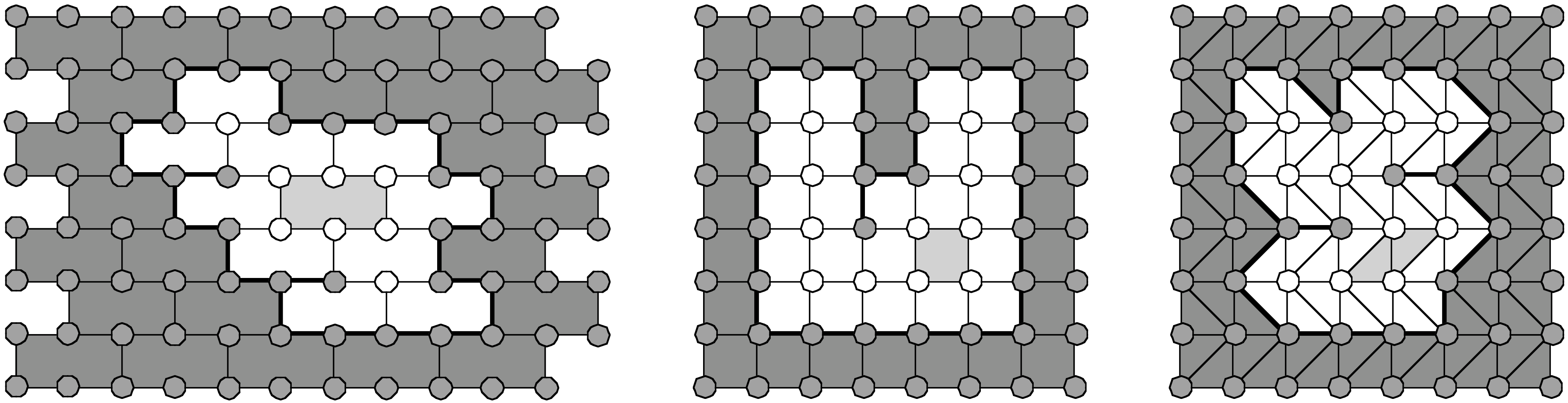
l = length of (inner or outer) border cycle
k = # invalid edges assigned to border cycle
$f = \square_2 - 1$

$$\boxed{k = \square + \frac{\square}{\square} l}$$

29 = 3 + 3/6 x 52

32 = 4 + 4/4 x 28

44 = 6 + 6/3 x 19



Voss 1986: *total curvature theorem*

M = finite connected subset of an infinite regular tiling $S_{\square,\square}$

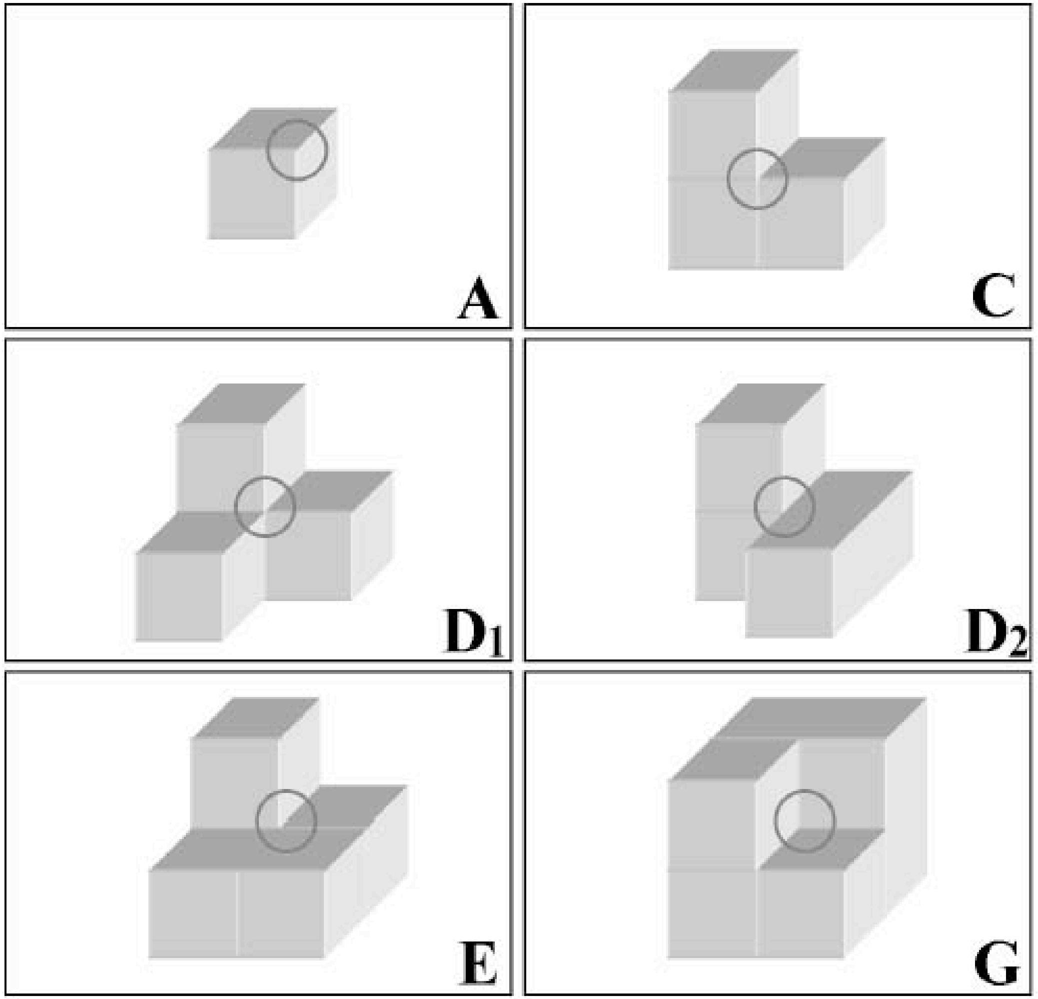
for any border cycle: $\pm 1 = \frac{k}{\Pi} \square \frac{l}{\Pi}$

outer border cycle: defined by positive sign

inner border cycle: defined by negative sign

Imiya and Eckhardt 1999: angles in an isothetic connected polyhedron

$$\boxed{(H_A + H_G) - (H_C + H_E) - 2(H_{D1} + H_{D2}) = 8}$$



H_A, H_c, H_{D1}, H_{D2}, H_E, H_G

= # A, C, D1, D2, E, G angles

of polyhedron H, respectively

$H_A - H_G < 0$ iff inner border

$H_A - H_G > 0$ iff outer border

Yip and Klette 2002: simple isothetic polyhedron

Voss 1986: *generalized Pick's theorem*

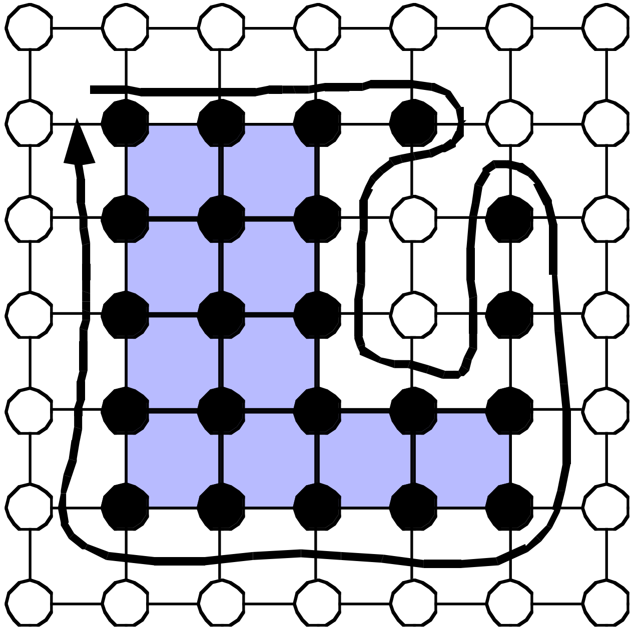
M = finite connected subset of an infinite regular tiling $S_{\square,\square}$
without proper holes, then

for the (outer) border cycle: $\square_0 = \frac{\square}{\square} f + l/2 + 1$

M = finite connected subset of an infinite regular tiling $S_{\square,\square}$
then

for any inner border cycle: $\square_0 = \frac{\square}{\square} f \,\square\, l/2 + 1$

(see G. Pick's area theorem $A = i + b/2 - 1$ from 1899 for the orthogonal grid)



$\square_0 = 22 \quad f = 10 \quad l = 22$

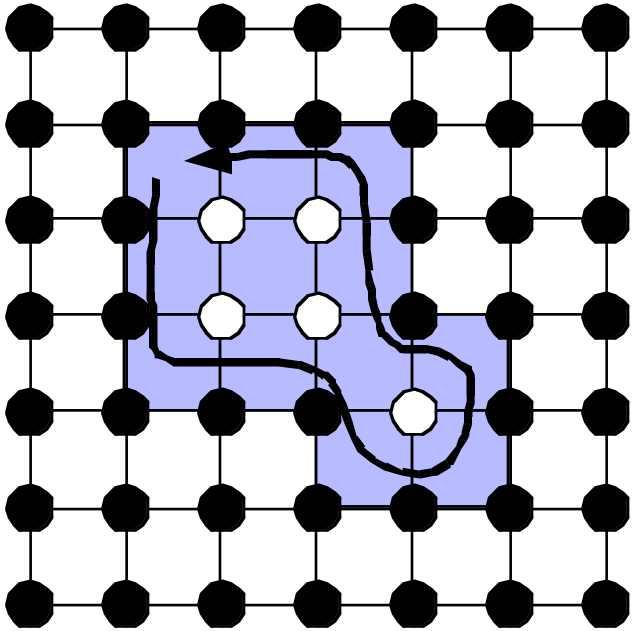
$22 = 10 + 22/2 + 1$

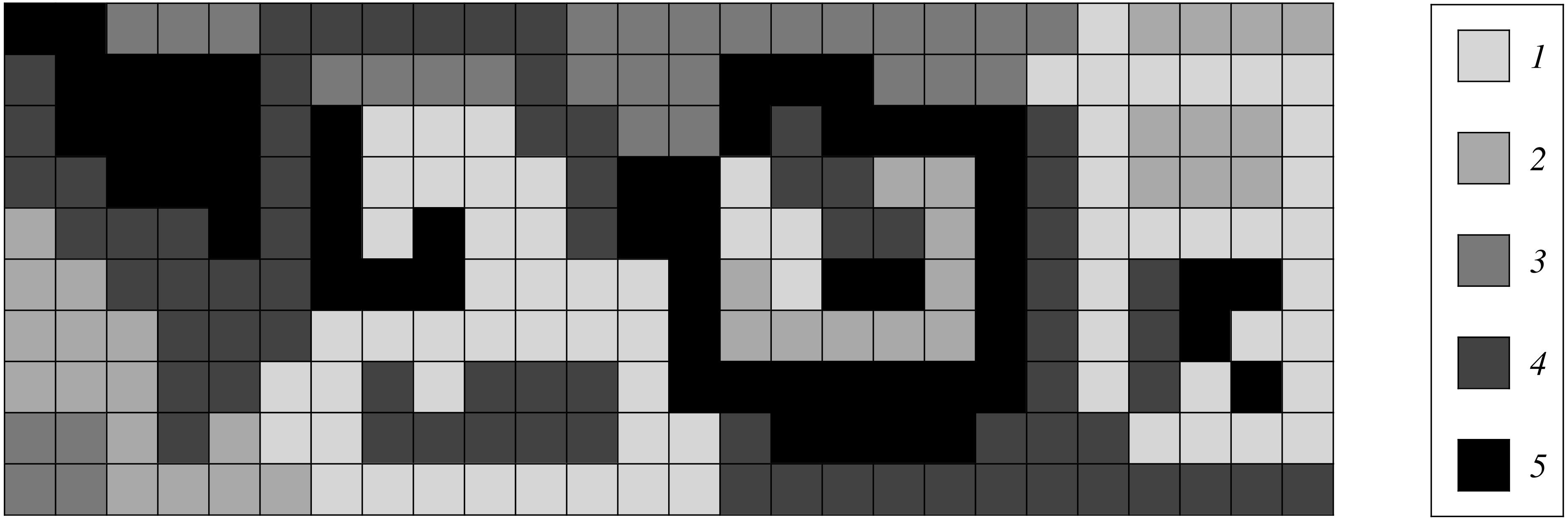
outer border cycle: set is connected, no proper hole, but one improper hole

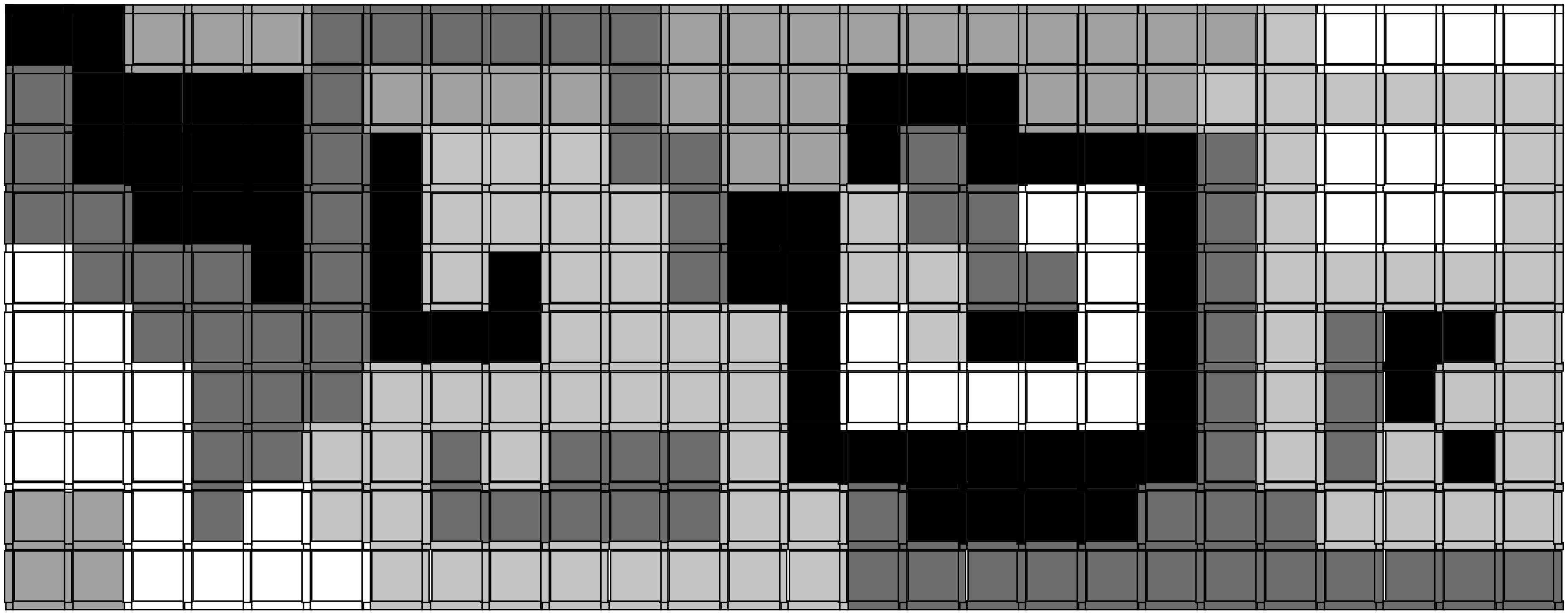
inner border cycle defining two proper holes

$\square_0 = 5 \quad f = 12 \quad l = 16$

$5 = 12 - 16/2 + 1$




1
2
3
4
5



R. Descartes (Cartesius): one convex polyhedron with $\square_0$, $\square_1$, $\square_2$

L. Euler

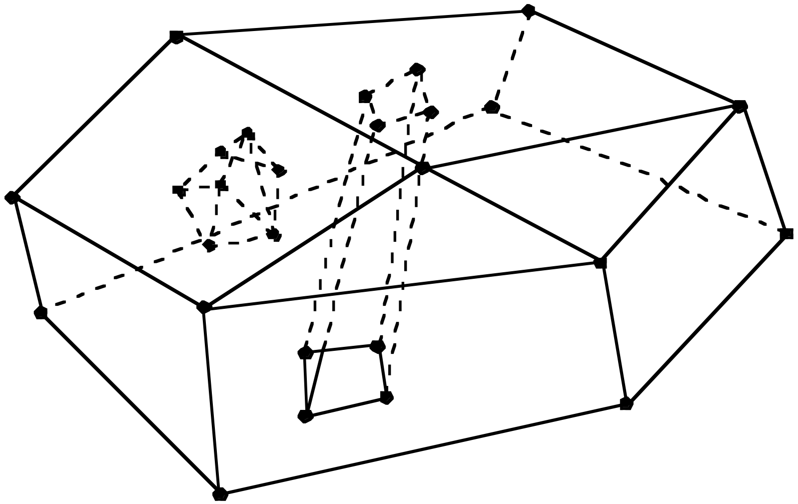
$$\boxed{\square_0 - \square_1 + \square_2 = 2}$$

first proof: 1794 by A.-M. Legendre

A. Cauchy 1813: *D* polyhedral cells within one convex polyhedron

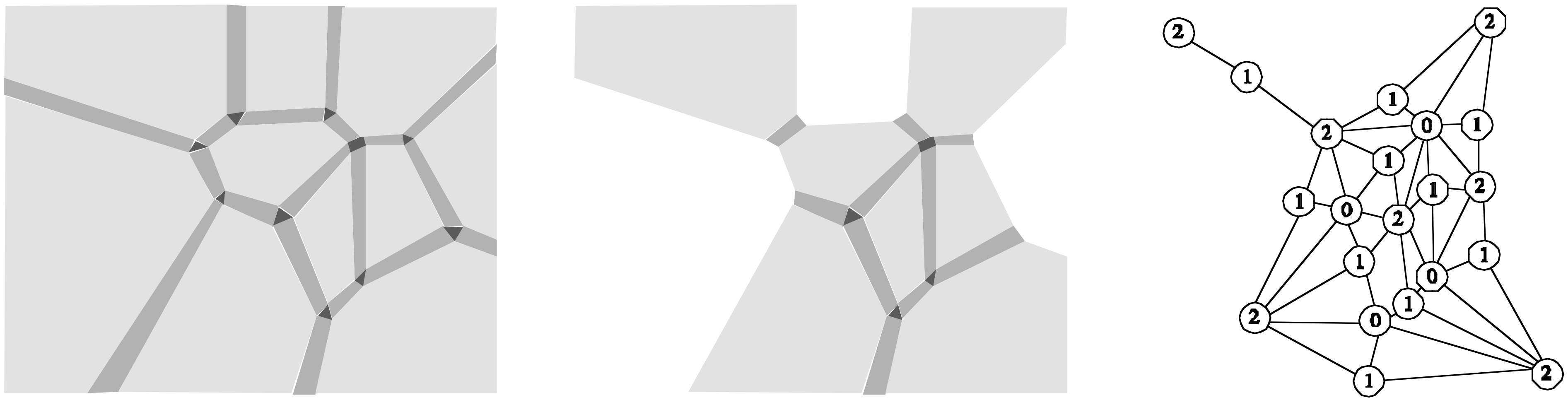
$$\boxed{\square_0 - \square_1 + \square_2 = D + 1}$$

A.-J. Lhuilier 1812: *b* \`bubbles', *t* \`tunnels' and *p* \`entrances/exits'



$$\boxed{\square_0 - \square_1 + \square_2 = 2\,(b - t + 1) + p}$$

wrong



incidence structure $G = [S, I, dim]$

countable set S

incidence relation I on S (reflexive and symmetric)

function $dim : S$ into $\{0, 1, ..., m\}$

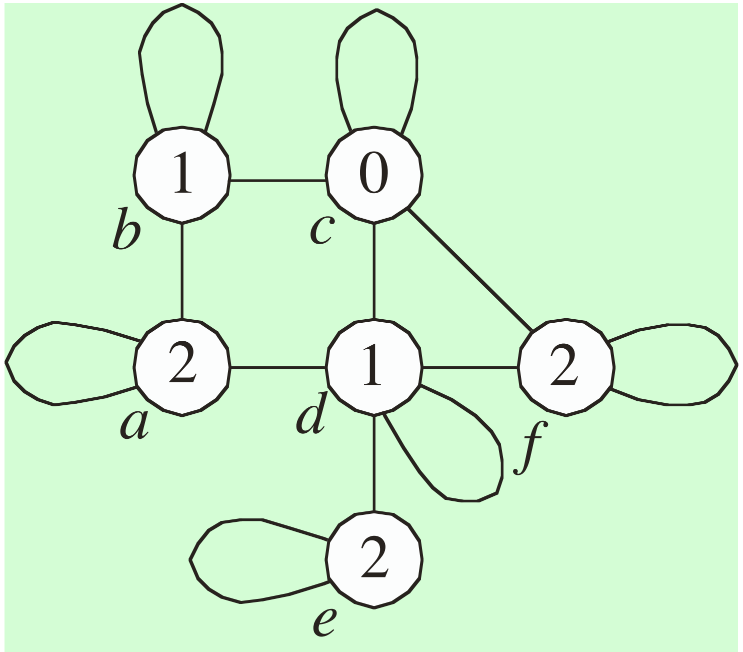
defining **classes** of i-nodes c by $dim(c) = i$

$ind(G)$ = maximum value of $dim(c)$

principal node c if $dim(c) = ind(G)$

all principal nodes = **core** of G

marginal node otherwise



$c_1 \, A_\mathrm{i} \, c_2$ iff $c_1 \neq c_2$

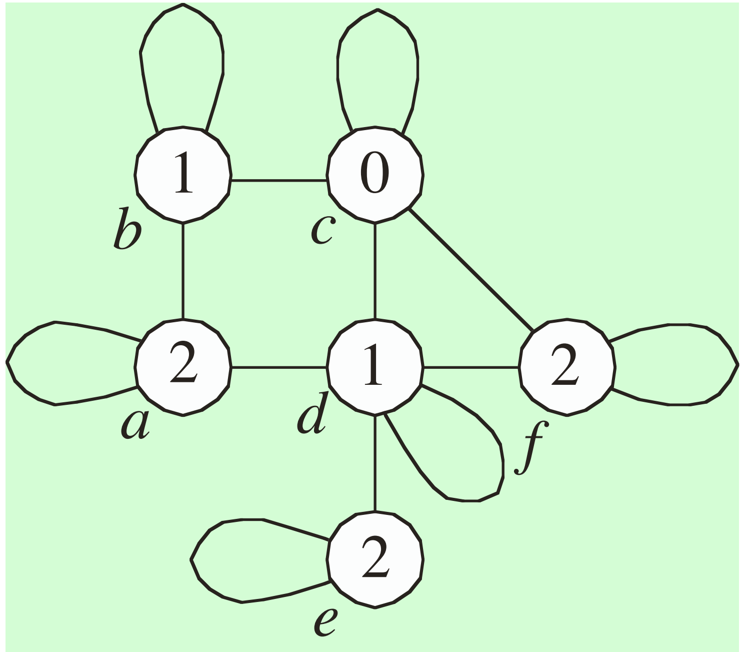
and ex. i-node c

$c_1 \square I(c) \square c \square I(c_2)$

i-adjacent, i-connected, i-path,

i-components, complementary i-components

adjacent iff ex. i and i-adjacent let $n = ind(G)$



$G = [S, I, dim]$ is

incidence pseudograph iff

I1: $I(c)$ always finite

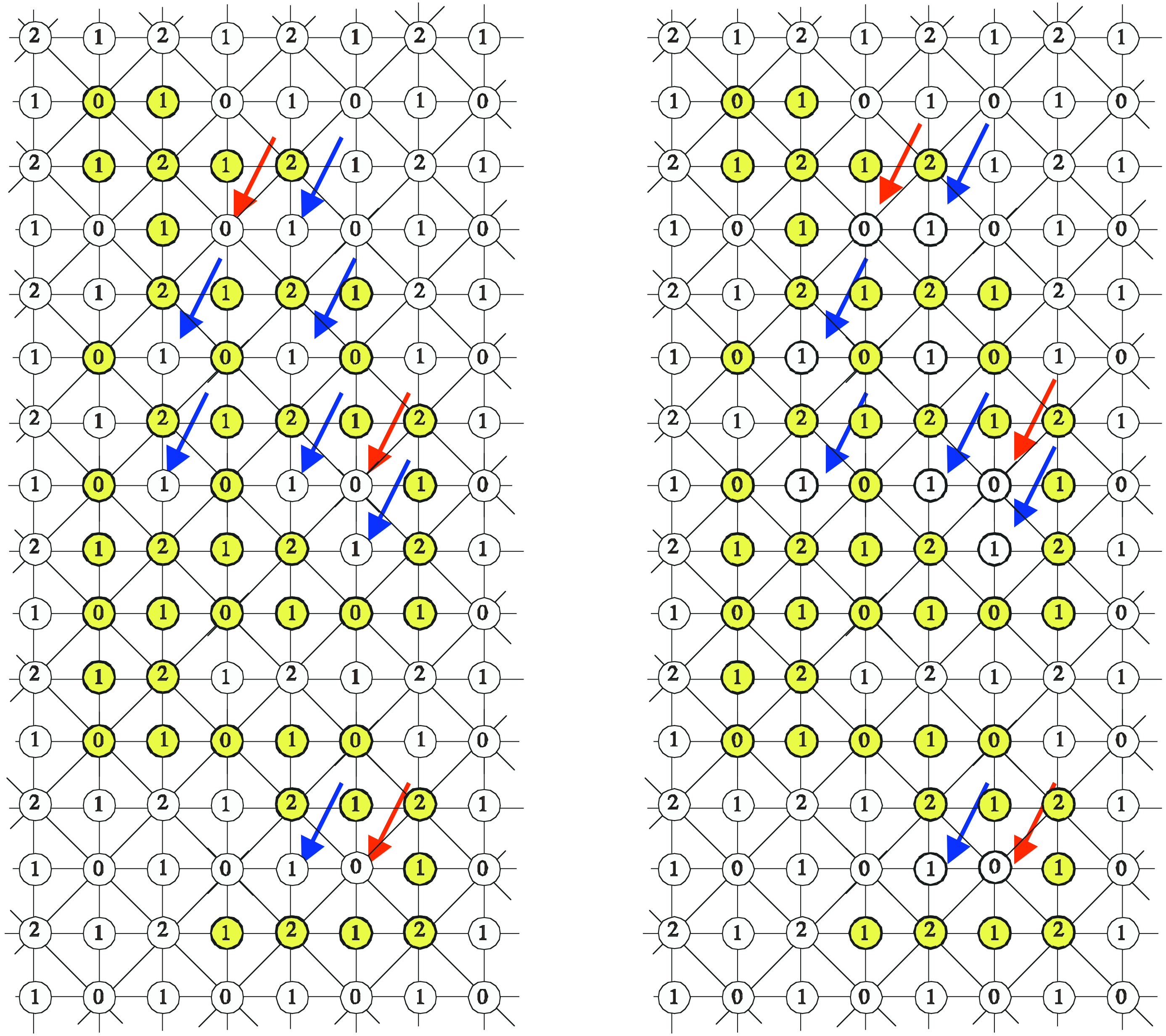
I2: set of principal nodes in G is $(n\text{-}1)$-connected

I3: $M \subseteq S$ finite: at most one infinite complementary $(n\text{-}1)$-component of principal node

I4: $c' \in I(c)$ and $c \neq c'$ then $dim(c) \neq dim(c')$

I5: $dim(c) < n$ then c incident with at least one n-node

incomplete components of 2-nodes

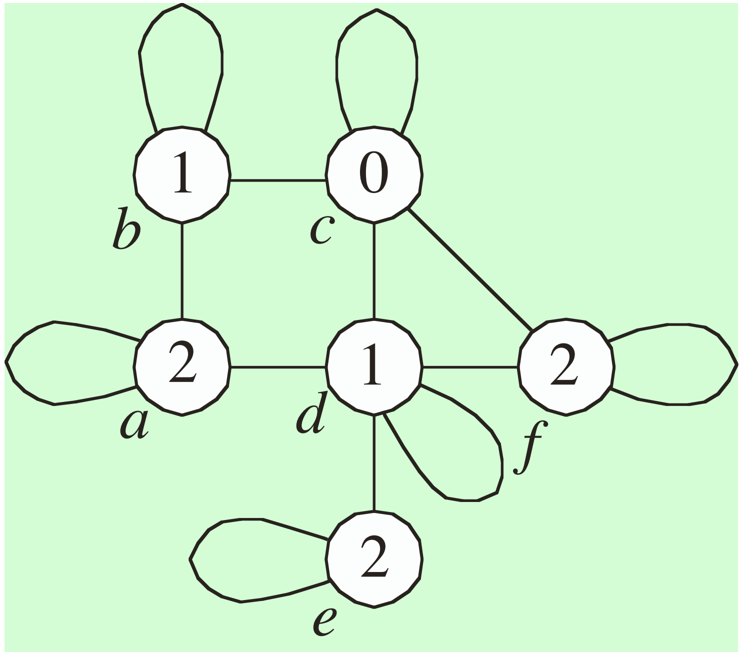


completion M^+ of M

(a) $M \subseteq M^+$

(b) if $c' \in M^+$ for all $c' \in I(c)$ with $dim(c') > dim(c)$

then $c \in M^+$

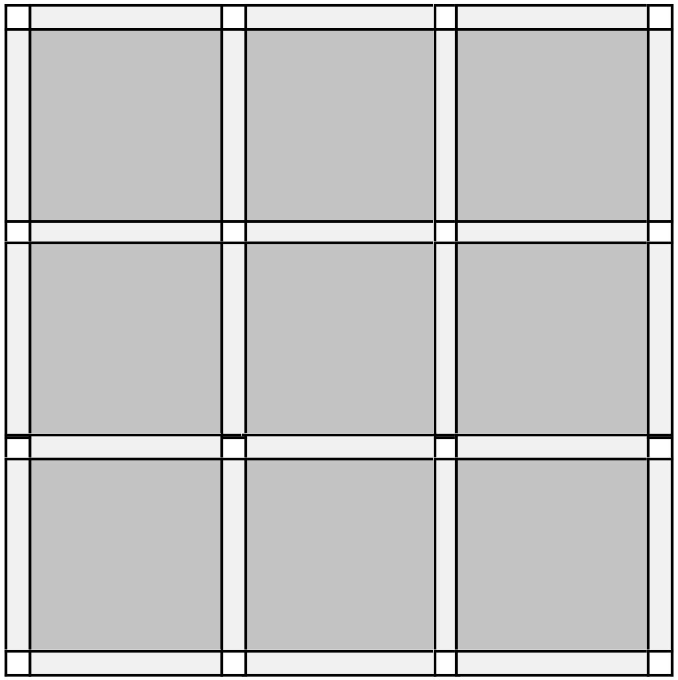


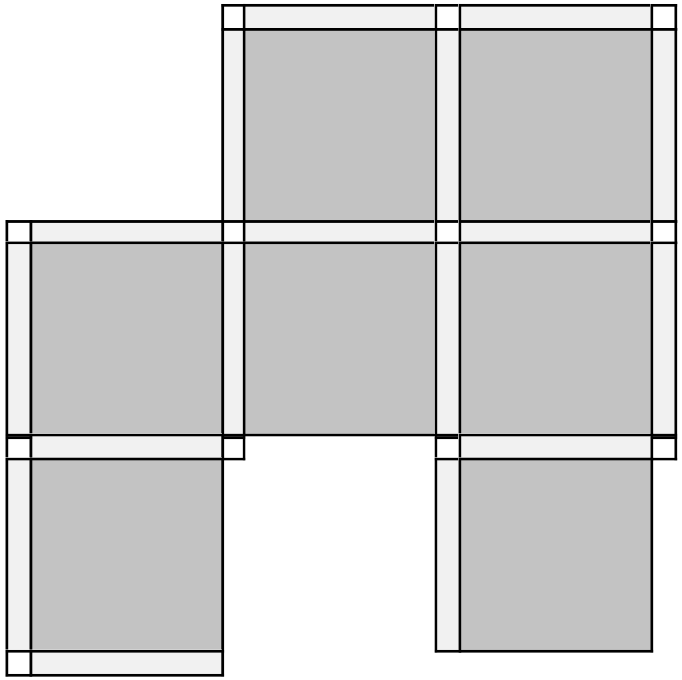
M **complete** iff $M = M^+$

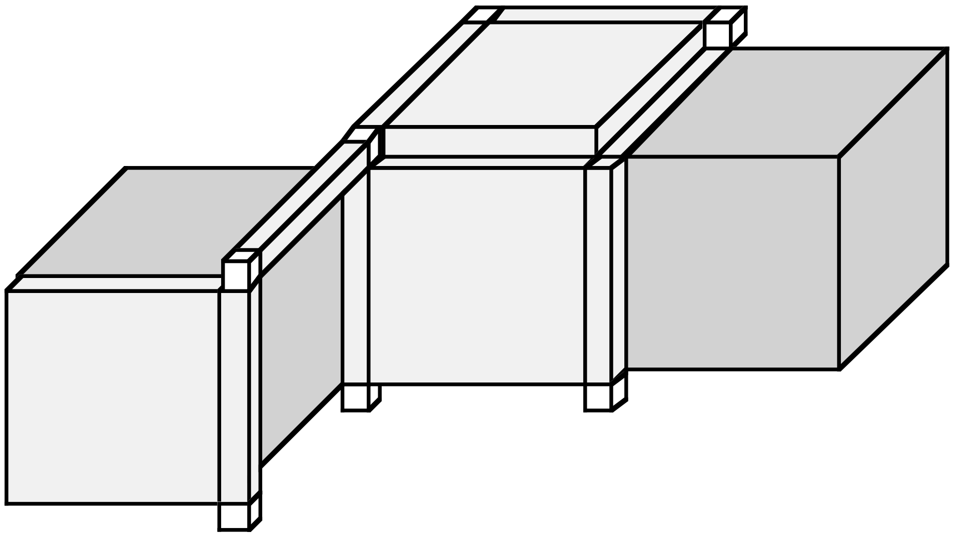
component C of set M

= nonempty core, core connected, C is complete

every i-node in C is incident with principal node in C







region M = finite component

$c \in M$ **inner node** iff $I(c) \subseteq M$

otherwise: **border node**

inner set M° and **border** ∂M

POSET TOPOLOGY

$M \subseteq S$ **closed**

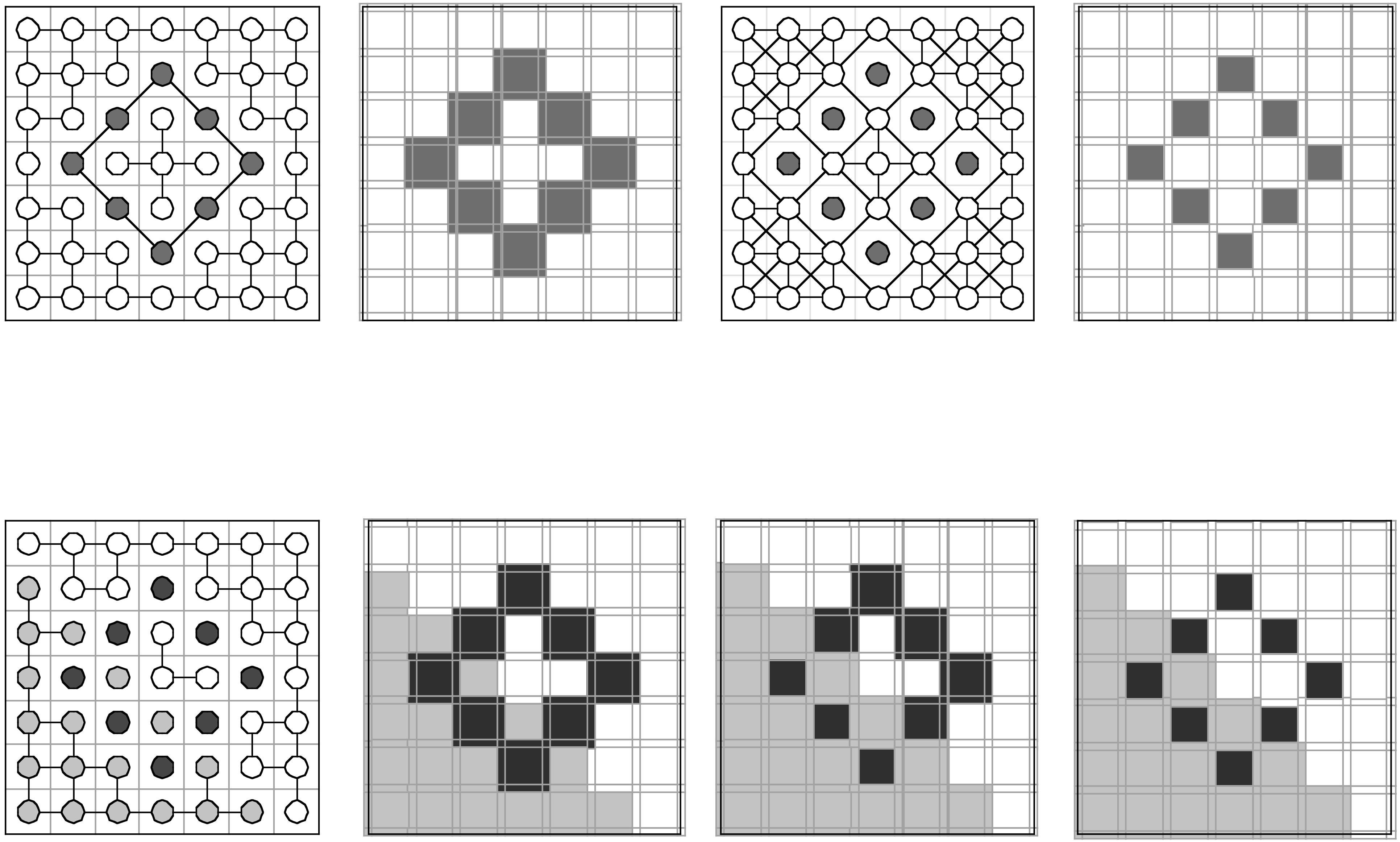
$c \in M$ and $c' \in I(c)$ with $dim(c') < dim(c)$

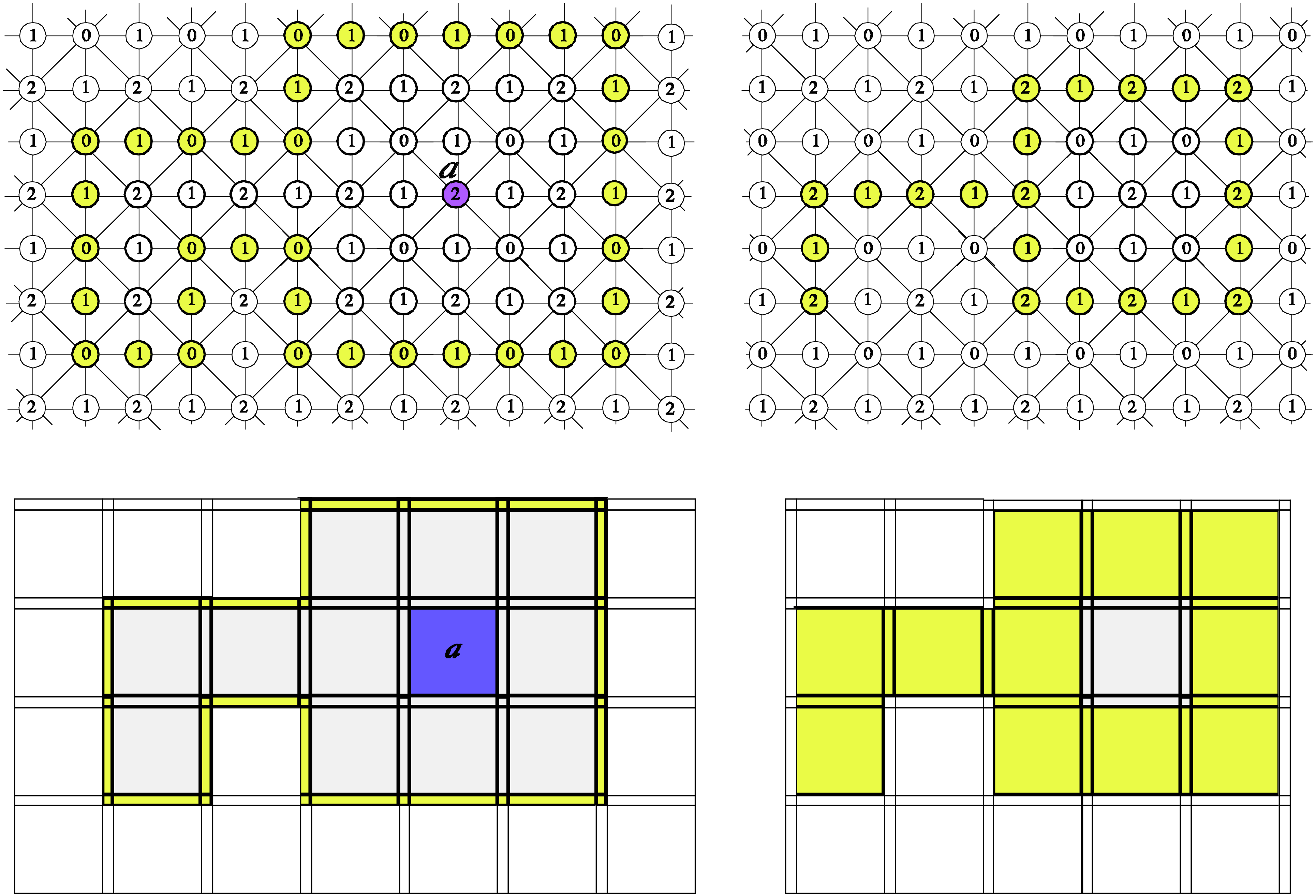
then $c' \in M$

$M \subseteq S$ **open** iff $\overline{M} = S \setminus M$ is closed

(note: if closed or open, then complete)

frontier of a set = border of its closure




a

incidence counts

$$a_{ij}(c) = \begin{cases} card\{c' \in S : dim(c') = j \wedge \{c, c'\} \in I\} \\ \qquad\qquad \text{if } i = dim(c) \\ 0 \qquad \text{otherwise} \end{cases}$$

MATCHING THEOREM

$$\sum_{c \in S} a_{ij}(c) = \sum_{c \in S} a_{ji}(c) \quad \text{for} \quad 0 \le i, j \le n$$

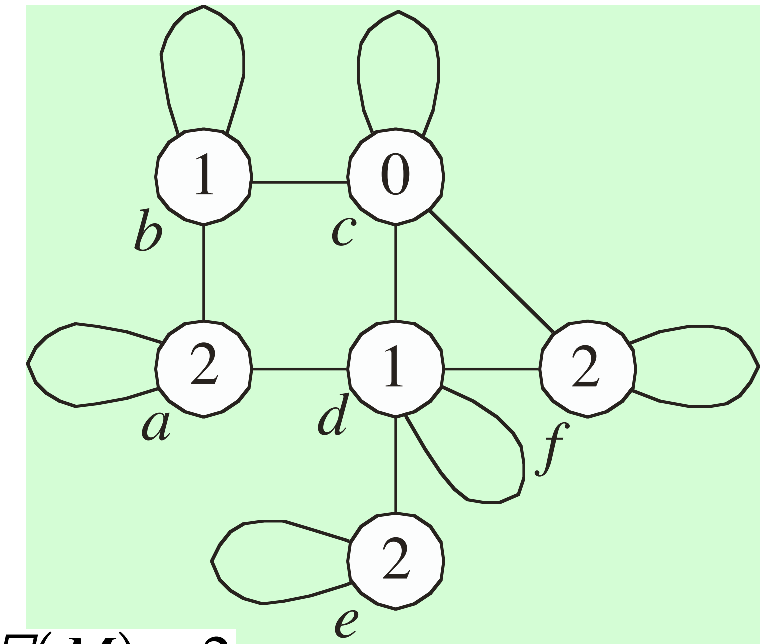
incidence grid (IG) = regular incidence pseudograph

$a_{ij}(c) = a_{ij}$ for all $c \in S$, with $dim(c) = i$

class cardinalities of a finite subset M

$$\alpha_i^M = card\{c : c \in M \wedge dim(c) = i\}$$

from Matching Theorem: $\alpha_i a_{ik} - \alpha_k a_{ki} = 0$ for IG S, $0 \le i \le n$
(and any $0 \le k \le n$)



$\chi(M) = 2$

Euler characteristic

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \alpha_i$$

from Matching Theorem:

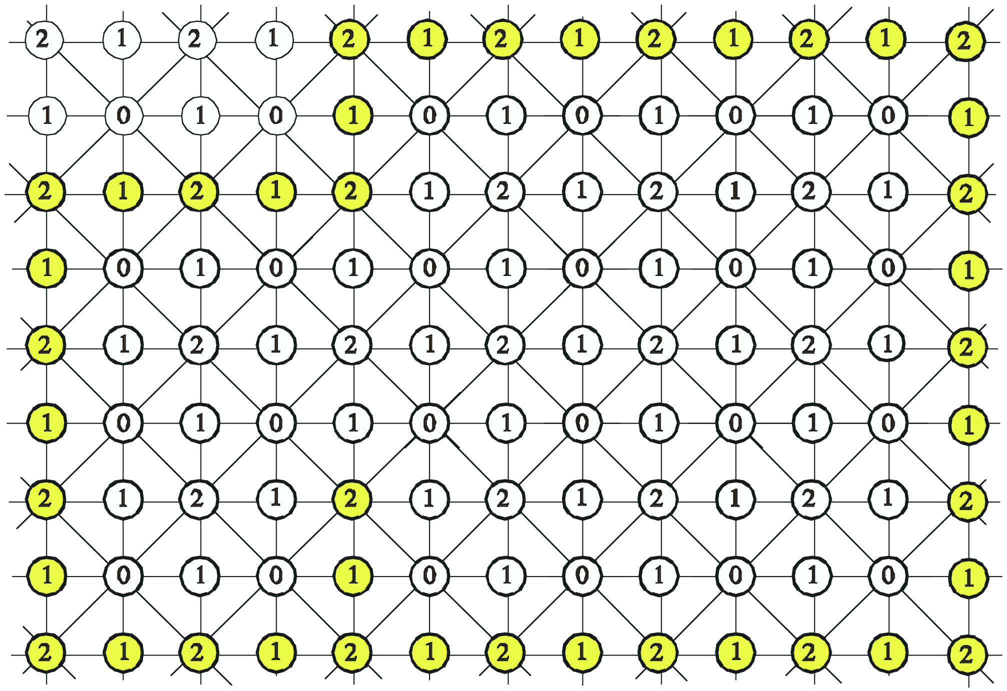
$$\frac{\chi(M)}{\alpha_k} = \sum_{i=0}^{n} (-1)^i \frac{a_{ki}}{a_{ik}}$$

for finite n-dim. IGs

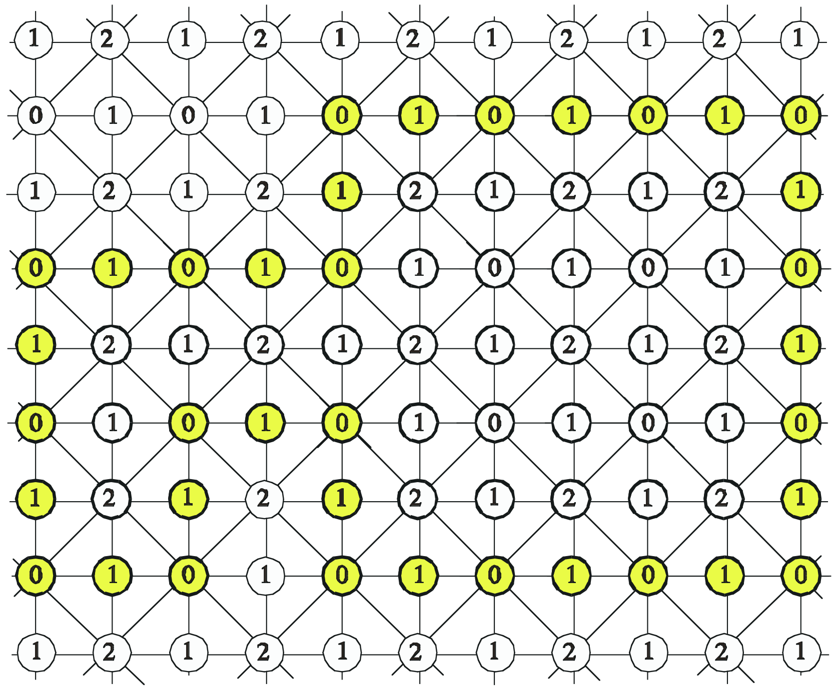
a node $c \in S$ is **invalid** w.r.t. M iff

$c \notin M$ but there is $c' \in M$ with $c' \in I(c)$

boundary of M = set of all invalid nodes



boundary of a closed region
= its border (frontier)



boundary of an open region
= border (frontier) of its closure

boundary counts

$$b_{ij}^{M}(c) = \begin{cases} card\{c' \in I(c) : dim(c') = j \wedge c' \; invalid\} & \text{if } i = dim(c) \text{ and } c \in M \\ 0 & \text{otherwise} \end{cases}$$

total boundary counts for M

$$b_{ij}^{M} = \sum_{c \in S} b_{ij}^{M}(c)$$

n-dimensional IG's

$$a_{ij} = \begin{cases} 2^{j-i}\binom{n-i}{n-j} & \text{if } i < j \\ 1 & \text{if } i = j \\ 2^{i-j}\binom{i}{j} & \text{if } i > j \end{cases}$$

B. Rosenfeld, I. Jaglom 1971
R. Klette 1972

Corollary: $$\sum_{j=0}^{n} (-1)^j \frac{a_{ij}}{a_{ji}} = 0$$

REGION MATCHING THEOREM

M open or closed region in n-dim. IG

$\Sigma_i\, a_{ij} - b_{ij} = \Sigma_j\, a_{ji}$ for $i < j$ if closed, or for $i > j$ if open
$\Sigma_i\, a_{ij} = \Sigma_j\, a_{ji}$ for $i = j$
$\Sigma_i\, a_{ij} + b_{ji} = \Sigma_j\, a_{ji}$ for $i > j$ if closed, or for $i < j$ if open

K. Voss 1993 for open regions

Let M be a finite union of pairwise disjoint closed (or pairwise disjoint open) regions.

The Euler characteristic of M is

$$\chi(M) = \frac{1}{2n} \cdot \sum_{i=1}^{n} (-1)^{i+1} b_{i,i-1}$$ for open regions

and

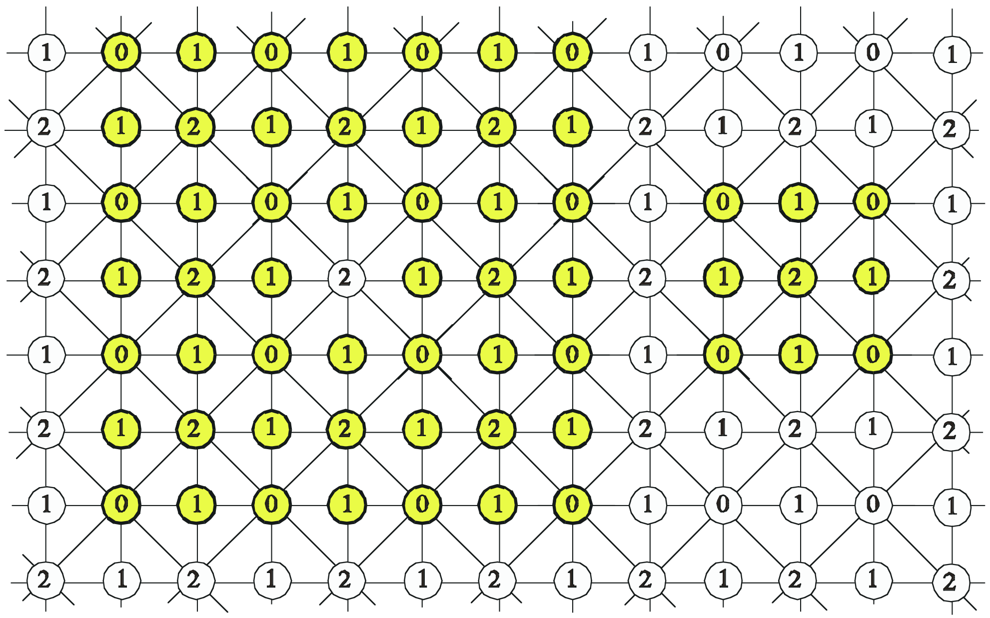
$$\chi(M) = \frac{1}{2n} \cdot \sum_{i=0}^{n-1} (-1)^{i+1} b_{i,i+1}$$ for closed regions.

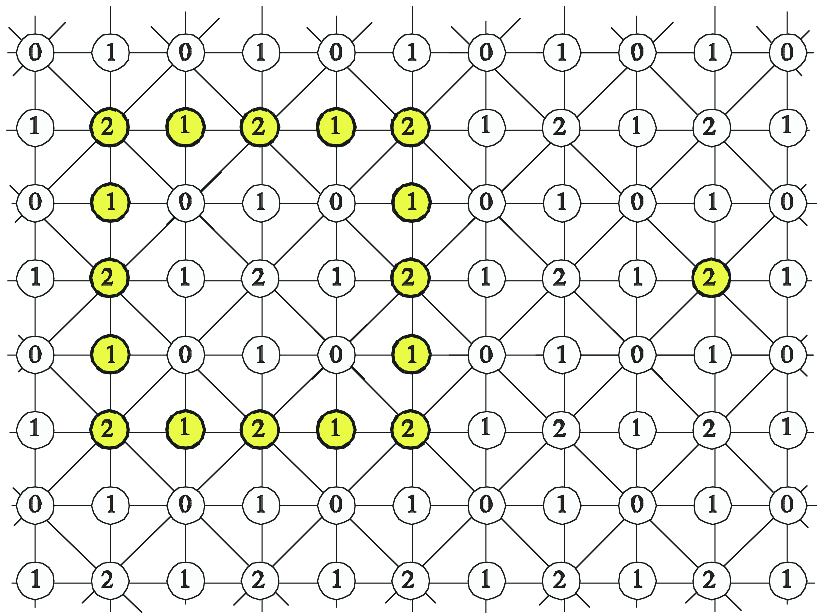
K. Voss 1993 for open regions

Region Matching Theorem allows to replace boundary counts by class cardinalities and (globally known) incidence counts:

$$b_{i,i\square 1} = \square_{i\square 1} a_{i\square 1,i} \square\square_i a_{i,i\square 1}$$ for open regions

$$b_{i,i+1} = \square_i a_{i,i+1} \square\square_{i+1} a_{i+1,i}$$ for closed regions





	grid point model	grid cell model
elements of adjacency structures		
elements of incidence structures