Digital Topology for Multi-Level Images

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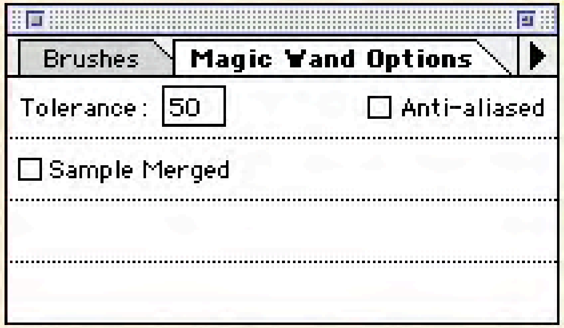
basic models/theories in digital topology:

- good pairs for binary images, 4-, 6-, 8-neighborhoods, ...
- adjacency graph models, neighborhood structures, ...
- poset topology, Khalimsky plane, Kovalevsky plane, ...
- digital spaces, topological digital spaces (Herman)
- inter-pixel boundaries, half-integer grid, ...
- oriented adjacency graphs, combinatorial maps, ...
- theory of n-dimensional cell complexes, ...
- combinatorial topology, ...
- ...

Adobe Photoshop™

(4 millions of legal installations)



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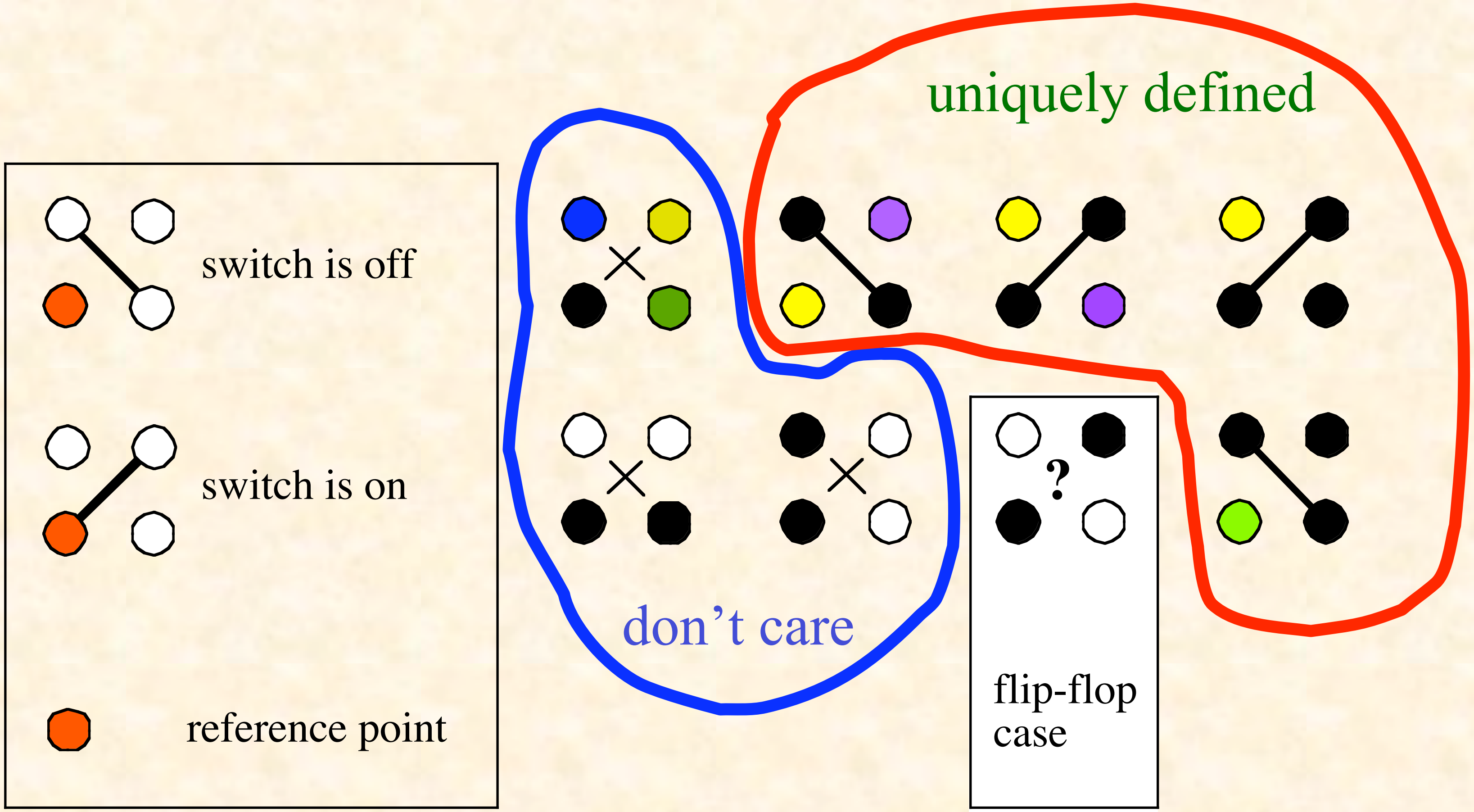
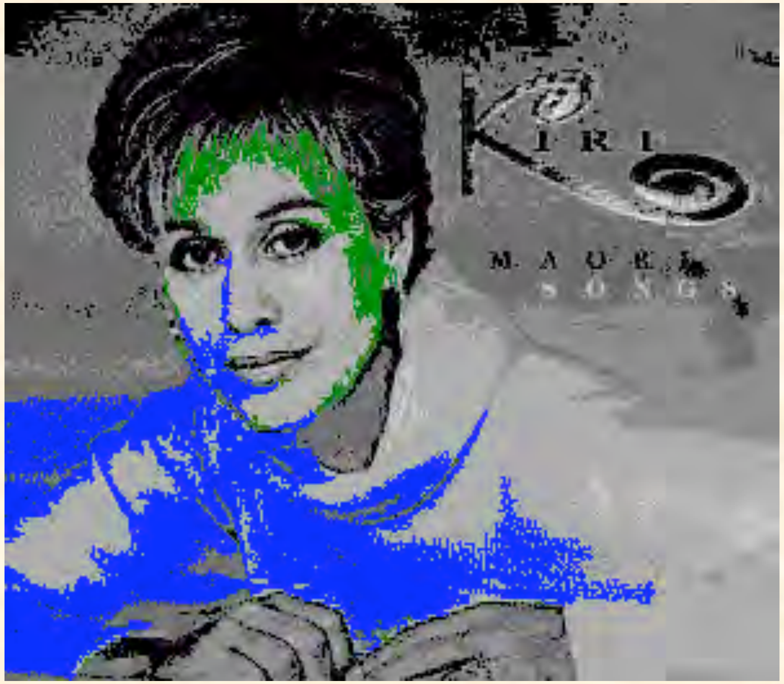
what they use in the non-anti-aliased case:

adjacency = 4-adjacency only
pixel = labeled grid square
object-related segmentation >>> post-processing

Switches-approach



see, e.g., strongly normal digital picture spaces, GADSs



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(i) a 4-connected region (no switches)

(ii) expanded by uniquely defined switches (no don't cares, no flip-flops)

(iii) and finally with all switches, including flip-flops

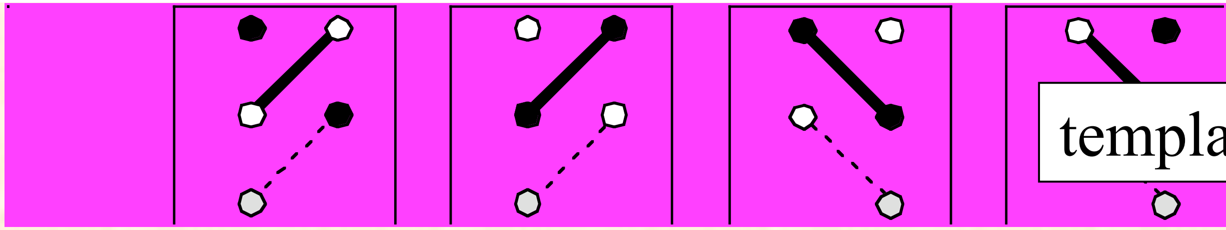

0.5 % flip-flops


KIRI
MAORI
SONGS
0.38 % flip-flops

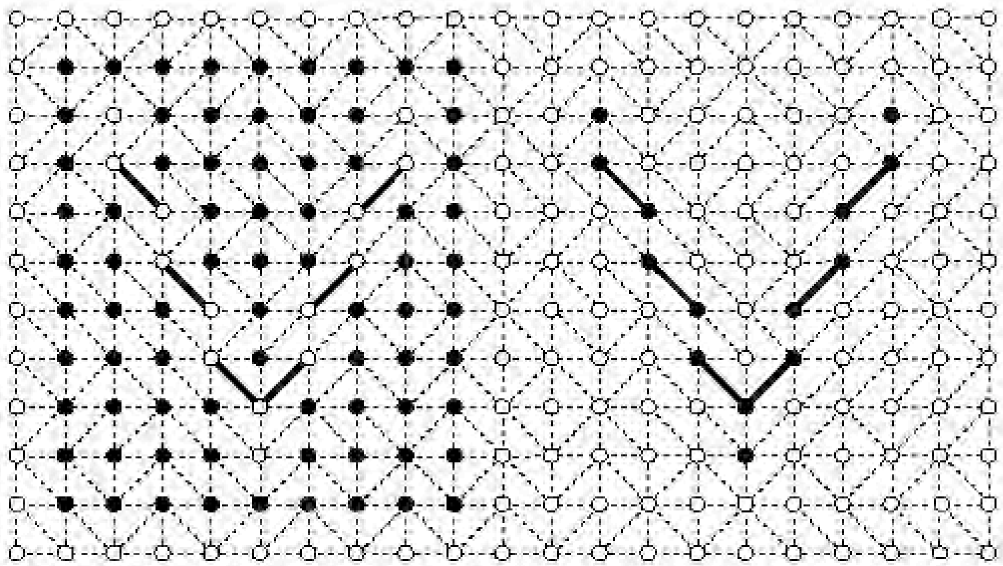

0.38 % flip-flops

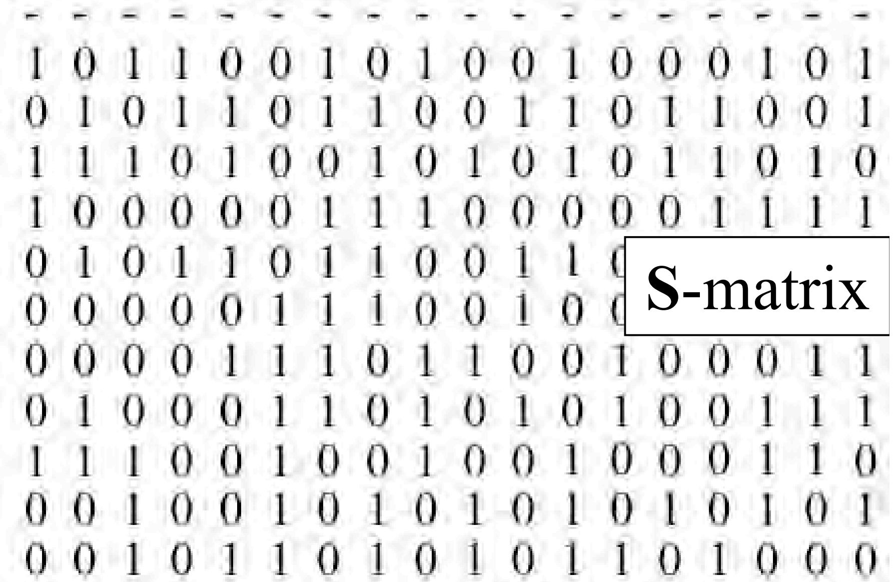

0.22 % flip-flops

Valid adjacencies are between adjacent grid points which are labeled by identical image values.

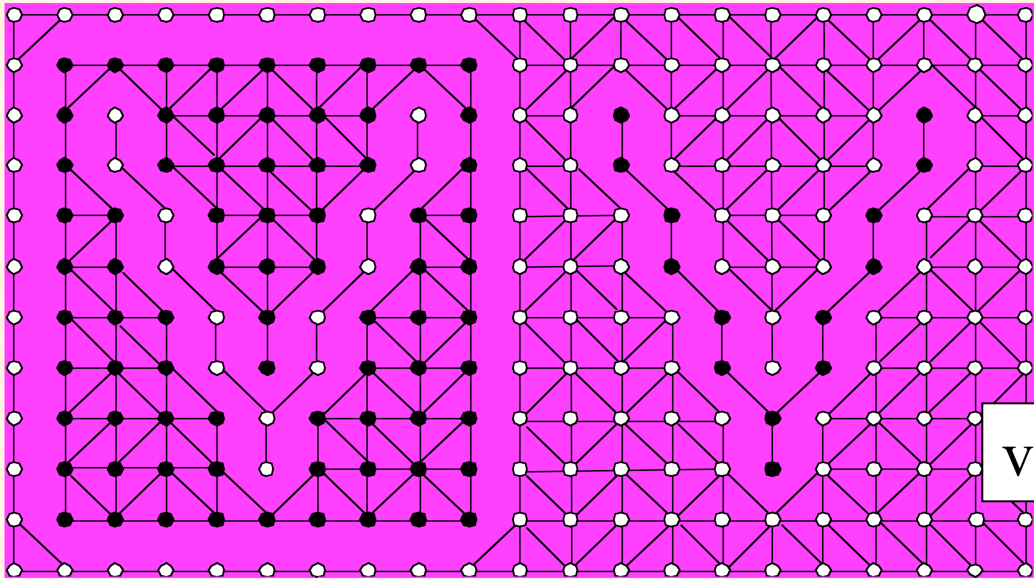


templates for flip-flops





S-matrix



valid adjacencies

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OPTION 1: SWITCH-APPROACH

1. **templates for flip-flops**

2. **S-matrix (constant, or using templates)**

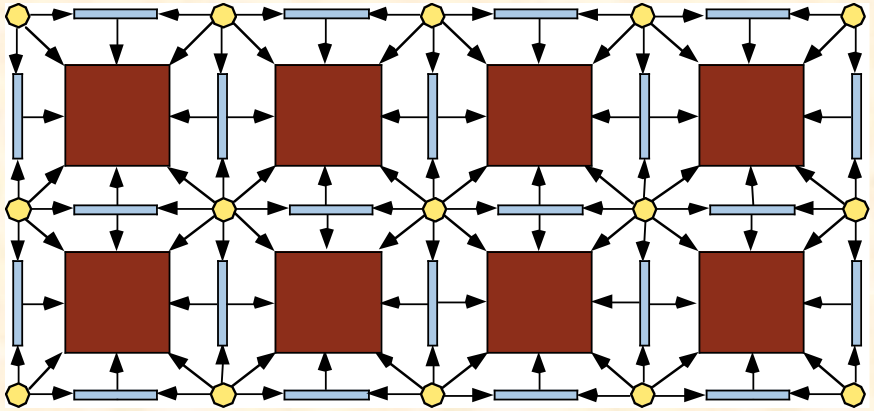
3. **remaining valid adjacencies**

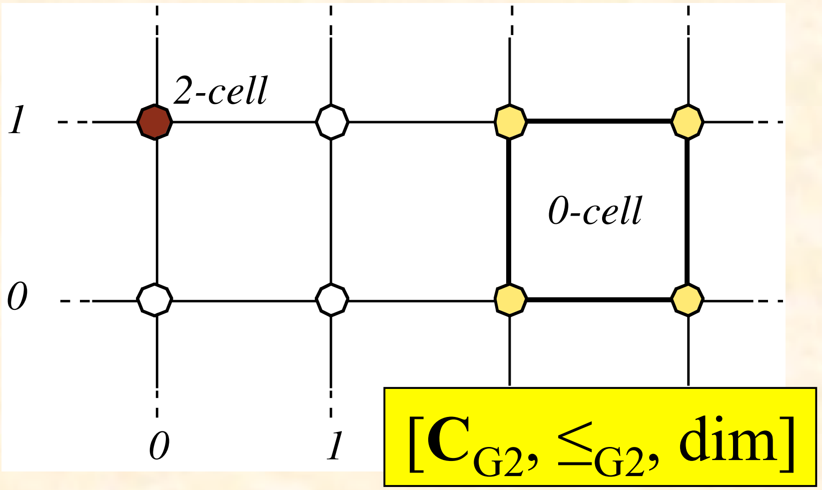
Aleksandrov-Hopf 1935

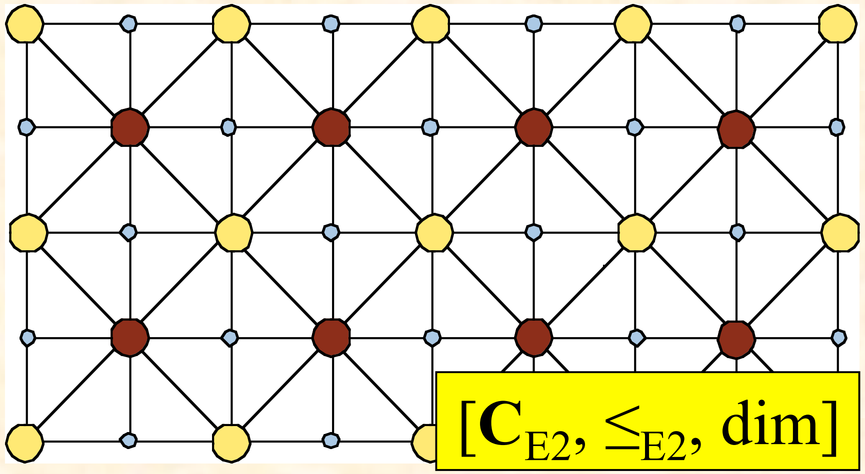
Khalimsky 1986

Kovalevsky 1989

• • •







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Φ : mapping of topological space $\mathbf{C}_1$ into topological space $\mathbf{C}_2$

Φ is **continuous** iff

$\Phi^{-1}(M) = \{p \text{ in } \mathbf{C}_1 : \Phi(p) \text{ in } M\}$ is open in $\mathbf{C}_1$,

for any open subset M of $\mathbf{C}_2$

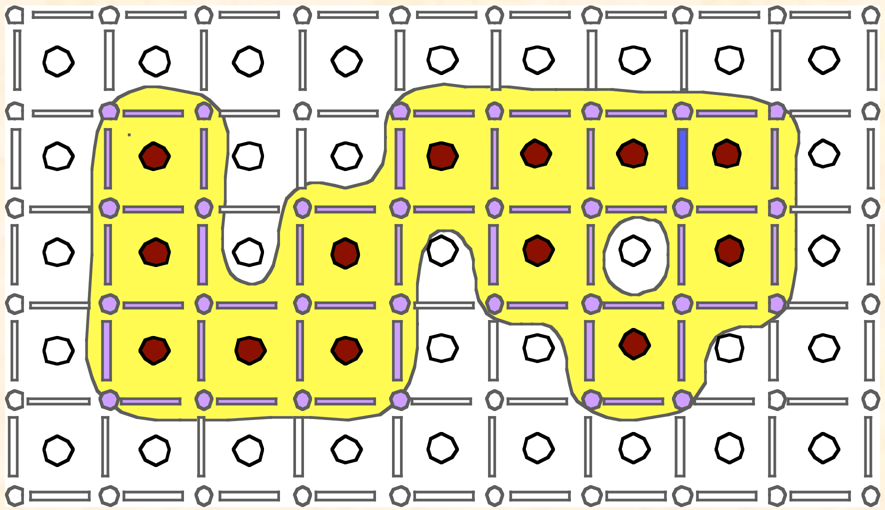
(H. Poincare, 1895)

Φ is a **homeomorphism**

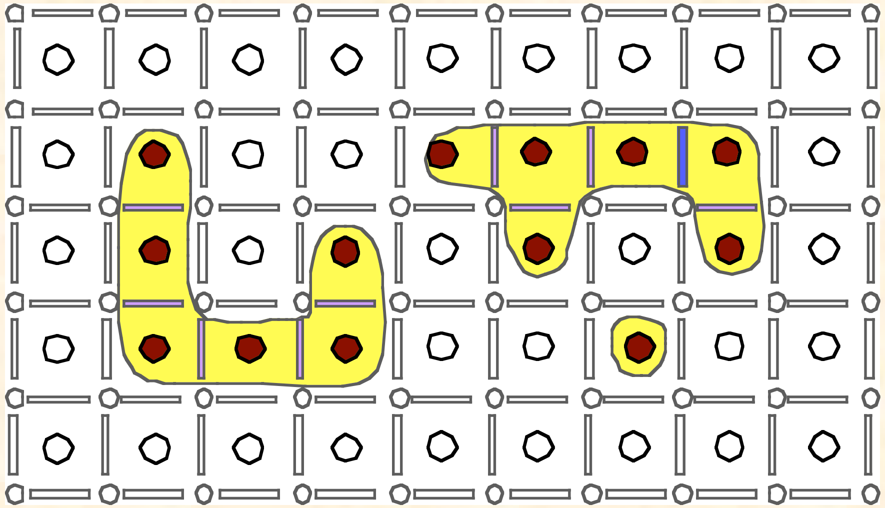
iff it is one-one, onto $\mathbf{C}_2$, continuous, and Φ^{-1} is continuous as well.

1. complexes $[\mathbf{C}_{G2}, \leq_{G2}, \dim]$ and $[\mathbf{C}_{E2}, \leq_{E2}, \dim]$ are isomorphic
2. Khalimsky and Kovalevsky plane are homeomorphic

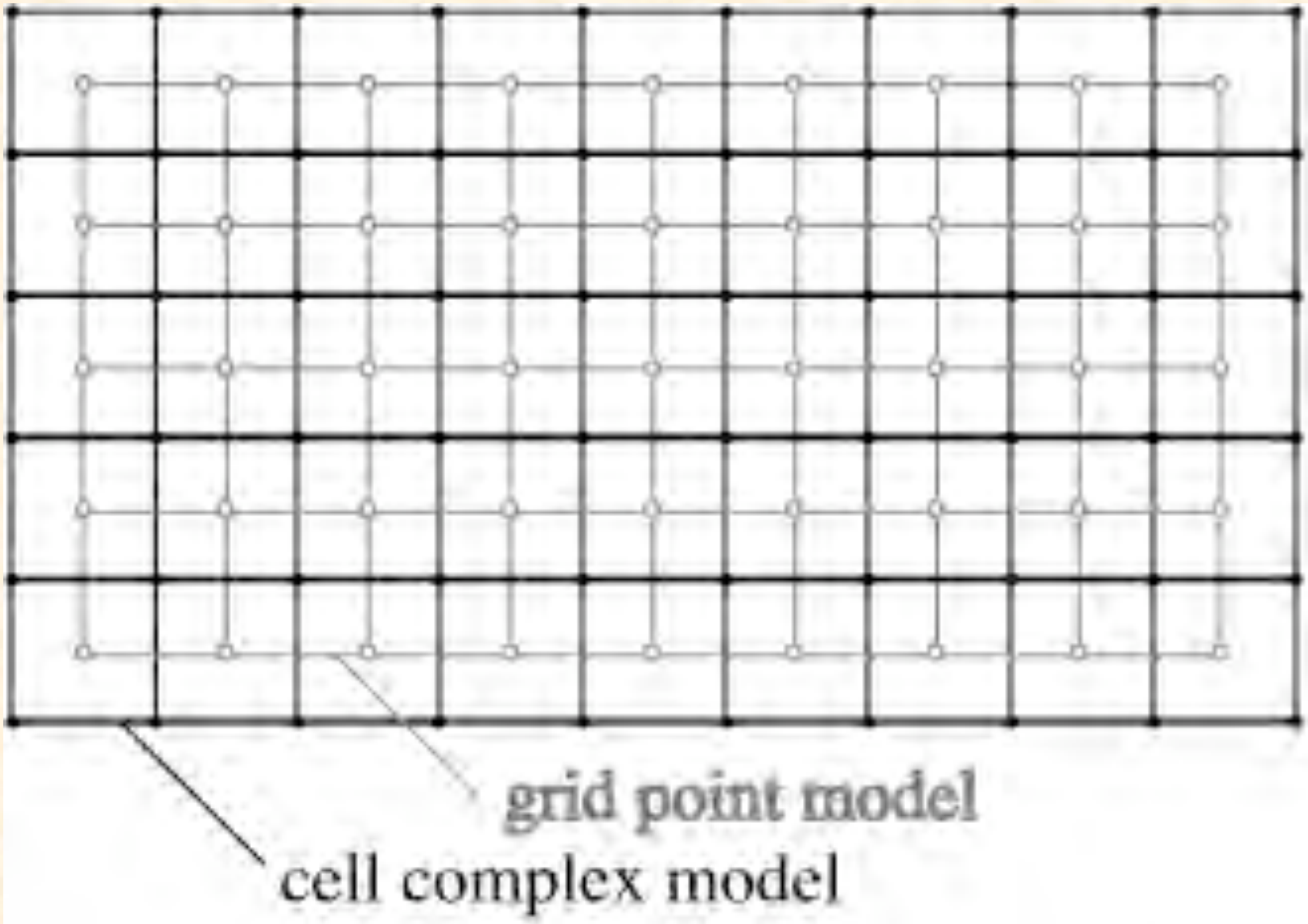
Topological spaces of isomorphic posets are homeomorphic.



closed



open

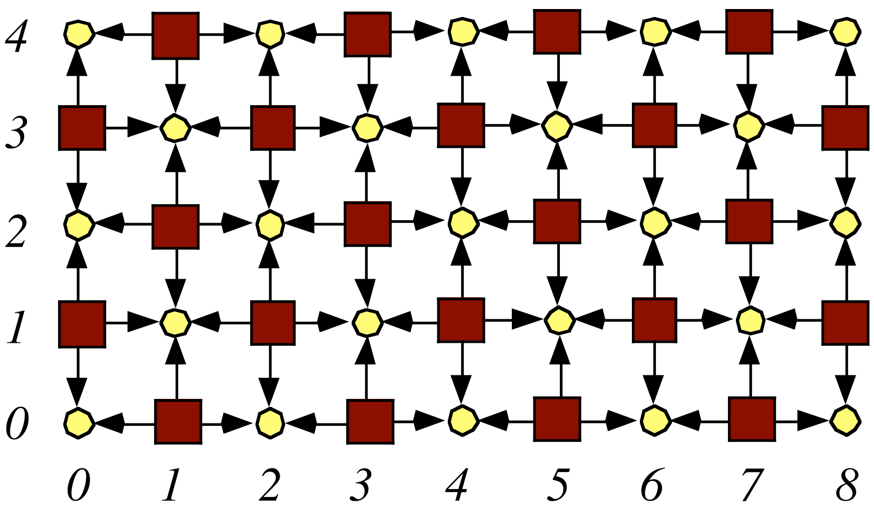

grid point model
cell complex model

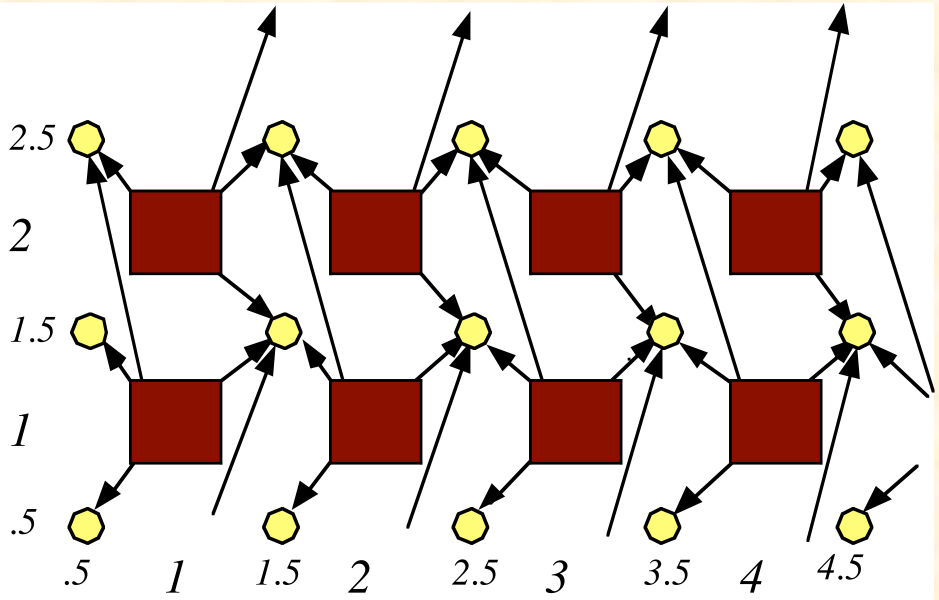
OPTION 2: POSET-APPROACH

1. specify poset topology

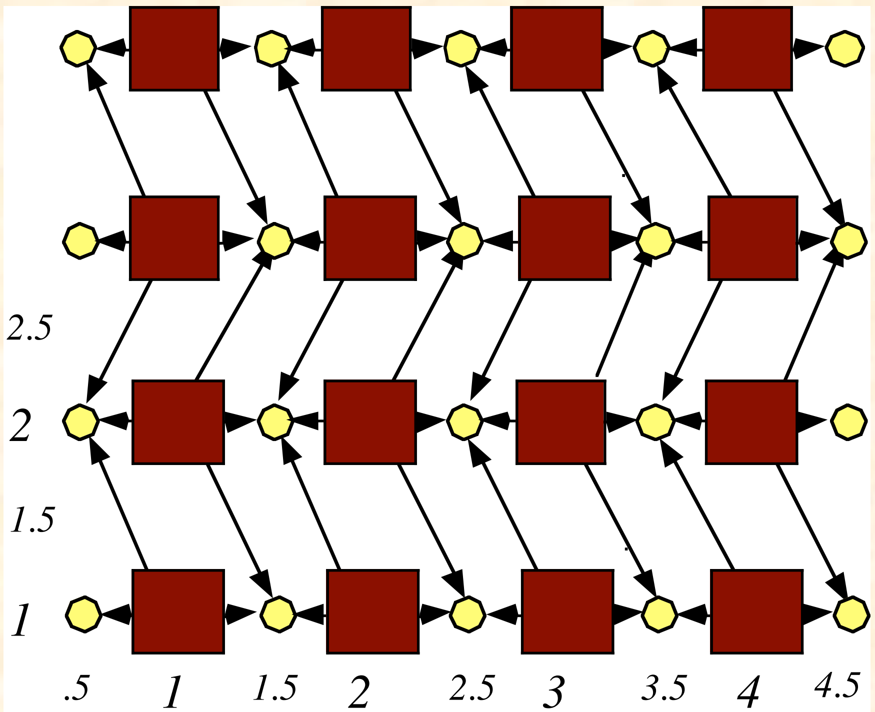
2. define data structure and
   design cell-structure algorithms

3. regions defined as in Euclidean topology





Wyse et al. 1970



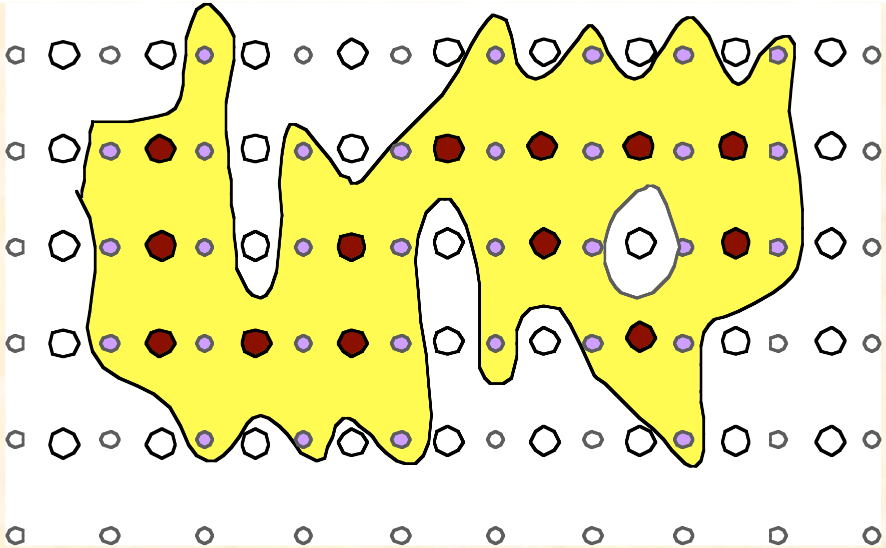
OPTION 3: 4-TOPOLOGY

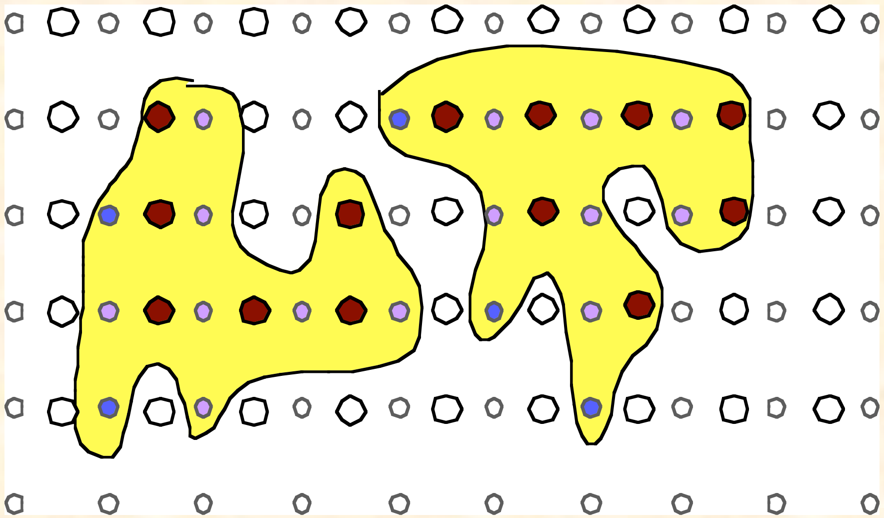
1. map all odd grid points into pixel positions

2. define data structure and
   design 4-topology algorithms

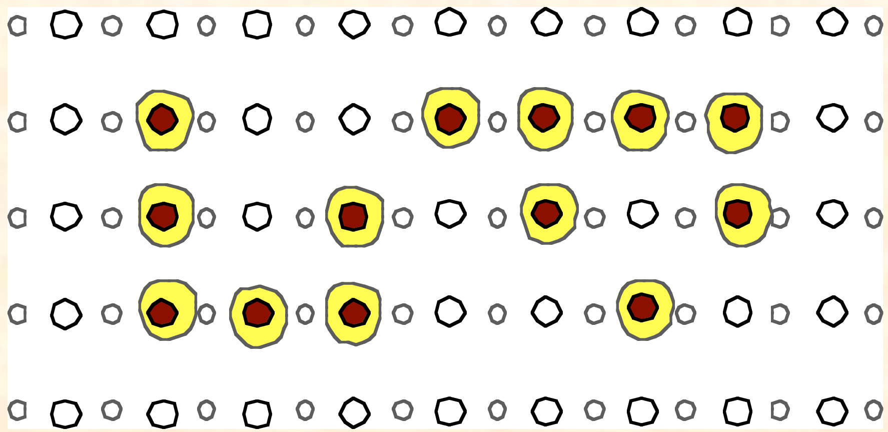
3. regions defined as in Euclidean topology

closed





neither closed nor open



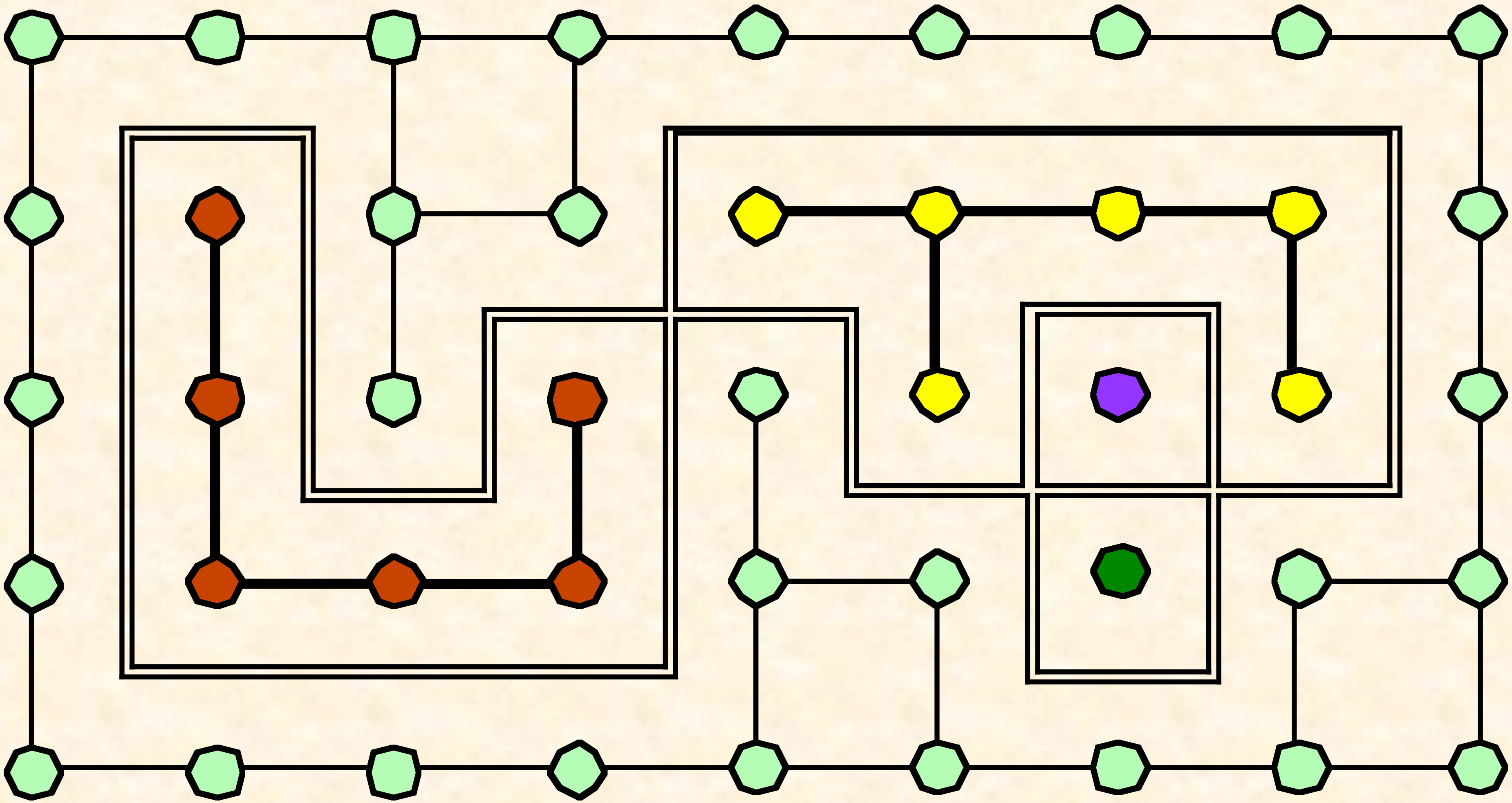
open

OPTION 4: oriented 4-adjacency

1. stay (in 2D) with 4-adjacency model **ONLY**

2. use local orientation cycles at pixel positions (similar to Freeman codes)

3. regions defined as in Clifford's marble example



Theory of oriented adjacency graphs

Klaus Voss, Reinhard Klette, Peter Hufnagl, Albrecht Hübler, ...
1985 ...

in German: in journal **Bild und Ton** (in existence till 1992 only)

book by Voss in 1988 (Akademie-Verlag, Berlin)

in English: TR (Klette/Voss) at CfAR, College Park, in 1987
published 1991 in
Pattern Recognition and Image Analysis

book by Voss in 1993 (Springer, Berlin)

recently: CITR-TR-101 (Klette, Oct. 2001) set of points **C**, adjacency relation A (irreflexive, symmetric)

neighborhood relation N (reflexive)

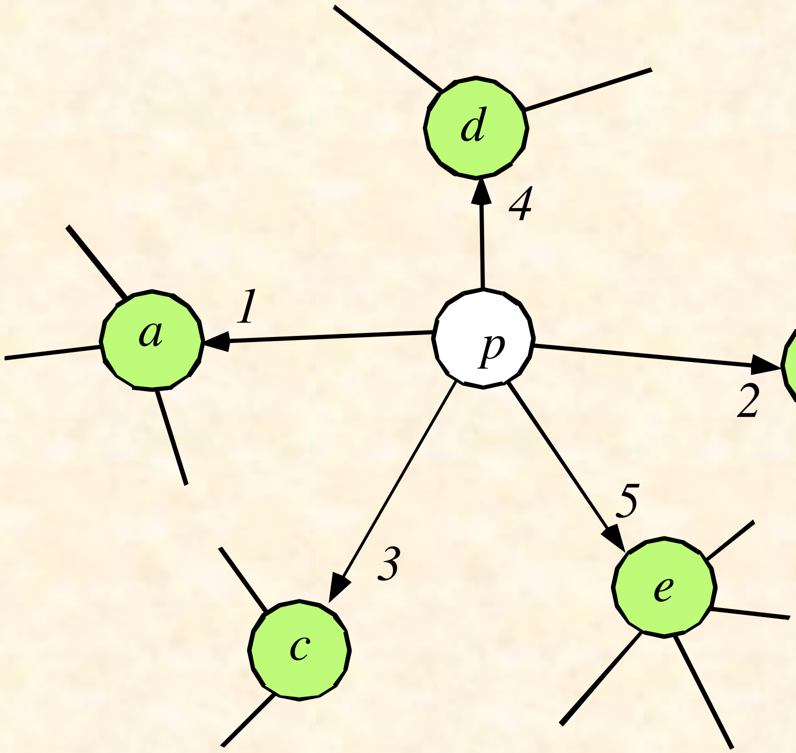
oriented adjacency graphs [**C**, A, ξ]

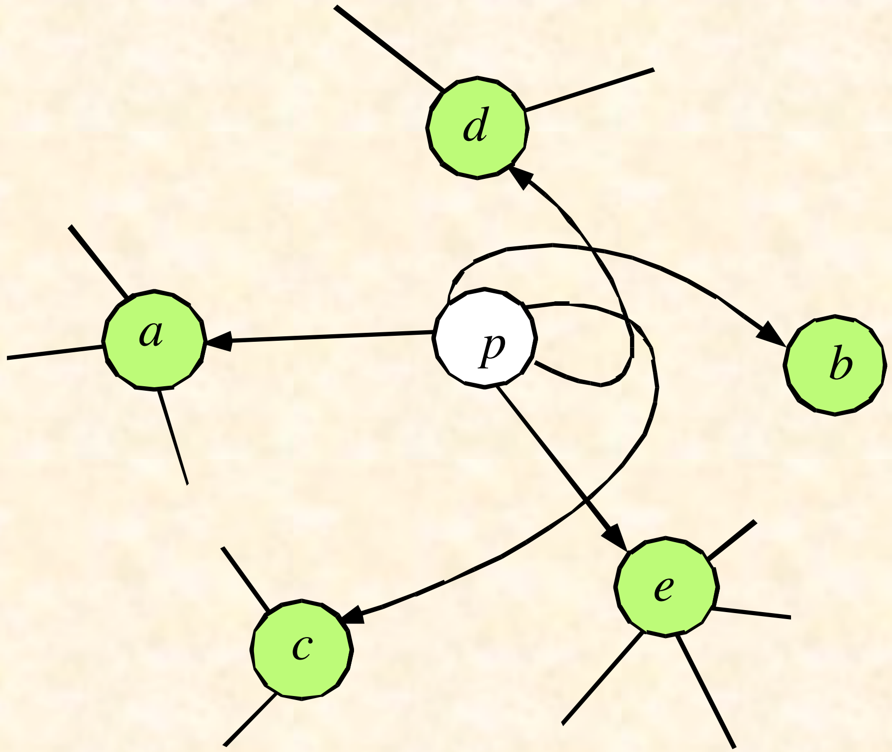
axioms :

- A(p) is finite for any p in **C.**
- [**C**, A] is a connected undirected graph.
- Any finite subset M of **C** possesses at most one infinite complementary component.
- Any directed edge generates a periodic path with respect to ξ .

generalization of oriented tilings or combinatorial maps

local circular order $\xi(p) = [a, b, c, d, e]$ of all points in the adjacency set $A(p)$





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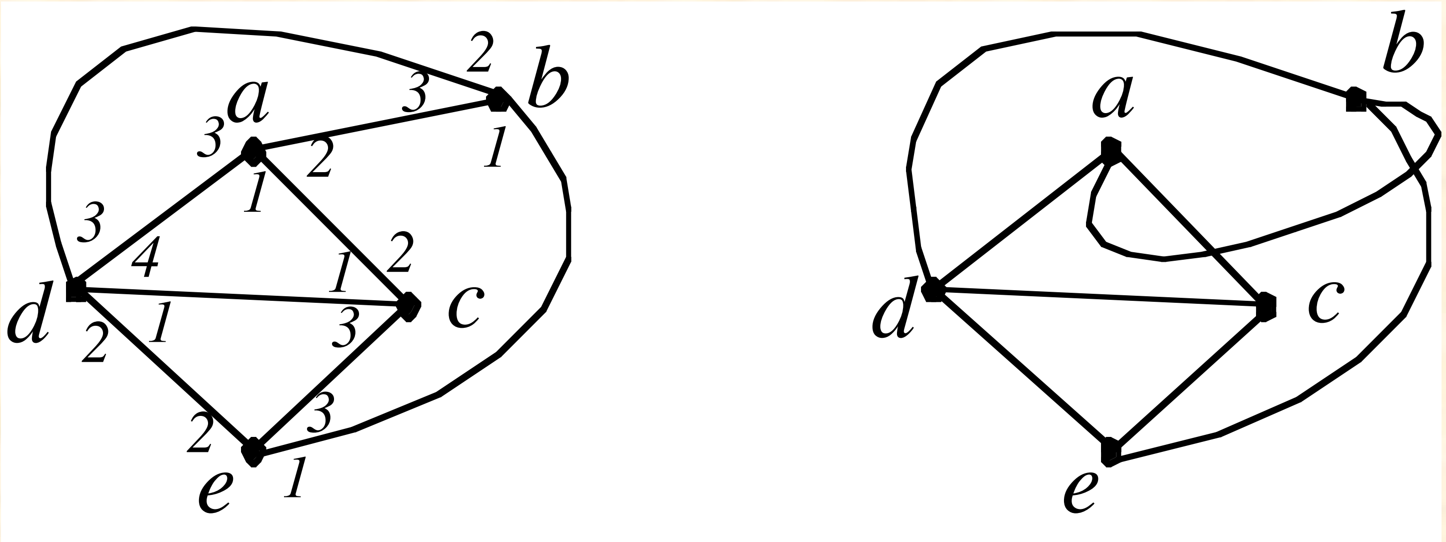
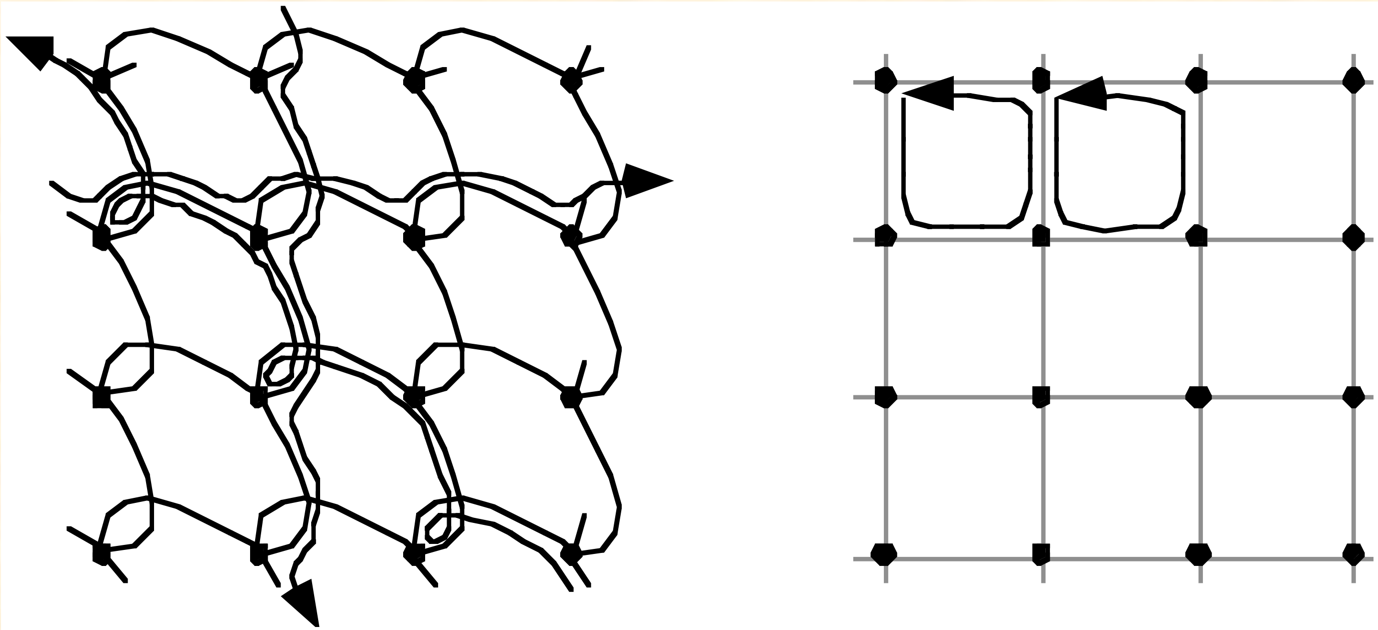


the undirected graph needs not to be planar

- LEFT: numberings of local circular orders
- RIGHT: drawing convention: clockwise order of outgoing edges

$\xi(a) = [c, b, d]$ $\xi(b) = [e, d, a]$ $\xi(c) = [d, a, e]$
$\xi(d) = [c, e, b, a]$ $\xi(e) = [b, d, c]$

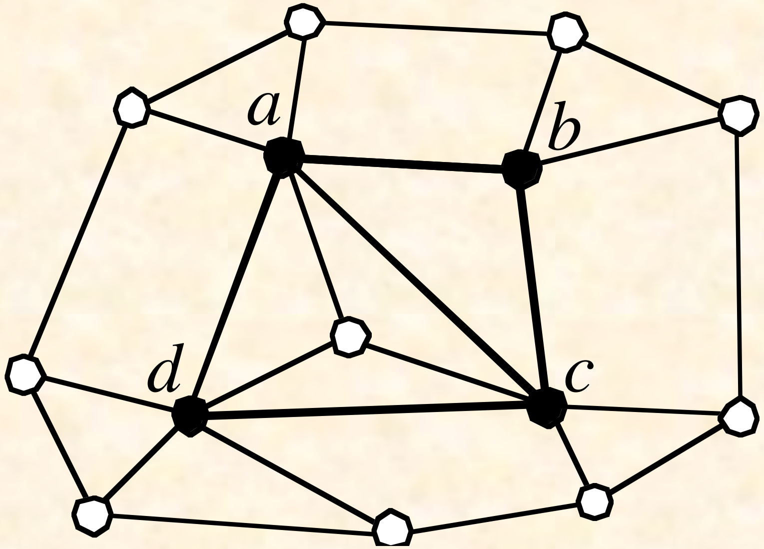
directed edge (d,a) generates circuit $\xi(d,a) = \langle d, a, c, e, b \rangle$
$\xi(a,d) = \langle a, d, c \rangle$, ...



not an oriented adjacency graph (infinite paths)

cycle = generated circuit

$M \subseteq \mathbf{C}$ generates **restricted local circular orders** $\xi_M(a) = [b,c,d]$



$(a,b),(b,c),(c,d),(d,a)$

$(b,a),(a,c),(c,b)$

$(d,c),(c,a),(a,d)$

<b,a,c> is cycle in [**C**, A, ξ] : **original** or **atomic cycle**

<a,b,c,d> and <d,c,a> are not cycles in [**C**, A, ξ] : **border cycles**

oriented adjacency graph [**C**, A, ξ]: $\sum_{p \in \mathbf{C}} \nu(p) = 2\alpha_1$ $\sum_{g} \lambda(g) = 2\alpha_1$

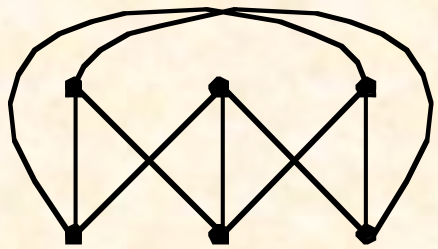
α_0 = card (**C**)

α_1 = card (A)

$\nu(p)$ = card(A(p))

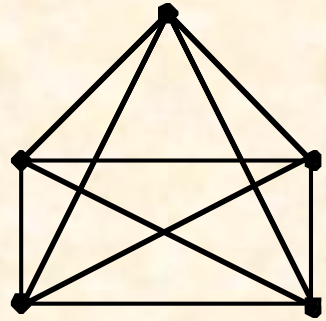
$\lambda(g)$ = length of cycle g

α_2 = # cycles



$\alpha_0 = 6$
$\alpha_1 = 9$
$\alpha_2 = 3$

$\chi = 0$

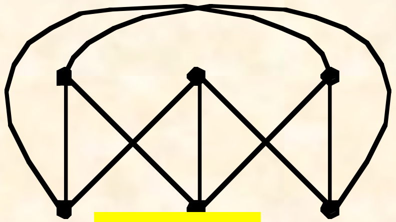


$\alpha_0 = 5$
$\alpha_1 = 10$
$\alpha_2 = 3$

$\chi = -2$

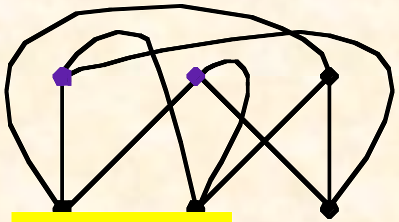
Euler characteristic $\chi = \alpha_0 - \alpha_1 + \alpha_2$

000-000



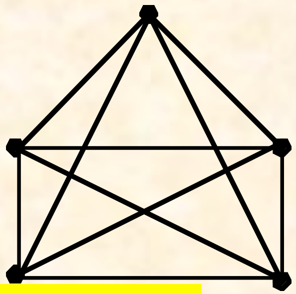
$\chi = 0$

110-000



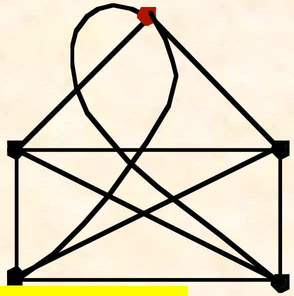
$\chi = -2$

00000



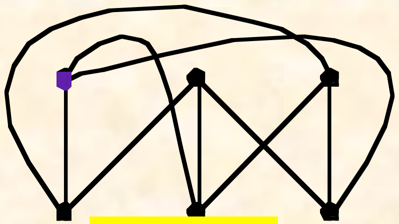
$\chi = -2$

30000



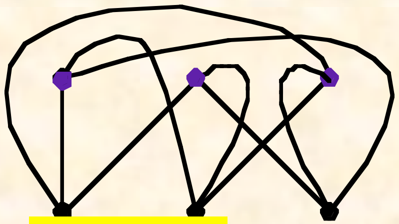
$\chi = -4$

100-000

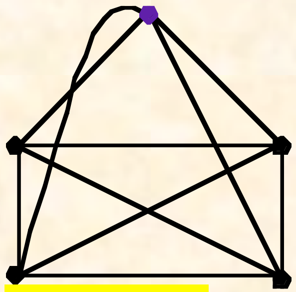


$\chi = -2$

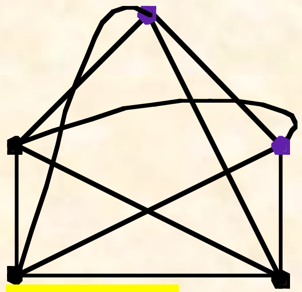
111-000



$\chi = 0$

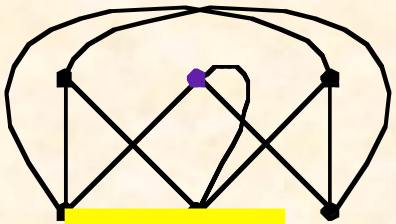


$\chi = -4$



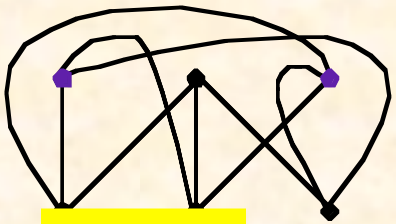
$\chi = -4$

010-000

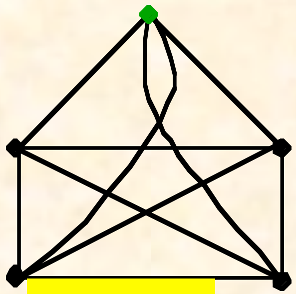


$\chi = -2$

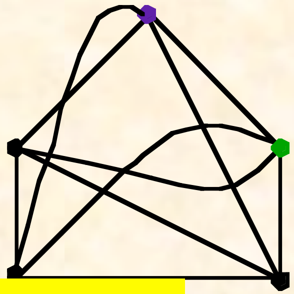
010-000



$\chi = -2$



$\chi = -2$



$\chi = -4$

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orientable surfaces: **orientable triangulations or tilings**

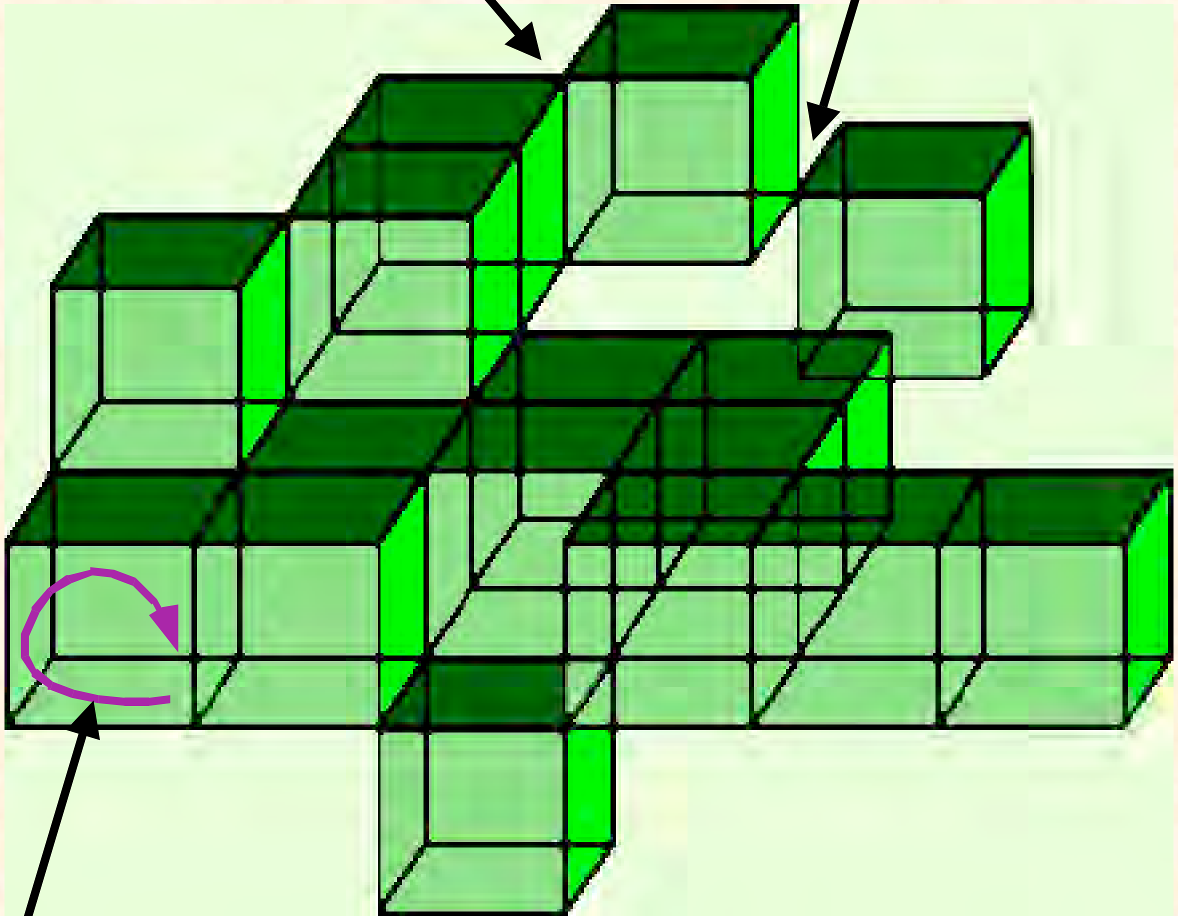
Listing 1861, Aleksandrov 1956, ...

pure two-dimensional tiling is **strongly connected** iff any two tiles may be connected via a chain of edge-adjacent tiles

Orientation of a single **edge** in a strongly connected tiling of an orientable surface specifies the orientation of all tiles in this tiling.

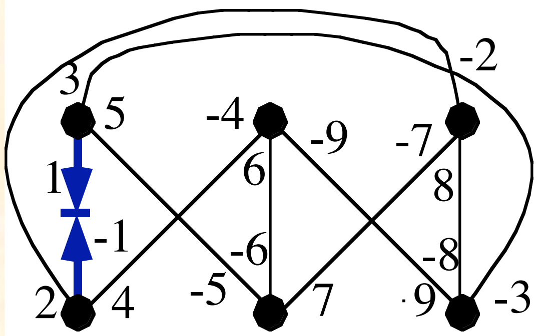
Euler characteristic $\chi = \alpha_0 - \alpha_1 + \alpha_2$ of a tiling is equal to 2 iff surface homeomorphic to unit sphere (Jordan surface) still strongly connected

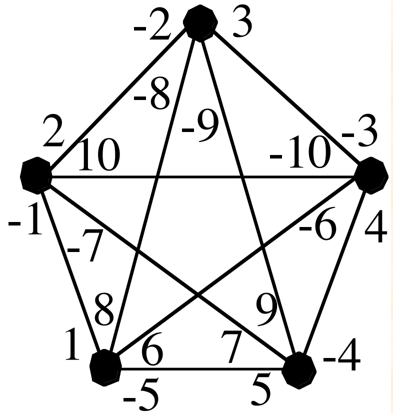
not strongly connected



initial orientation of one of the square tiles combinatorial maps: each directed edge = two **darts**

Edmonds 1960, Tutte 1963





anti-clockwise

α = (1,–1)(2,–2)(3,–3)(4,–4)(5,–5)(6,–6)(7,–7),(8,–8),(9,–9)

σ = (5,3,1)(-4,6,-9)(-7,8,-2)(-1,2,4)(-6,-5,7)(-8,9,-3)

(1,2,-7,-6,-9,-3)(-1,5,7,8,9,-4)(-2,4,6,-5,3,-8)

$\varphi = \sigma \circ \alpha =$

clockwise

σ = (2,10,-7,-1)(-2,3,-9,-8)(-3,4,-6,-10)(9,-4,5,7)(8,6,-5,1)

(-2,10,-3,-9,-4,-6,-5,7,-1,8)(1,2,3,4,5)(6,-10,-7,9,-8)

$\varphi = \sigma \circ \alpha =$

combinatorial maps = finite oriented adjacency graphs,
but different ways of presentations (global permutations vs. local adjacency cycles) and different directions of studies

(**different results!**)

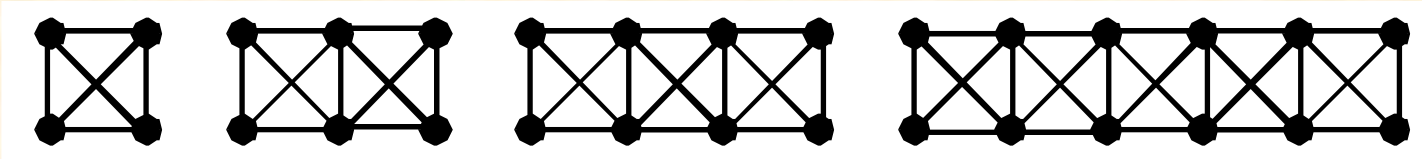
both theories developed without knowing from each other
(resolved in February 2002 at least)

here: further review of theory of oriented adjacency graphs

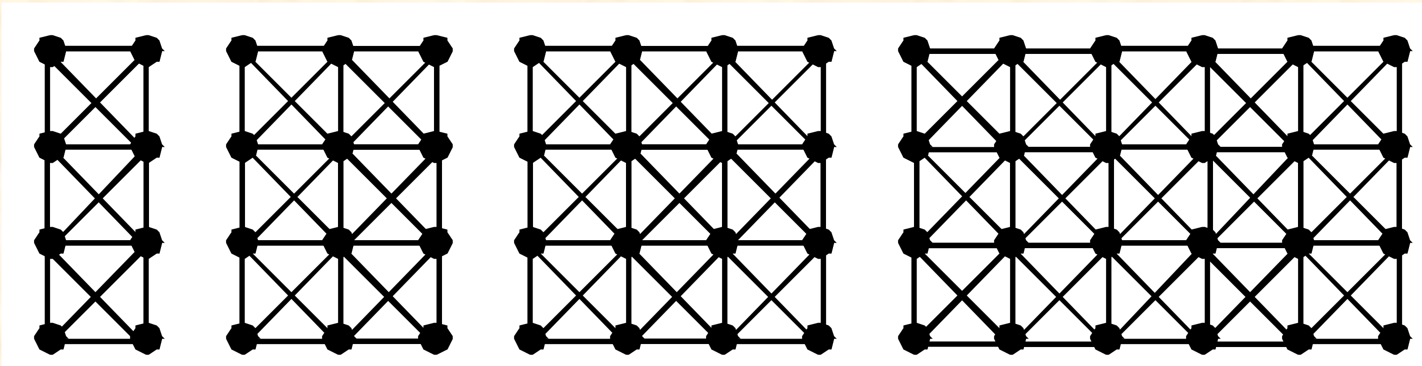
$\chi \leq 2$ for any finite oriented adjacency graph (Voss/Klette 1986)

finite: **planar** iff $\chi = 2$
infinite: **planar** iff any non-empty finite connected subgraph planar

(π,κ,μ):	(4,6,2)	(6,11,3)	(8,16,4)	(12,26,6)	(2n,5n-4,n)
χ :	0	-2	-4	-8	**-2(n-2)**



(π,κ,μ):	(8,16,4)	(12,29,7)	(16,42,10)	(24,68,16)	(4n,13n-10,3n-2)
χ :	-4	-10	-16	-28	**-2(3n-4)**



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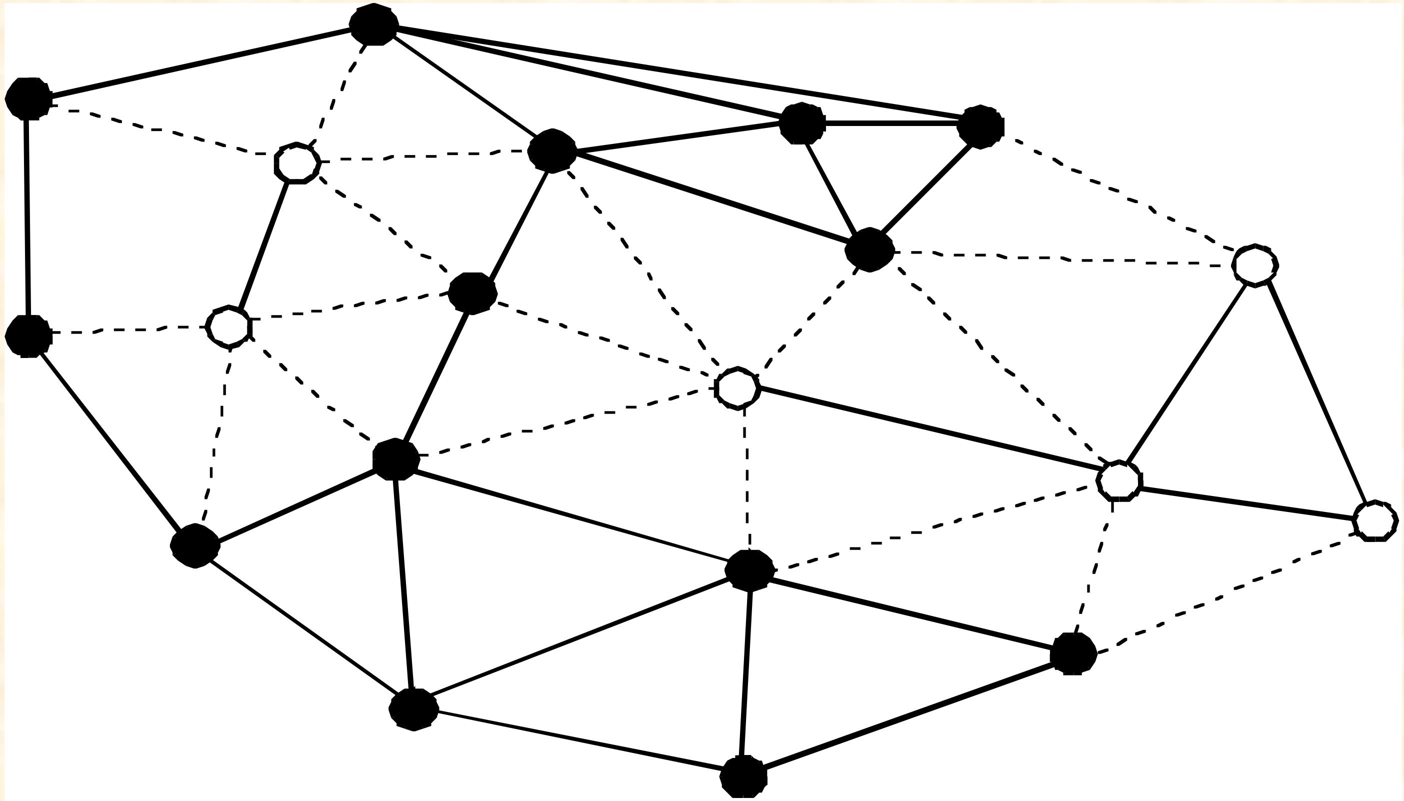
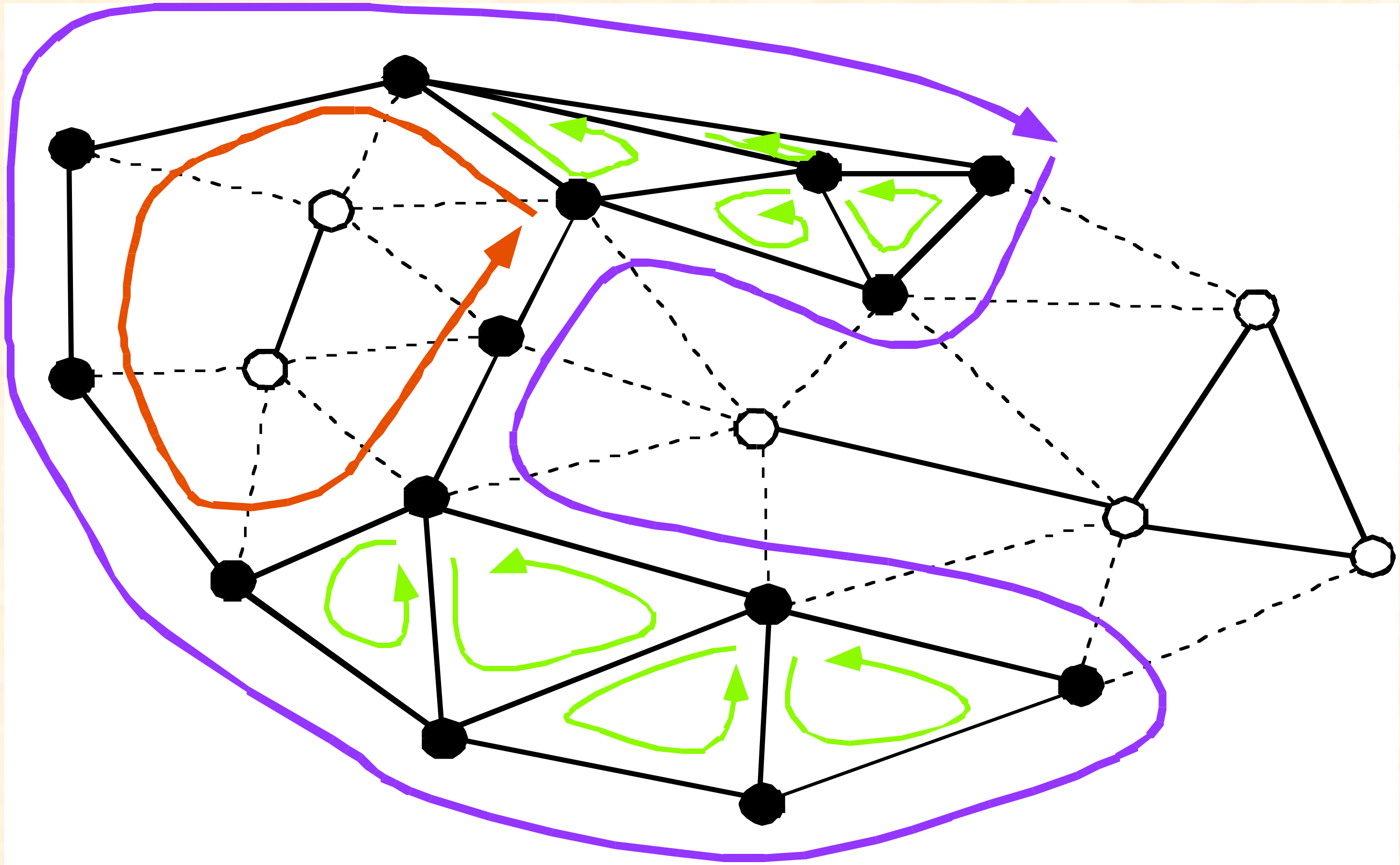
[**C**, A, ξ]: $\alpha_0 = 20$ $\alpha_1 = 46$ $\alpha_2 = 28$ $\chi = 2$



[**C**, A, ξ_M]: $\alpha_0 = 14$ $\alpha_1 = 22$ $\alpha_2 = 10$ $\chi = 2$



[**C**, A, ξ_M]: 8 original cycles

2 border cycles

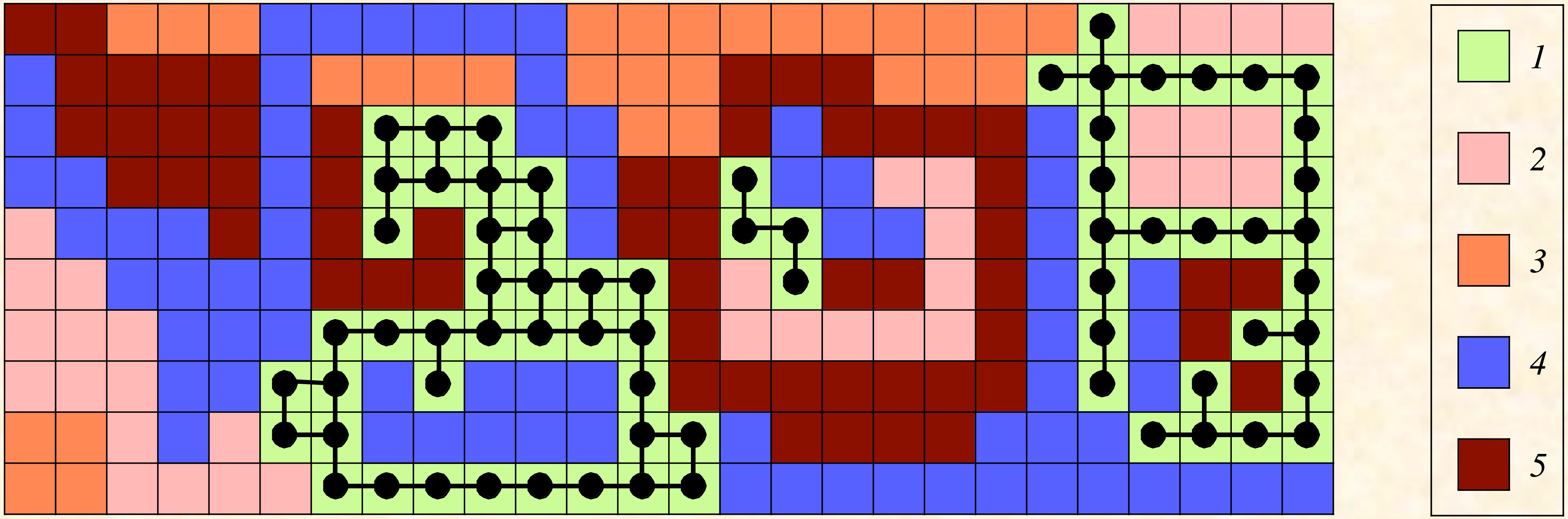
undirected invalid edges **assigned to a border cycle**

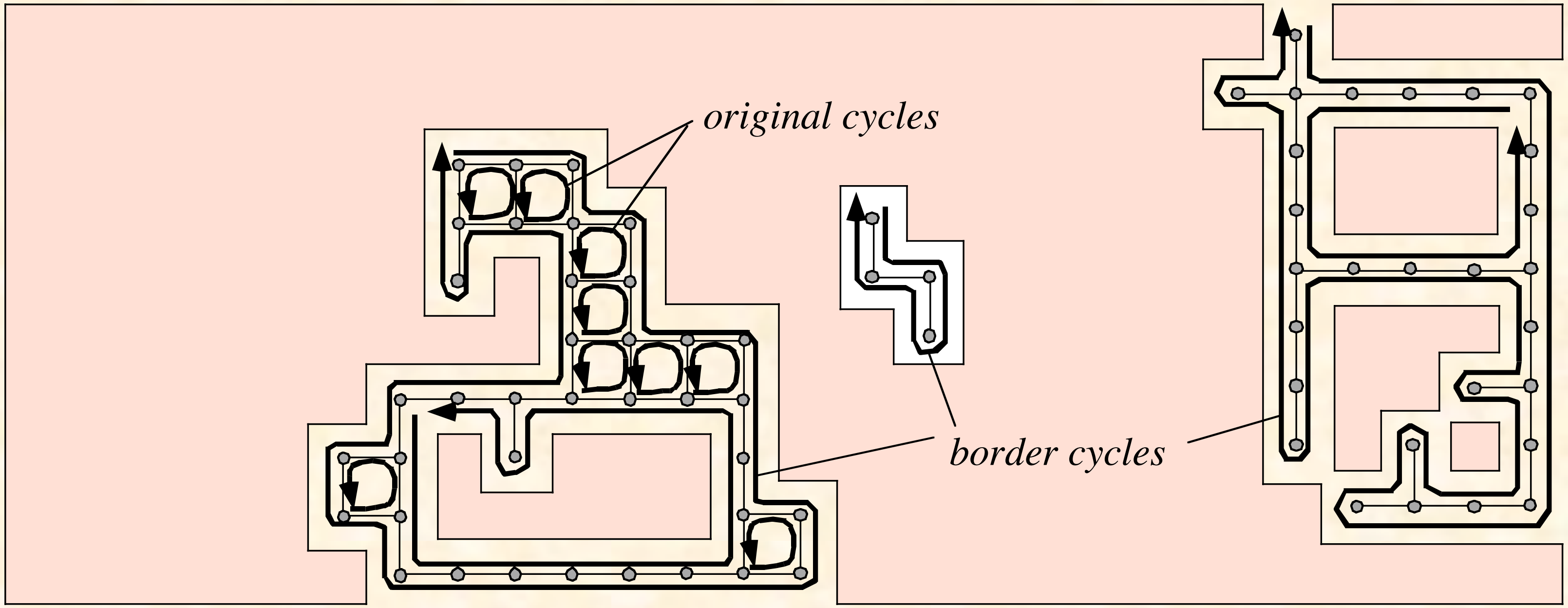
Voss and Klette 1986: *separation theorem*

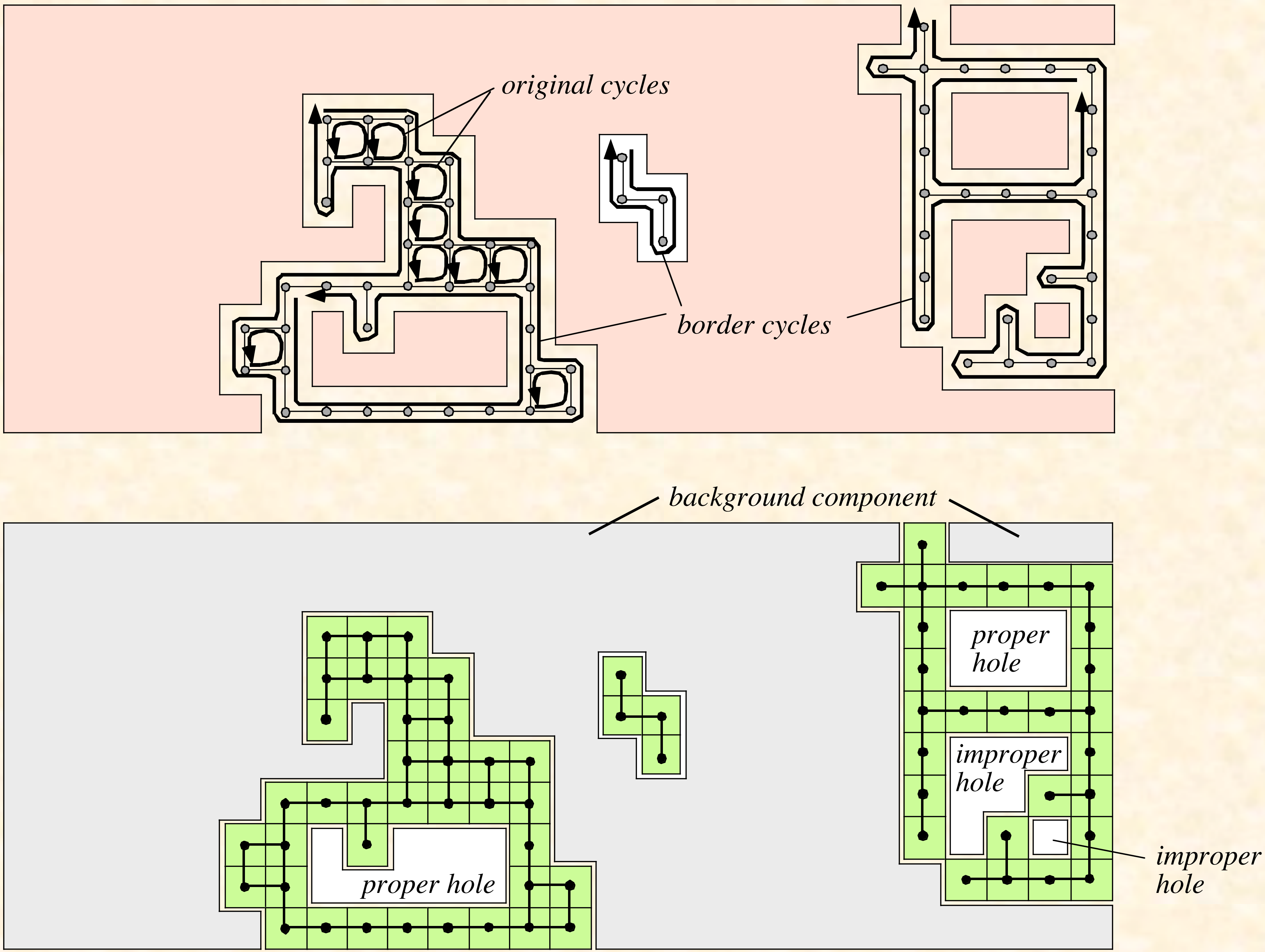
Let [**C**, A, ξ] be a planar oriented adjacency graph.

Let M be a non-empty finite connected proper subset of **C**.

By deleting all undirected invalid edges assigned to one of the border cycles of M, [**C**, A, ξ] splits into **at least** two non-connected substructures.


1
2
3
4
5

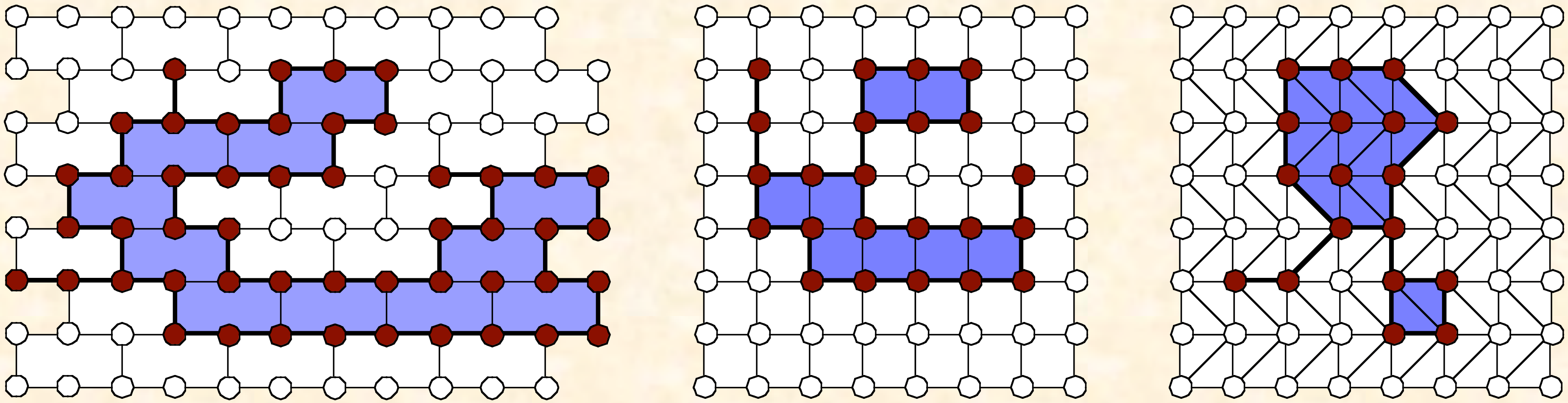

original cycles
border cycles


original cycles
border cycles
border cycles
background component
proper hole
proper hole
improper hole
improper hole

the uniquely defined **outer border cycle** of M separates one (infinite) background component and a finite number of improper holes from M

any **inner border cycle** of M separates a finite number of proper holes from M

mesh = planar oriented adjacency graph
regular mesh = $\nu(p)$ and $\lambda(g)$ constants



left : $\nu = 3, \lambda = 6, \alpha_0 = 49, \alpha_1 = 59, \alpha_2 = 12$, $l = 52$, $k = 29$, f =
middle : $\nu = 4, \lambda = 4, \alpha_0 = 23, \alpha_1 = 30, \alpha_2 = 9$, $l = 28$, $k = 32$, $f = 8$
right : $\nu = 6, \lambda = 3, \alpha_0 = 18, \alpha_1 = 32, \alpha_2 = 16$, $l = 19$, $k = 44$, f =

l = length of (inner or outer) border cycle
k = # invalid edges assigned to border cycle
$f = \alpha_2 - 1$

$$\boxed{k = \nu + \frac{\nu}{\lambda} l}$$

29 = 3 + 3/6 x 52

32 = 4 + 4/4 x 28

44 = 6 + 6/3 x 19

Voss 1986: *total curvature theorem*

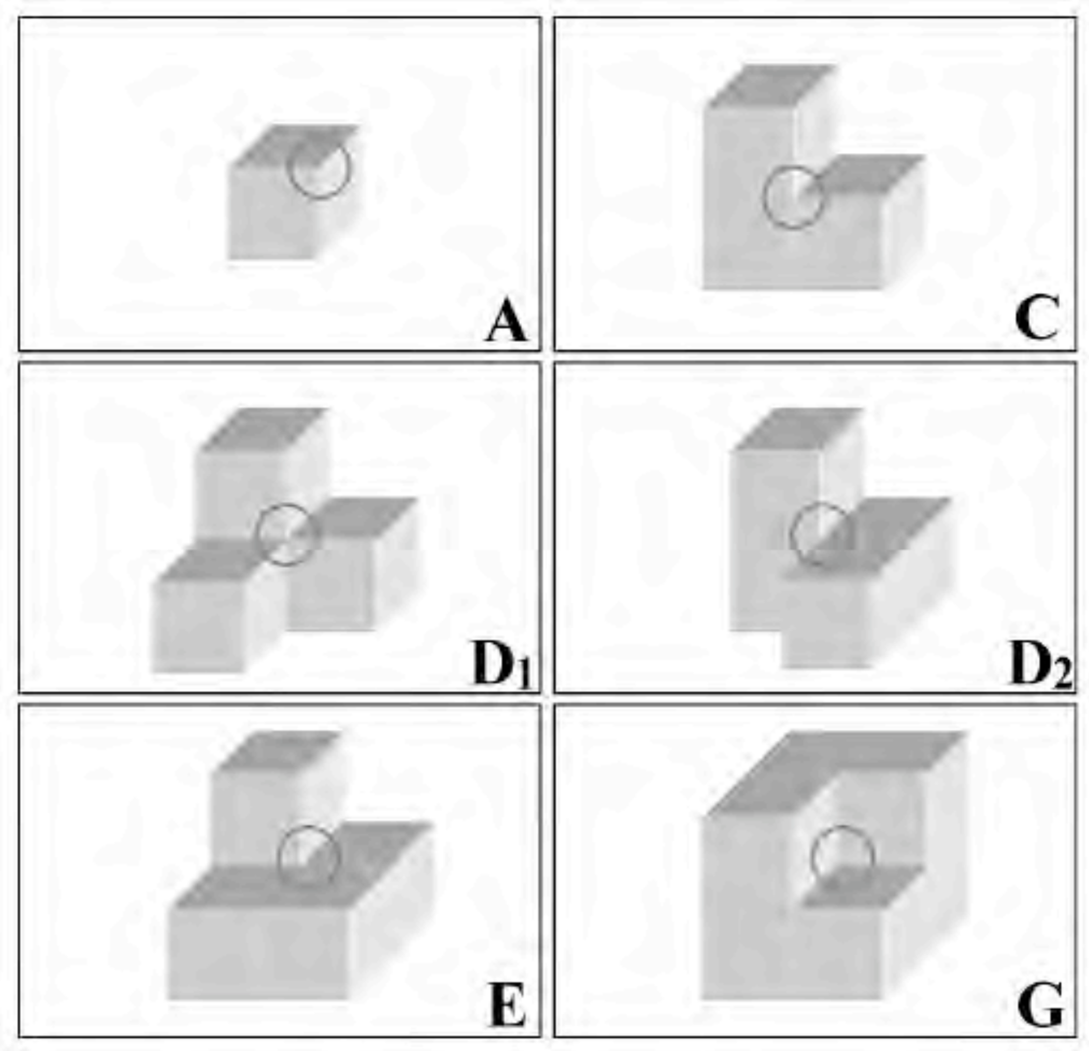
M = finite connected subset of an infinite regular mesh $\mathbf{C}_{\nu,\lambda}$

For any border cycle: $$\pm 1 = \frac{k}{\nu} - \frac{l}{\lambda}$$

outer border cycle: positive sign, **inner border cycle**: negative sign

Yip/Klette 2001: generalized to orthogonal grid in 3D
(based on classification and counts of possible edges)

Six kinds of angles in an isothetic simple polyhedron



H_A - $H_G < 0$ iff inner border

H_A - $H_G > 0$ iff outer border

H_A, H_c, H_{D1}, H_{D2}, H_E, H_G = # A, C, D1, D2, E, G angles of polyhedron H, respectively

Yip and Klette 2001:

$$(H_A + H_G) - (H_C + H_E) - 2(H_{D1} + H_{D2}) = 8$$

for any isothetic simple polyhedron H

R. Klette

Voss 1986: *generalized Pick's theorems*

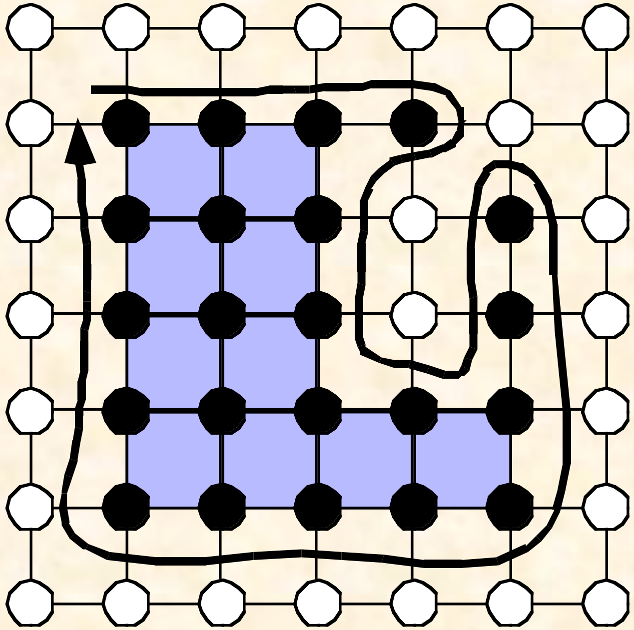
M = finite connected subset of an infinite regular mesh $\mathbf{C}_{\nu,\lambda}$ without proper holes, then

for the (outer) border cycle: $\alpha_0 = \frac{\nu}{\lambda} f + l/2 + 1$

M = finite connected subset of an infinite regular mesh $\mathbf{C}_{\nu,\lambda}$ then

for any inner border cycle: $\alpha_0 = \frac{\nu}{\lambda} f - l/2 + 1$

(see G. Pick's area theorem $A = i + b/2 - 1$ from 1899 for the orthogonal grid)

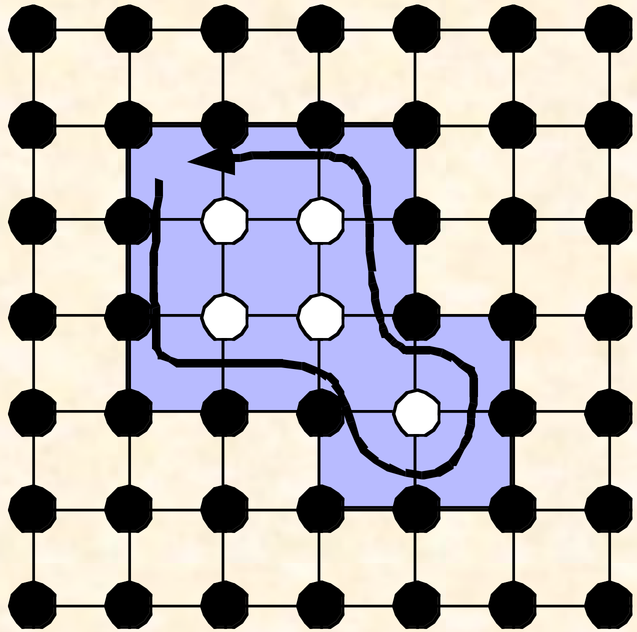


$\alpha_0 = 22 \quad f=10 \quad l=22$

$22 = 10 + 22/2 + 1$

outer border cycle: set is connected, no proper hole, but one improper hole

inner border cycle defining two proper holes



$\alpha_0 = 5 \quad f=12 \quad l=16$

$5 = 12 - 16/2 + 1$

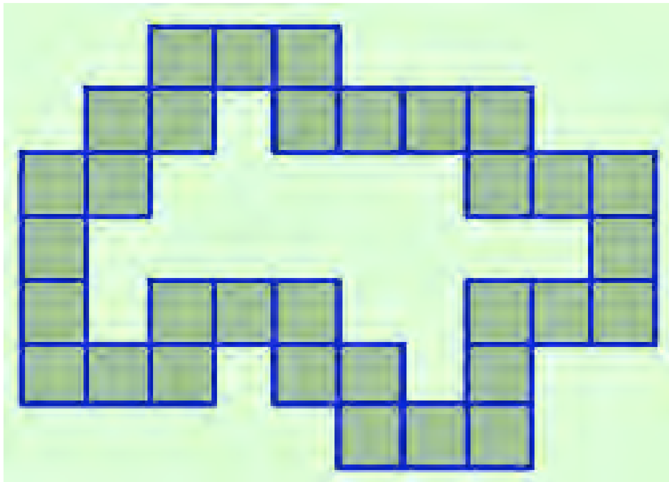
Brief review

- image analysis based on 4-adjacency only (2D): Univ. Jena, TU Berlin, Univ. Auckland, Academy Bratislava, ...

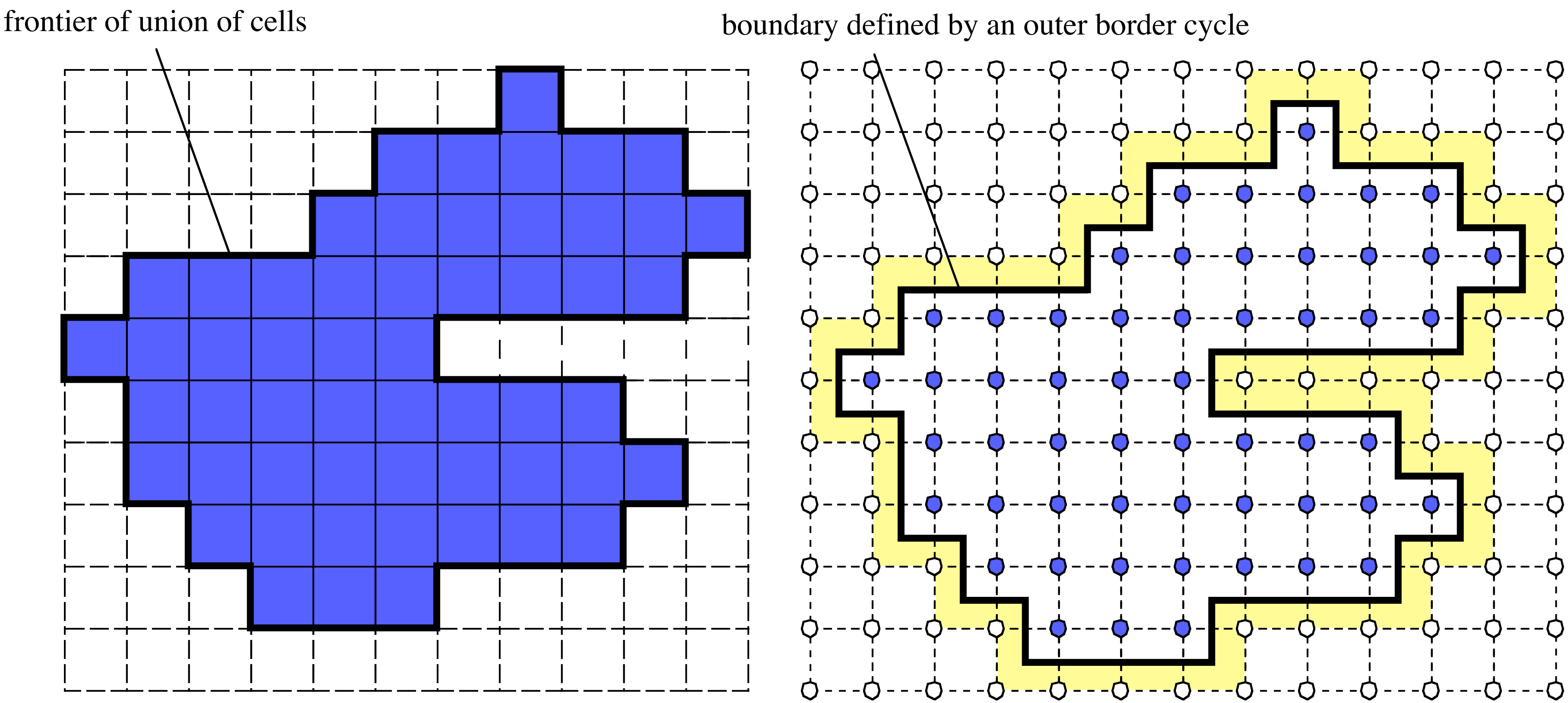
- 6-adjacency only in 3D

<u>example 1:</u> (Urysohn) curve
= one-dimensional continuum

(see Sloboda/ Zatko/ Klette, VG'98)

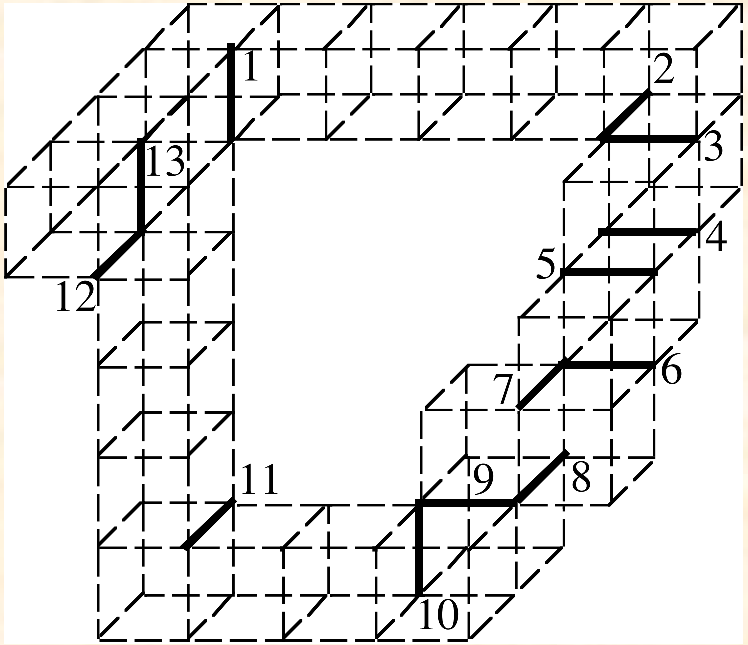


example 2: simple curves in 2D as frontiers of cell complexes (see Kovalevsky et al.)



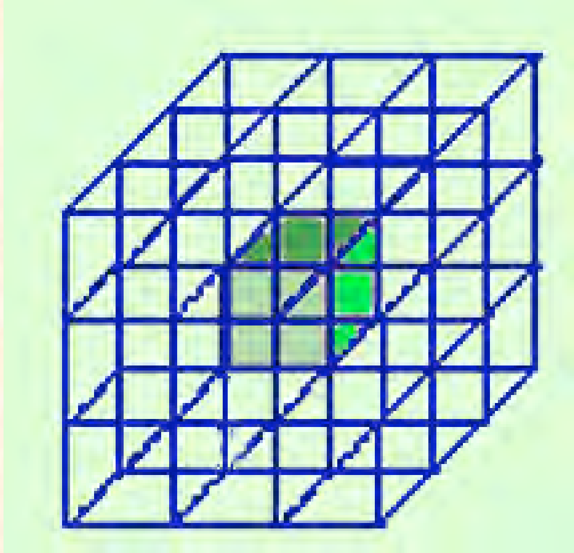
DSS approximation, skeletonization, ...

Bülow and Klette 2000/2001: iterative algorithm with measured time complexity $O(n)$



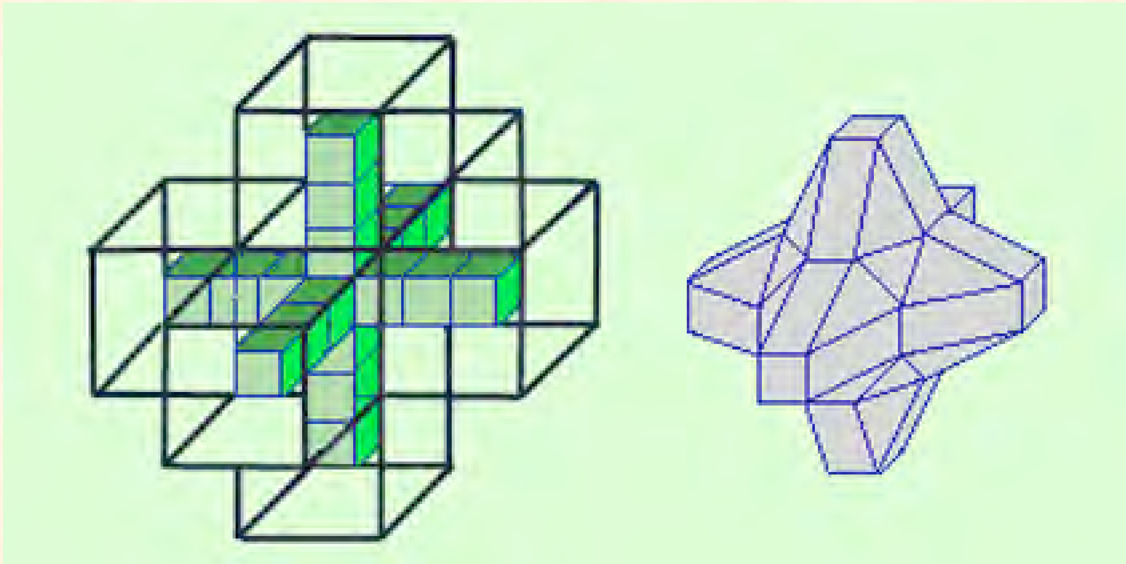
Vertices of MLP always on critical edges.

two-dimensional grid continua:



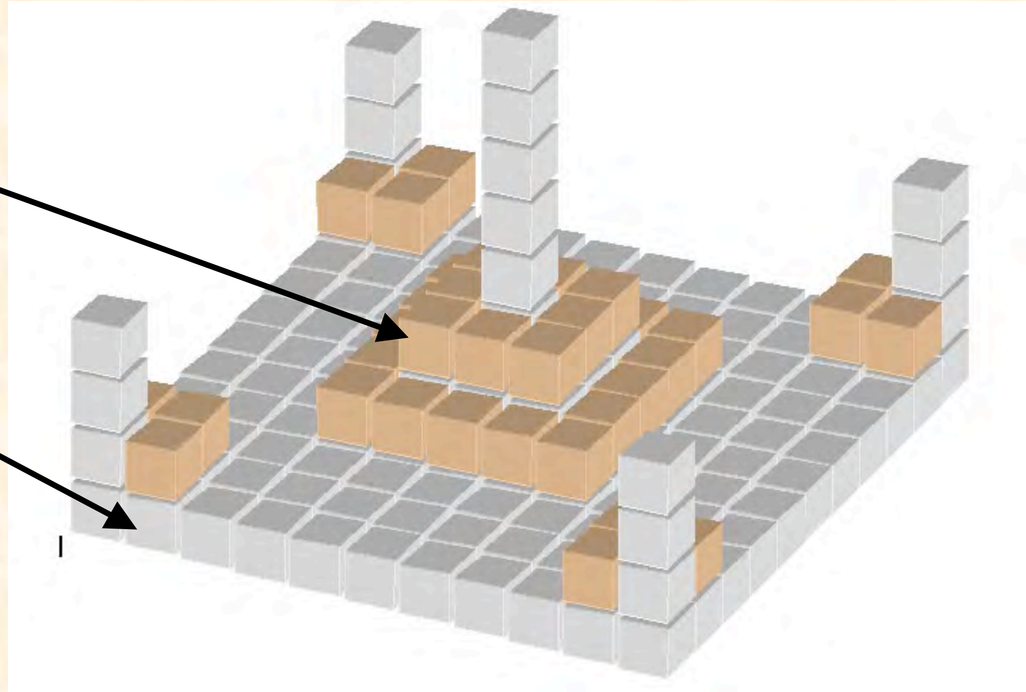
Sloboda and Zatko 2000:

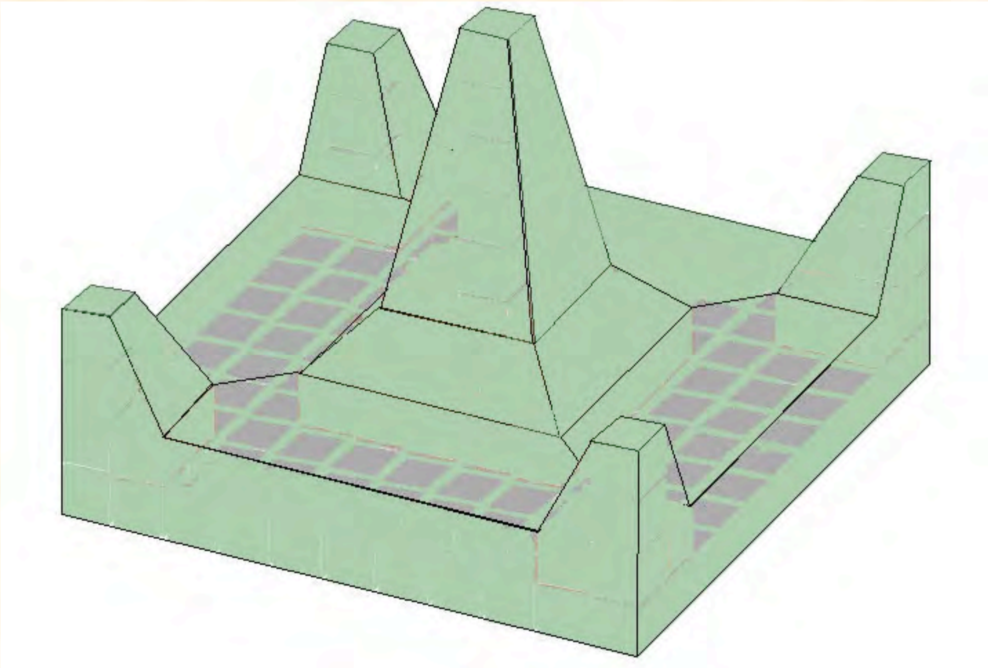
multigrid-convergent surface area estimator: contents of relative convex hull (using inner and outer Jordan digitization)



Klette and Yu 2001:
approximation algorithm

$J^-(S)$ and part of $J^+(S)$





$CH_{J+}(J^-(S))$

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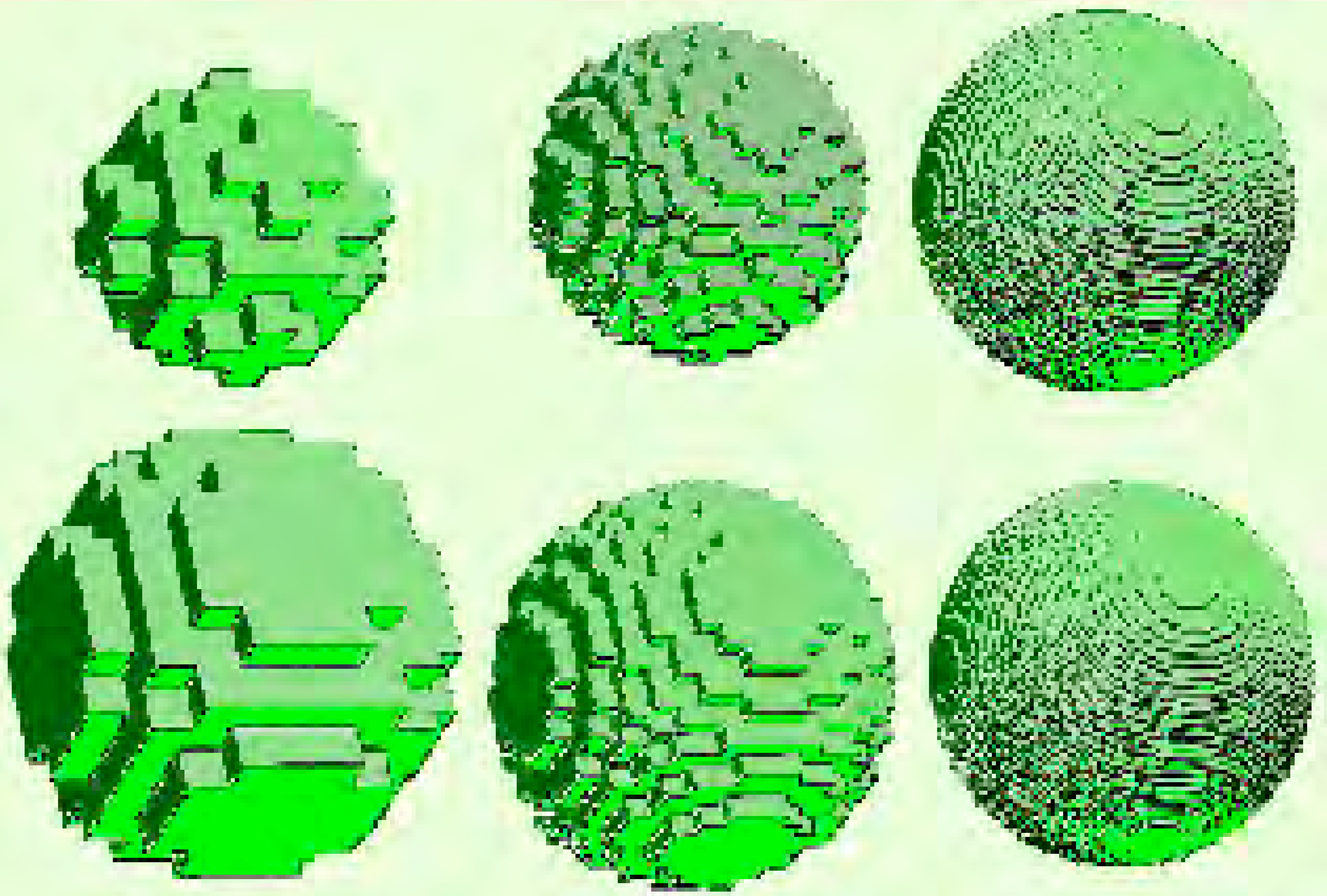


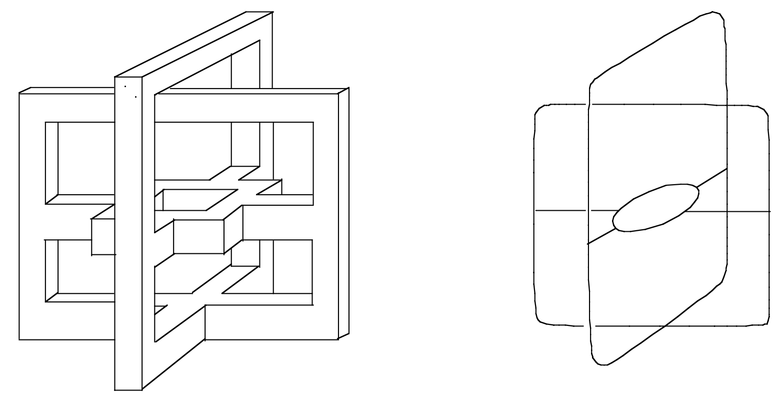
R. Klette

The global picture

boundaries = space between black and white marbles, the potential site of polygonal (2D) or polyhedral (3D) approximations

things to do:

- finish the measurement chapter
- algorithms towards further topological analysis
  - isotopy test 
  - fundamental group calculation (classification of 3D objects)
  - analysis of elementary curves (e.g. list of branching indices)



Listing 1861