Digital Straightness

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digital geometry =

digitized Euclidean geometry

in regular grid or cell spaces

1961 **H. Freeman** **1963** **J. Bresenham**

digital straight lines - digitization and synthesis

1974 **A. Rosenfeld**

digital straight segments - characterization by chord property

Contents of Talk

- **GEOMETRY, NUMBER THEORY, and THEORY OF WORDS**

  Definitions and tangential lines

  Self-similarity studies in pattern recognition

  Periodicity studies in the theory of words

  Counts and spirographs

- **ALGORITHMS for DSS RECOGNITION**

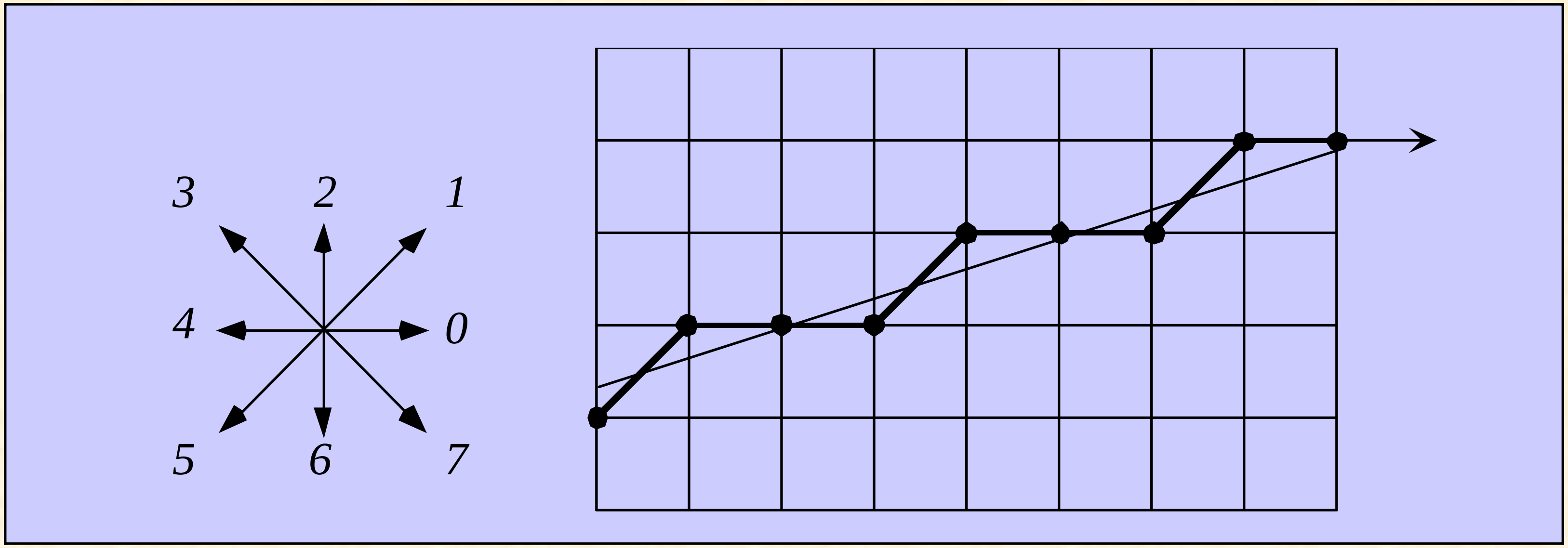
  1982 : the first online, linear-time recognition algorithm

  Review of some algorithms

- **A FEW OPEN PROBLEMS**

GRID - INTERSECTION DIGITIZATION

1961 **H. Freeman**



(real) ray $\gamma_{\alpha,\beta} = \{(x, \alpha x + \beta) : 0 \leq x < +\infty\}$

$0 \leq \alpha \leq 1$ (symmetry of the grid)

$$I_{\alpha,\beta} = \{(n, I_n) : n \geq 0 \wedge I_n = \lfloor \alpha n + \beta + 0.5 \rfloor\}$$

$i_{\alpha,\beta}(n) = I_{n+1} - I_n$ *digital ray* $i_{\alpha,\beta} = i_{\alpha,\beta}(0)i_{\alpha,\beta}(1)i_{\alpha,\beta}(2)$☹

defines a set of grid points $G(i_{\alpha,\beta})$, generated by $\gamma_{\alpha,\beta}$

right infinite word in $\{0,1\}^\omega$ $i_{0,\beta} = 0^\omega$ $i_{1,\beta} = 1^\omega$

<u>Theorem 1.</u> (**Rosenfeld 1974**) A digital ray is an irreducible 8-arc.

$0 \leq \beta \leq 1$ ($\beta - \beta'$ integer $\Rightarrow i_{\alpha,\beta} = i_{\alpha,\beta'}$)

Theorem 2. (**Bruckstein 1991**) For irrational α, $I_{\alpha,\beta}$ uniquely determines both α and β . For rational α , $I_{\alpha,\beta}$ uniquely determines α, and β is determined up to an interval.

rational digital ray OR *irrational digital ray*

for specification of intercepts of rational digital rays:

see spirograph theory of **Dorst and Duin 1984**

Theorem 3. (**Brons 1974**) Rational digital rays are periodic v^{ω} and irrational digital rays are aperiodic.

$v \in \{0,1\}^{*}$ = *basic segment* of rational digital ray

FINITE SEGMENTS in pattern recognition

DSS (digital straight segment)

v factor of $u = v_1 v v_2$

non-empty factor of (rational or irrational) digital ray

4-DSS (digital 4-straight segment)

non-empty factor of (rational or irrational) digital 4-ray

CSS (cellular straight segment)

non-empty factor of cellular ray

ALTERNATIVE DEFINITIONS

arithmetic geometry **1991** **J.-P. Reveilles**

$$D_{a,b,c,d} = \{(i,j) \in \mathbf{N}^2 : i \le m \wedge c \le ai - bj \le c + d\}$$

integers a, b, c, d, m $\quad gcd(a,b) = 1$

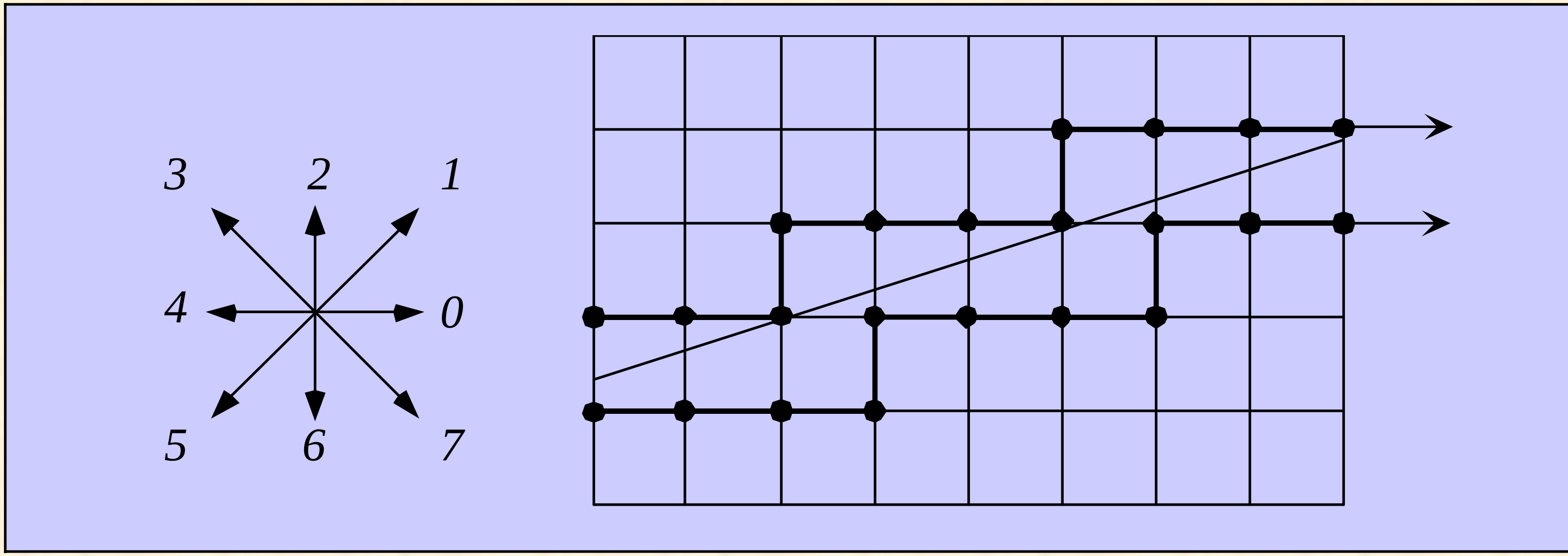
digital bar slope *a/b*, lower bound *c*, arithmetical width *d*

Theorem 4. (**Reveilles 1991**) A digital bar of arithmetical width $max\{|a|, |b|\}$ is a set $G(w)$ of gridpoints of a DSS w, and vice-versa.

upper tangential line $y = \alpha x + \beta$ $\quad \alpha = a/b \quad \beta = -c/b$

lower tangential line $y = \alpha x + \beta - \left(1 - \frac{1}{b}\right)$

Corollary 1. A word $w \in \{0,1\}^*$ is a DSS iff $G(w)$ lies on or between two parallel lines having a distance <1 in y-direction.



$$U_{\alpha,\beta} = \{(n, U_n) : n \geq 0 \wedge U_n = \lceil \alpha n + \beta \rceil\}$$

$$L_{\alpha,\beta} = \{(n, L_n) : n \geq 0 \wedge L_n = \lfloor \alpha n + \beta \rfloor\}$$

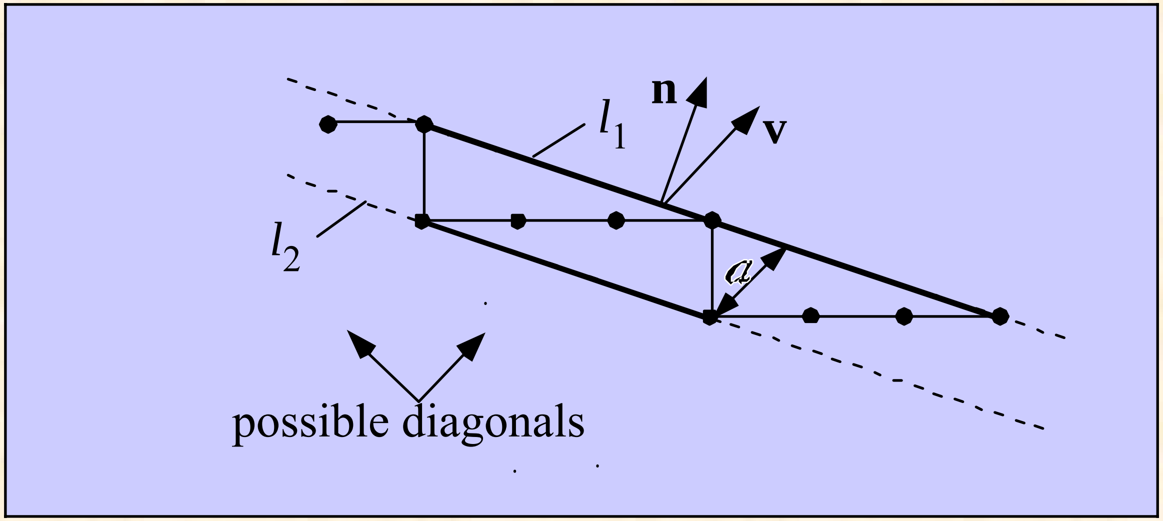
$$u^{\mathrm{o}}_{\alpha,\beta}(n) = \begin{cases} 0, & \text{if } U_n = U_{n+1} \\ 02, & \text{if } U_n = U_{n+1} - 1 \end{cases}$$

$$l^{\mathrm{o}}_{\alpha,\beta}(n) = \begin{cases} 0, & \text{if } L_n = L_{n+1} \\ 02, & \text{if } L_n = L_{n+1} - 1 \end{cases}$$

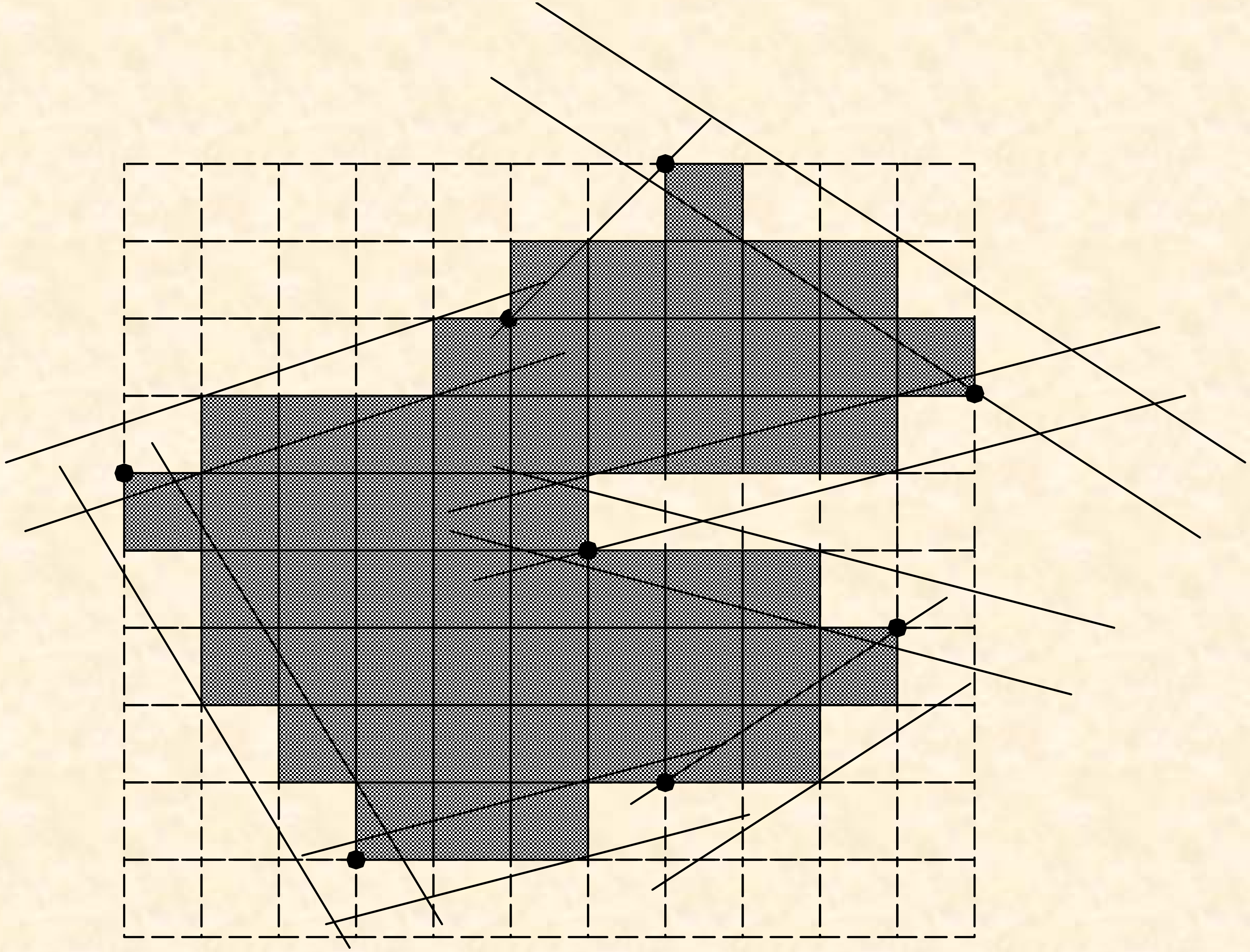
upper digital 4-ray

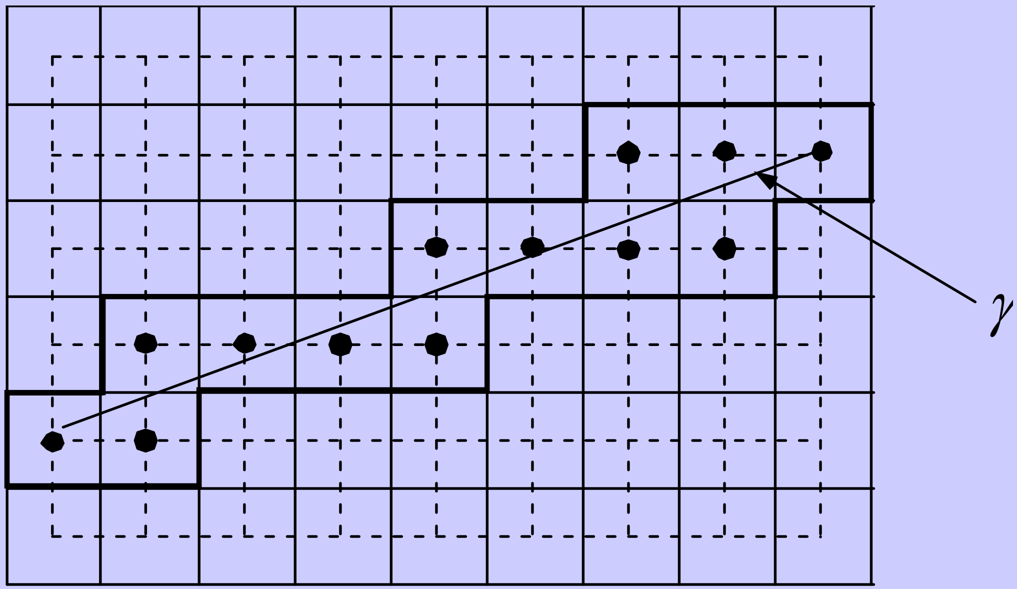
lower digital 4-ray

Theorem 5. (**Kovalevsky 1990**) A word $w \in \{0,1\}^*$ is a 4-DSS iff $G(w)$ lies on or between two parallel lines having a main diagonal distance $< \sqrt{2}$.



main diagonal (of a line) = the grid diagonal which maximizes the dot product with the normal to the line

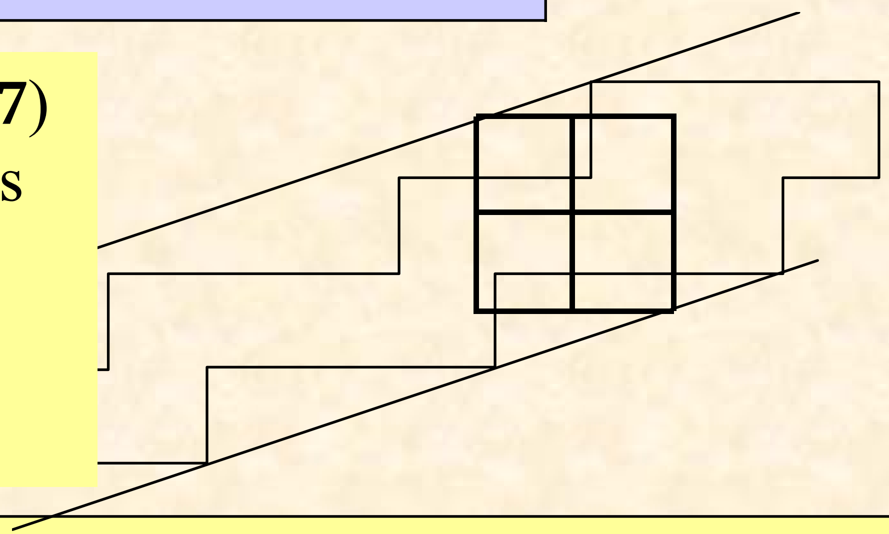




cellular straight line every cell intersects γ

γ contained in union of cells

<u>Theorem 6.</u> (**Fam & Sklansky 1977**)
An edge-connected set F of cells is cellularly straight iff there exists a direction θ with $width_\theta(F) \leq width_\theta(R_{2x2})$.



SELF-SIMILARITY STUDIES in pattern recognition

1970 **H. Freeman**

"…

(F1) at most two types of elements can be present, and these can differ only by unity, modulo eight;

(F2) one of the two element values always occurs singly;

(F3) successive occurrences of the element occuring singly are **as uniformly spaced as possible**."

1974 A. Rosenfeld

set G of grid points satisfies the **chord property** iff for any two p,q in G and r on pq there is t in G such that $d_\infty(r,t) < 1$.

<u>Theorem 7.</u> (**Rosenfeld 1974**) A finite irreducible 8-arc u is a DSS iff $G(u)$ satisfies the chord property.

(shorter proof via Santalo's *Traversal Theorem* by **C. Ronse** in 1985)

"... (R1) the runs have at most two directions, differing by 45^0, and for one of these directions, the run length must be 1.

(R2) The runs can have only two lengths, which are consecutive integers.

(R3) One of the runs can occur only once at a time.

(R4) ..., for the run length that occurs in runs, **these runs can themselves have only two lengths, which are consecutive integers; and so on**."

1981 **A. Hübler, R. Klette, K. Voss**

Early example for the *linguistic method* for DSS recognition

linear-time off-line algorithm based on repeated

reduction operations for reducible words

$$R(u) = \begin{cases} \text{word which results from } u \text{ by replacing all} \\ \quad \text{factors of nonsingular letters by run lengths,} \\ \quad \text{and by deleting all other letters in } u \\ \text{letter } a, \quad \text{if } u = a^{\omega} \end{cases}$$

finite $u \in \{0,1,...,7\}^*$ or infinite word $u \in \{0,1,...,7\}^{\omega}$ reducible iff contains no singular letter, OR any factor of u containing only nonsingular letters is of finite length

$w \in \{0,1,..,7\}^{\omega}$ satisfies **DSL property** iff $w_0 = w$ and $w_{n+1} = R(w_n)$ reducible, $n \geq 0$, and w_n satisfies

(L1) at most two letters a and b in w_n, and if two, then $|a-b| = 1$

(L2) if two, then at least one of them is singular

l(w) = run length of nonsingular letters left of first singular letter

r(w) = run length of nonsingular letters right of last singular letter

$w \in \{0,1,..,7\}^{*}$ satisfies **DSS property** iff $w_0 = w$ satisfies (L1) and (L2), and any nonempty sequence $w_{n+1} = R(w_n)$, $n \geq 0$, satisfies (L1) and (L2) and

(S1) if w_{n+1} contains only letter a, or letters a and $a+1$, then $l(w_n) \leq a+1$ and $r(w_n) \leq a+1$,

(S2) if a and $a+1$, and a is nonsingular in w_{n+1}, then if $l(w_n) \leq a+1$ then w_{n+1} starts with a, and if $r(w_n) \leq a+1$ then w_{n+1} ends with a.

(both formulations as in **Hübler** 1989)

1981 **A. Hübler, R. Klette, K. Voss**

DSS recognition algorithm using
"finite 8-arc DSS iff satisfies DSS property" without proof

1982 **L. D. Wu**

DSS recognition algorithm (with a flaw) using
"finite 8-arc DSS iff satisfies DSS property" with

Theorem 8. (**Wu 1982**) A finite 8-arc is a DSS iff it satisfies the DSS property.

Theorem 9. (**Hübler 1989**) A two-sided infinite 8-arc is a digital straight line iff it satisfies the DSL property.

Wu’s proof: very long, many case discussions, minor (reparable) flaws

material for a brief and accurate proof: see number theory book by

B.A. Venkov 1970 :

discussion of *continued fractions* for Bernoulli sequences

1991 **A. Bruckstein**

continued fraction arguments based on Venkov’s book

1991 **K. Voss**

independent proof of Wu’s theorem for rational slopes based on continued fractions

PERIODICITY STUDIES in the theory of words

forthcoming book:

M. Lothaire (ed.). *Algebraic Combinatorics on Words.* Cambridge Univ. Press, to appear.

let w be word over $\{0,1\}$

$F_n(w) =$ set of all factors of w of length n

$$P(w,n) = card(F_n(w))$$

Example: infinite periodic word w with period k, then $P(w,n) \leq k$

note: rational digital rays are infinite periodic words (Theorem 3)

Theorem 10. (**Coven & Hedlund 1973**) Equivalent for an infinite word w:

(*i*) w is eventually periodic, i.e. $w = uv^{\omega}$,

(*ii*) $P(w,n) = P(w,n+1)$ for some $n \geq 0$,

(*iii*) $P(w,n) < n+k-1$ for some $n \geq 1$, where k is the number of letters appearing in w ,

(*iv*) $P(w,n)$ is bounded.

i.e. any aperiodic infinite word w has $P(w,n) \geq n+1$, for all $n \geq 0$

Sturmian word infinite word w over binary alphabet with $P(w,n) = n+1$, for all $n \geq 0$

i.e. Sturmian words are aperiodic words of minimum complexity

height $h(w),\ w \in \{0,1\}^*$, is number of letters equal to 1 in w

balance of two words u,v of the same length = $\delta(u,v) = |h(u) - h(v)|$

set X of words is **balanced** iff $|v| = |u|$ implies $\delta(u,v) \leq 1$
for all u,v in X

an infinite word is balanced iff its set of all factors is balanced

slope $\pi(u) = h(u)/|u|$

we have: infinite word w is balanced iff for all non-empty factors u,v of w, $|\pi(u) - \pi(v)| < \frac{1}{|u|} + \frac{1}{|v|}$

i.e. this sequence of slopes is a Cauchy sequence, i.e. it converges

Theorem 11. (**Morse & Hedlund 1940**) Let w be a (rational or irrational) digital ray with slope α . Then w is balanced of slope α .

the proof uses the inequality $|\pi(u)-\alpha|<\frac{1}{|u|}$

Corollary 2. Any digital straight segment is a factor of a rational digital ray.

Theorem 12. (**Morse & Hedlund 1940**) Equivalent for an infinite word w:

(i) w is Sturmian,

(ii) w is balanced and aperiodic,

(iii) w is an irrational digital ray.

note: there are balanced infinite words with rational slope which are not rational digital rays, e.g. 01^{ω}

COUNTS AND SPIROGRAPHS

Theorem 13. (**Mignosi 1991**) The number of balanced words of length n is

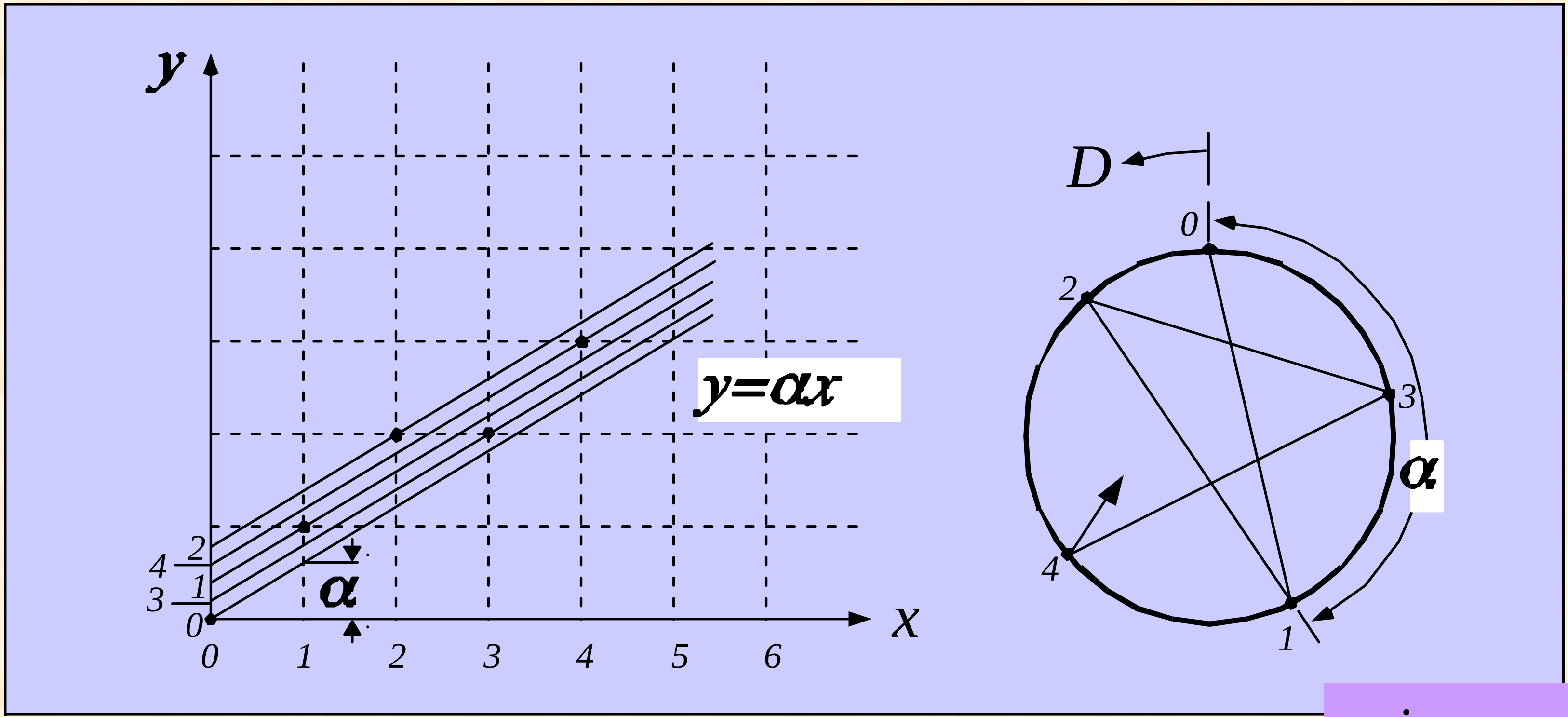
$$1+\sum_{i=1}^{n}(n+1-i)\phi(i)$$

where ϕ is Euler's totient function.

Farey series $F(n)$ of order n = ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed n

1976 **J. Rothstein & C. Weiman**

1-1 correspondence between $F(n)$ and all DSSs of length n passing through the origin



spirograph

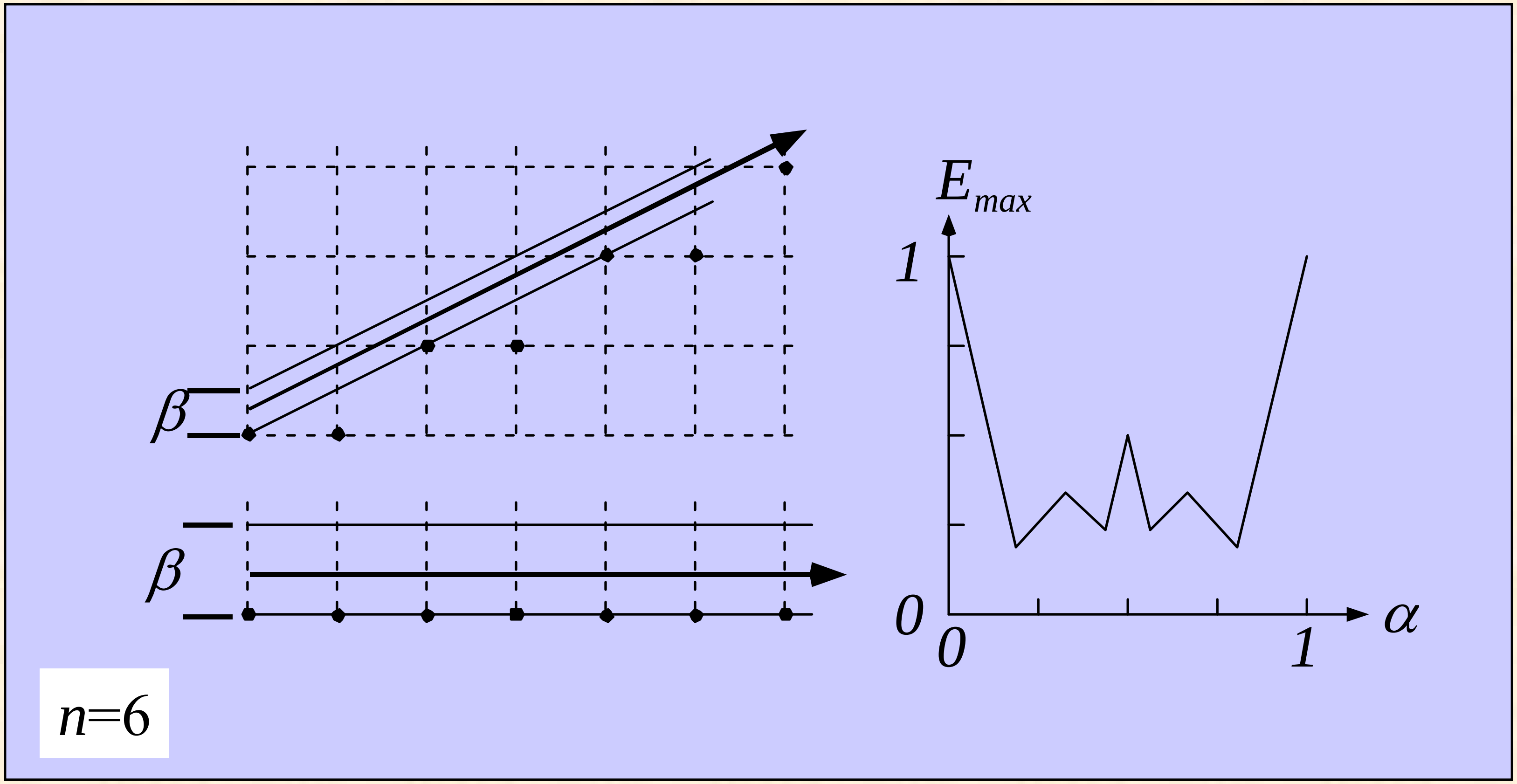
1984 **L. Dorst & R.P. Duin**

consider DSSs of length n generated by ray $\gamma_{\alpha,0}$, i.e. $y = \alpha x$

$0 < \alpha < 1$

task: specify the interval of possible intercepts β

(see Theorem 2)



<u>Theorem 14.</u> (**Dorst and Duin 1984**) The maximum intercept error $E_{max}(\alpha, n) = D_{right} + D_{left}$ is specified by distances D_{right} and D_{left} in spirograph $S(\alpha, n+1)$.

<u>Example:</u> if $\alpha = a/b$ in $F(n)$, then $E_{max}(a/b, n) = 1/b$

ALGORITHMS for DSS RECOGNITION

input: sequence of chain codes $i(0), i(1), ...$ $i(k) \in \{0,1\}, \quad k \geq 0$

off-line DSS recognition algorithm

decides for $u \in \{0,1\}^*$ whether u is a DSS or not

on-line DSS recognition algorithm

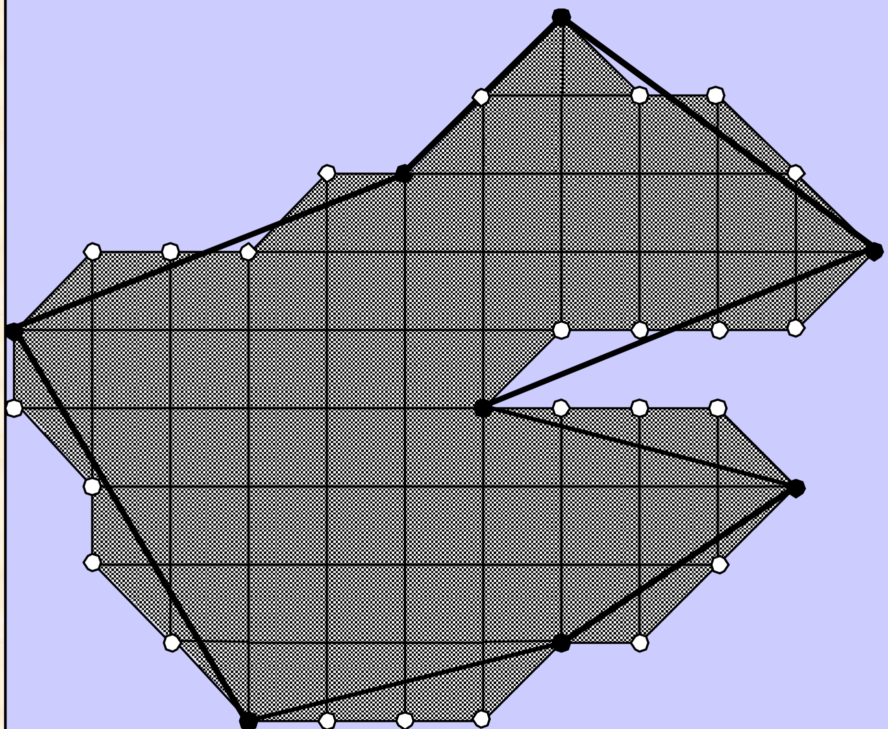
decides for any new $i(k+1)$ whether $i(0)...i(k)i(k+1)$ is still a DSS based on previous positive test for $i(0)...i(k)$

linear (time) algorithm

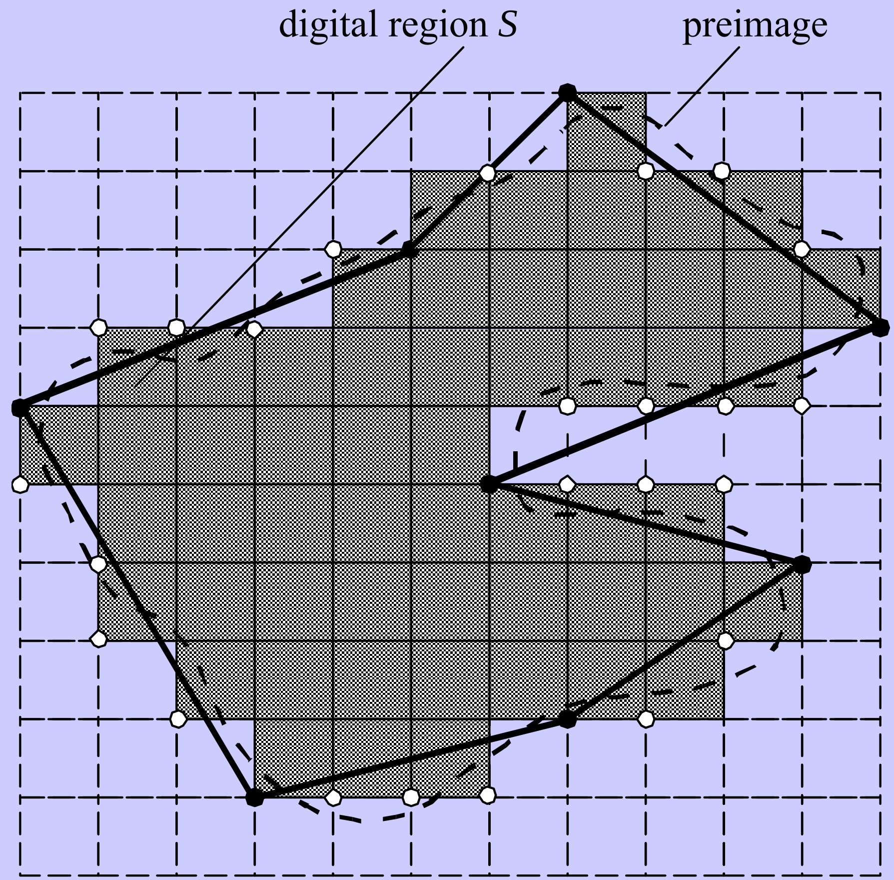
at most $O(|u|)$ basic steps for any input $u \in \{0,1\}^*$

clockwise

DSS approximation



4-DSS approximation



1976 **J. Rothstein & C. Weiman** RW_1976

first layer only of off-line linguistic DSS algorithm
(DSS property not yet known)

1981 **A. Hübler, R. Klette & K. Voss** HKV_1981

linear off-line DSS algorithm: linguistic approach

1982 **L.D. Wu** W_1982

linear off-line DSS algorithm: linguistic approach (minor flaw)

1982 **C.E. Kim** K_1982

brief sketch of linear off-line CSS algorithm
(based on Sklansky's convex hull algorithm)

1982 **E. Creutzburg, A. Hübler, & V. Wedler** CHW_1982a/b

two linear on-line DSS algorithms:
(a) linguistic approach and (b) geometric approach

1982: first linear-time online DSS recognition algorithm

$$
\begin{aligned}
CC_0 &= 11011101110111011110111011101111011101110 \\
&\quad 11110111011101110111101110111011110111 \\
s(0) &= 0, \quad n(0) = 1, \quad l(0) = 2, \quad r(0) = 3 \\
CC_1 &= 3334333433334334 \\
s(1) &= 4, \quad n(1) = 3, \quad l(1) = 3, \quad r(1) = 0 \\
CC_2 &= 2232 \\
s(2) &= 3, \quad n(2) = 2, \quad l(2) = 2, \quad r(2) = 1 \\
CC_3 &= \varepsilon
\end{aligned}
$$

CHW_1982a

syntactic code

	s	n	l	r
0	0	1	2	3
1	4	3	3	0
2	3	2	2	1

maximum *m* layers, with $n \geq \left(\frac{1}{2} + \frac{1}{4}\sqrt{2}\right)\left(1 + \sqrt{2}\right)^m - 2$

$k = 0$

1 if $T_1(k,d)$ then goto 10
if $T_2(k,d)$ then goto 20
if $T_3(k,d)$ then goto 30
if $T_4(k,d)$ then goto 40
if $T_5(k,d)$ then goto 50
goto 100

10 $n(k) = d, \quad l(k) = 1,$ return "yes"

20 if $T_{2.1}(k,d)$ then goto 21
if $T_{2.2}(k,d)$ then goto 22
goto 100

21 $l(k) = l(k) + 1,$ return "yes"

22 $s(k) = d,,$ return "yes"

30 $s(k) = n(k), \quad n(k) = d, \quad l(k) = 0, \quad r(k) = 2$
return "yes"

40 $r(k) = r(k) + 1,$ return "yes"

50 $d = r(k), \quad r(k) = 0, \quad k = k + 1,$ goto 1

100 for $m = 0$ until $k - 2$ do $r(m) = s(m+1)$
if $k \neq 0$ then $r(k-1) = d$
return "no"

$T_1(k,d) : n(k) = -1 \wedge s(k) = -1 \wedge$
$[k > 0 \rightarrow l(k-1) \leq d+1 \wedge r(k-1) \leq d+1]$

$T_2(k,d) : n(k) \neq -1 \wedge s(k) = -1 \wedge T_{2.1}(k,d) \wedge T_{2.2}(k,d)$

$T_{2.1}(k,d) : d = n(k)$

$T_{2.2}(k,d) : \mathbf{N}(k,d,n(k)) \wedge [k > 0 \rightarrow$
$\{l(k-1) \leq n(k) \vee (l(k-1) = d \wedge l(k) \neq 0\} \wedge$
$\{r(k-1) \leq n(k) \vee (r(k-1) = d \wedge r(k) \neq 0\}]$

$T_3(k,d) : d = s(k) \wedge r(k) = 0 \wedge$
$l(k) = 1 \wedge s(k+1) = -1 \wedge n(k+1) \leq 1 \wedge [k > 0 \rightarrow$
$l(k-1) \leq s(k) \wedge \{r(k-1) \leq s(k) \vee r(k-1) = n(k)\}]$

$T_4(k,d) : d = n(k) \wedge [s(k+1) = -1 \rightarrow r(k) \leq n(k+1)] \wedge$
$[s(k+1) \neq -1 \rightarrow r(k) + 1 \leq n(k+1) \vee$
$\{r(k) + 1 = s(k+1) \wedge r(k+1) \neq 0\}]$

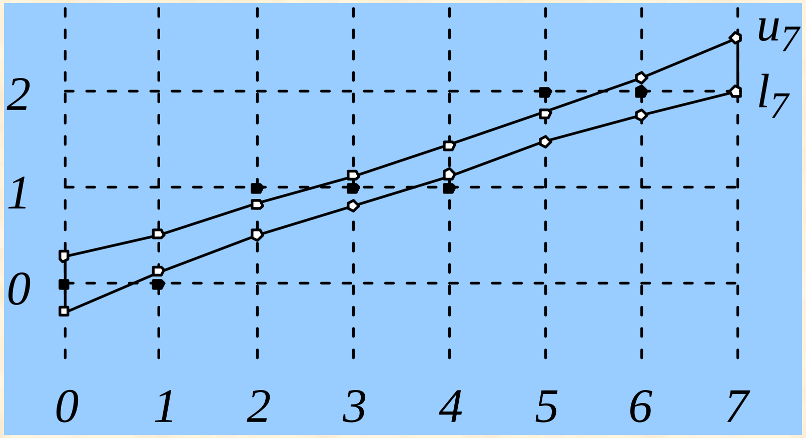
$T_5(k,d) : d = s(k) \wedge r(k) \neq 0$

CHW_1982a as published in 1982

O(*n*) steps for input sequences of length *n*, i.e. any new chain code requires *O*(1) operations ON AVERAGE

1982: 2nd linear-time online DSS recognition algorithm

CHW_1982b



digitization polygon union of all possible preimages

basic idea: continue as long as the digitization polygon is not empty

REVIEW OF SOME ALGORITHMS

possible **design principles**
(based on one of the `iff'-characterization theorems):

(C1) original definition of a DSS (grid-intersection digitization)

CHW_1982b

(C2) characterization by pairs of tangential lines

(C2.1a) - Theorem 4 (Reveilles), DSS
(C2.1b) - Corollary 1, DSS
(C2.2) - Theorem 5 (Kovalevsky), 4-DSS
(C2.3) - Theorem 6 (Fam&Sklansky), CSS

(C3) equivalence with chord property (Rosenfeld), DSS

(C4) DSS property (Hübler et al, Wu) - linguistic approach

CHW_1982a

1983 **S. Shlien** S_1983

linear off-line DSS algorithm: linguistic approach **(C4)**

1985 **T.A. Anderson & C.E. Kim** AK_1985

sketch of linear off-line DSS algorithm establishing **(C2.1b)**

1988 **E. Creutzburg, A. Hübler, O. Sykora** CHS_1988a

linear on-line DSS using **(C1)** for specifying a separability problem for monotone polygons

1988 **E. Creutzburg, A. Hübler, O. Sykora** CHS_1988b

linear on-line DSS algorithm based on **(C2.1b**), independent of AK_1985

1990 **V.A. Kovalevsky** K_1990

brief sketch of linear on-line DSS establishing **(C2.3)**

1991 **A.W.M.Smeulders & L. Dorst** SD_1991

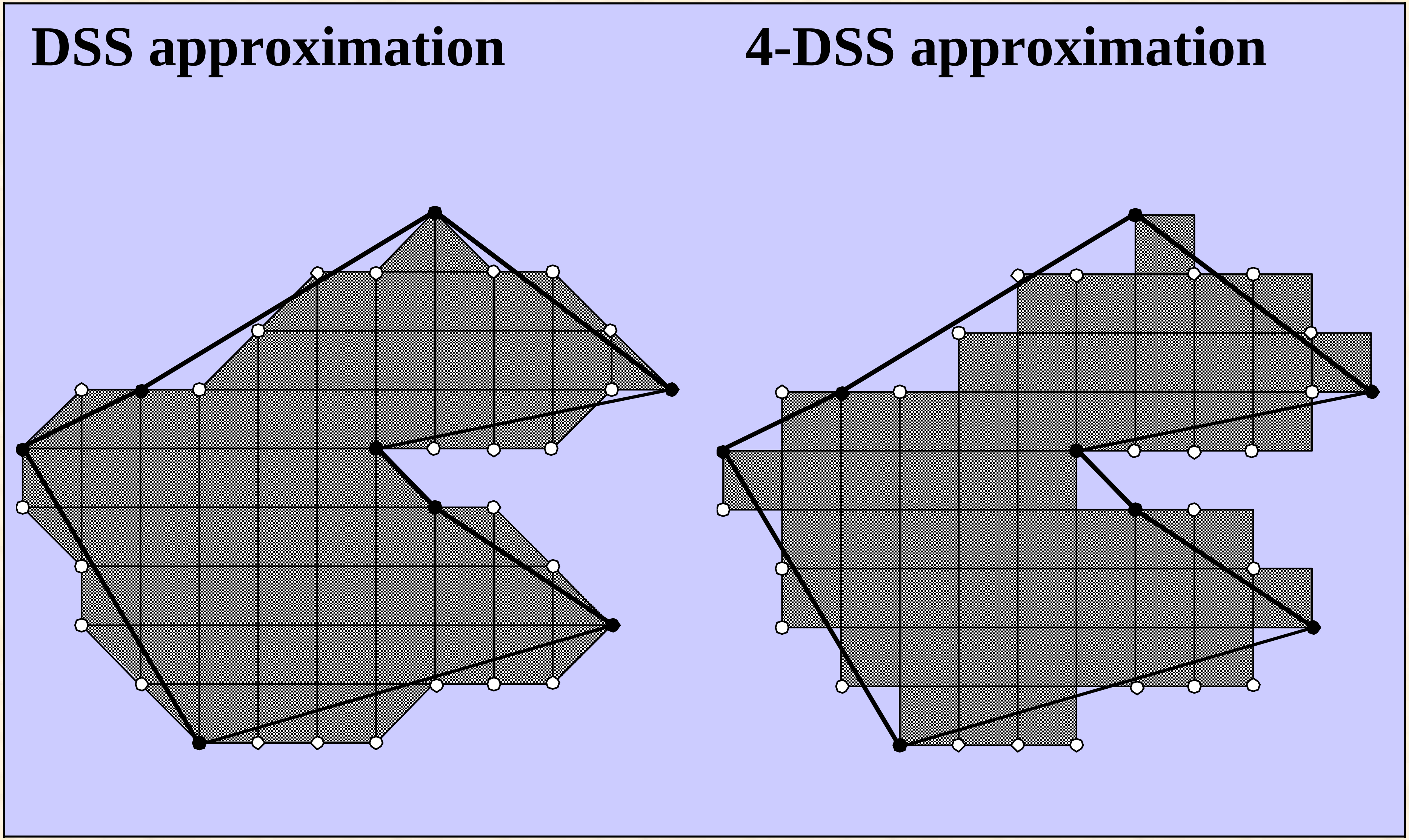
linear off-line DSS following **(C4)** and correcting W_1992

1995 **I. Debled-Rennesson & J.-P. Reveilles** DR_1995

linear on-line DSS establishing **(C2.1a)** correcting W_1992 many more -

and no comparative evaluation so far !!!!!!!

counter-clockwise (8 vertices as for clockwise)



Open Problems

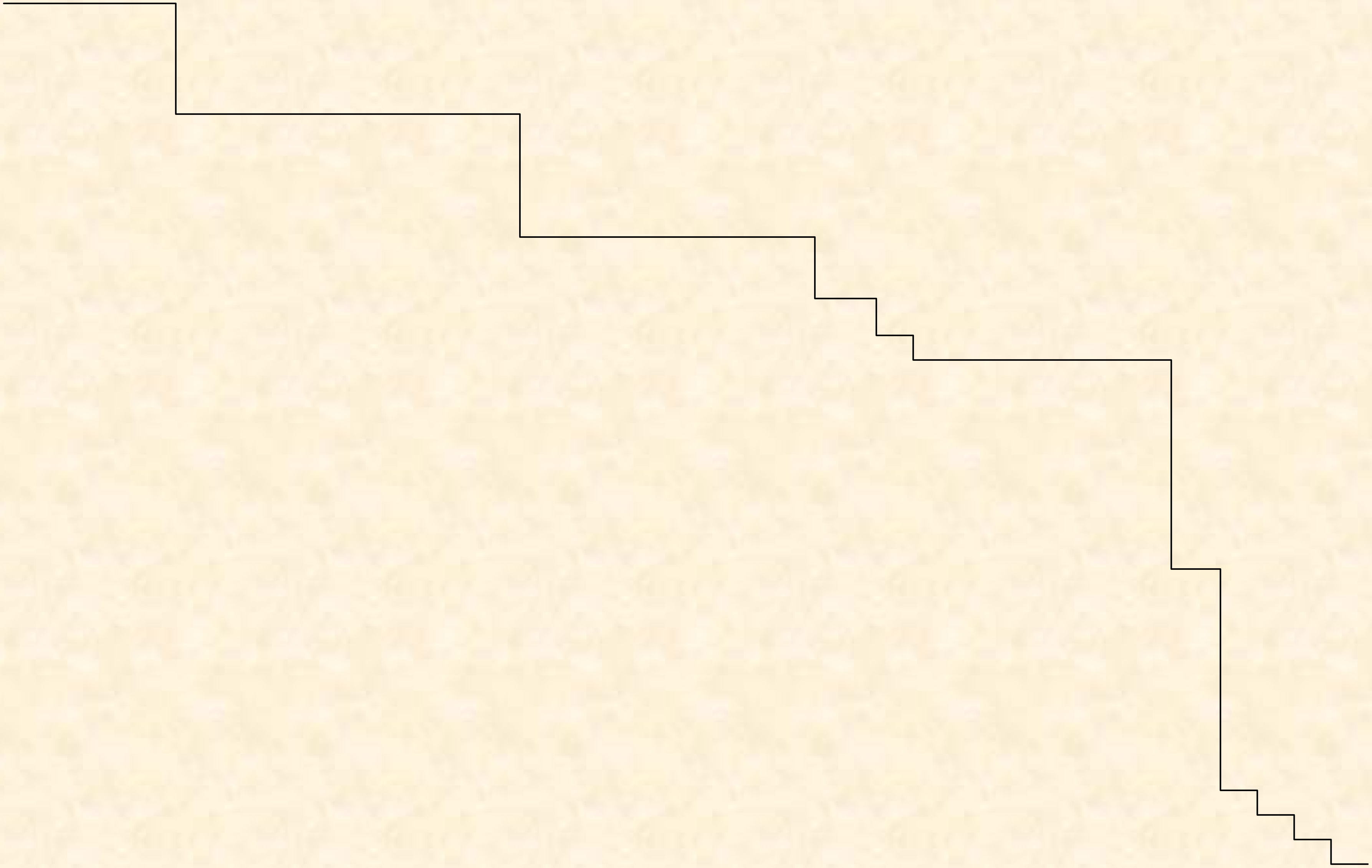
DSS ANALYSIS

- **segmenting a digital arc into a minimal number of DSSs**

ALGORITHMS

- **comparative evaluation**

  (measured run-time for DSS recognition)



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