Aspects of free groups

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Abstract. If n-tuples $\overline{g}, \overline{h}$ in a rank 2 free group satisfy the same existential formulas, then there is an automorphism taking $\overline{g}$ to $\overline{h}$.

Fix a prime p. Then F_k is, up to isomorphism, the only k–generated group having all finite k-generated p-groups as homomorphic images.

The theory of nonabelian free groups has no prime model.

1. Introduction

This works contributes to the understanding of the model theory and the structure of free groups. In the first two sections, we study the extent to which the first–order properties of an n–tuple $\overline{g}$ in a free group F determine its orbit. We show that in F_2, if n–tuples $\overline{g}, \overline{h}$ satisfy the same existential formulas, then there is an automorphism of F_2 taking $\overline{g}$ to $\overline{h}$. In particular, F_2 is ω–homogeneous (as defined in model theory). We present partial results along these lines for other free groups. In the third Section, we give a characterization of F_k. Fix a prime p. Then F_k is the only k–generated group having all finite k–generated p-groups as homomorphic images. In [**1**, Question (F14)], it was asked whether any finitely generated (f.g.) residually finite group G with the same finite homomorphic images as F_k is isomorphic to F_k. By our result, this is true if G has rank at most k.

Understanding f.g. groups through their finite images has been a recurrent theme. For instance, both Wehrfritz and Robinson proved that a f.g. solvable group is nilpotent iff all its finite images are nilpotent (see [**9**, Thm 15.5.3]).

A structure $\mathbf{A}$ is a *prime model* of a theory T if it is the least model of T, in the sense that $\mathbf{A}$ is elementarily embedded into any other model of T. For instance, $(\mathbb{N}, +, \times)$ is a prime model of $\mathrm{Th}(\mathbb{N}, +, \times)$. The concept of a prime model is basic in model theory. In the final section we observe how the recent result of Kharlampovich and Myasnikov [**5**] that all nonabelian free groups have the same first-order theory implies that this theory has no prime model. This contrasts with a result of the author [**8**, Cor. 5.3]: the free step-2 nilpotent group of rank 2 is a prime model of its theory.

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2. Test tuples are ∃–determined

We begin with some definitions. We use the first–order language with symbols $\{\circ, \ ^{-1}, 1\}$ for the group operation, inverse and the neutral element. An ∃–formula is a formula $\exists z_1 \ldots \exists z_r\ \psi$, where ψ is a quantifier free formula. If ψ contains no negation sign, the formula is *positive*. Let G be a countable group. If $\overline{g}, \overline{h} \in G^n$, we write $\overline{g} \equiv_\exists \overline{h}$ if $\overline{g}, \overline{h}$ satisfy the same ∃–formulas. An *n-orbit* of G is an orbit under the action of $\mathrm{Aut}(G)$ on G^n.
Given $\overline{g}$, if $\overline{h}$ is in the same n–orbit as $(\overline{g})$, then clearly $\overline{h}$ satisfies the same first–order formulas as $\overline{g}$. If the converse holds, $\overline{g}$ is called *first–order determined.* The following is a stronger property: $\overline{g}$ is ∃–*determined* if each $\overline{h}$ such that $\overline{h} \equiv_\exists \overline{g}$ is in the same n–orbit as $\overline{g}$.
Consider $G = F_k$. Next we define sets $T(\overline{g})$ and $T^*(\overline{g})$ (for $k = 2$) which capture how the n–tuple $\overline{g}$ can be expressed by terms of the group language applied to elements $u_1, \ldots, u_k$.

DEFINITION 2.1. *Let $G = F(x_1, \ldots, x_k)$ be the free rank k group and let $\overline{g} \in G^n$.*
- (i) $T(\overline{g}) = \{t_1, \ldots, t_n \in G : \exists u_1, \ldots, u_k \in G \bigwedge_{1 \le i \le n} g_i = t_i(u_1, \ldots, u_k)\}$.
- (ii) *If $k = 2$, let* $T^*(\overline{g}) = \{t_1, \ldots, t_n \in G :$
  $\exists u_1, u_2 \in G\ ([u_1, u_2] \neq 1\ \&\ \bigwedge_{1 \le i \le n} g_i = t_i(u_1, u_2))\}$.

Clearly, "$t_1, \ldots, t_n \in T(\overline{g})$" can be expressed by a positive ∃–formula about $\overline{g}$, and "$t_1, t_2 \in T^*(\overline{g})$" can be expressed by an ∃-formula.

LEMMA 2.2.
- (i) *$T(\overline{g}) \subseteq T(\overline{h})$ iff there is an endomorphism α of F_k such that $\alpha(g_i) = h_i$ $(1 \le i \le n)$.*
- (ii) *In case $k = 2$, $T^*(\overline{g}) \subseteq T^*(\overline{h})$ iff there is an monomorphism α of G such that $\alpha(g_i) = h_i$ $(1 \le i \le n)$.*

Proof. The direction from right to left is trivial for both (i) and (ii) . To prove the other direction, choose a base $b_1, \ldots, b_k$ for F_k. For (i), choose $t_1, \ldots, t_n$ such that $g_i = t_i(b_1, \ldots, b_k)$ for each i. Since $T(\overline{g}) \subseteq T(\overline{h})$, we may pick $u_1, \ldots, u_k \in G$ such that $h_i = t_i(u_1, \ldots, u_k)$. If α is the endomorphism of G given by $\alpha(b_i) = u_i$, then $\alpha(g_i) = h_i$ $(1 \le i \le n)$.
For (ii), since $T^*(\overline{g}) \subseteq T^*(\overline{h})$, we can choose $u_1, u_2 \in G$ as above such that, in addition, $[u_1, u_2] \neq 1$, so that α is a monomorphism. ♢

The following was studied by Turner [**10**]

DEFINITION 2.3. *A tuple $\overline{g} = g_1, \ldots, g_n$ of elements of a group G is called a* test tuple *if each endomorphism $\varphi : G \to G$ fixing all elements g_i is an automorphism. The tuple is called a* test tuple for monomorphisms *if each monomorphism $\varphi : G \to G$ fixing all elements g_i, $1 \le i \le n$, is an automorphism.*

For instance, if $\overline{g}$ generates G, or, more generally, a subgroup of finite index of G, then $\overline{g}$ is a test tuple.

PROPOSITION 2.4. *Suppose $\overline{g}$ is a test tuple in F_k. Then $\overline{g}$ is ∃–determined.*

Proof. If $\overline{g} \equiv_\exists \overline{h}$, then $T(\overline{g}) = T(\overline{h})$. By Lemma 2.2, we can choose endomorphisms α, β of F_k such that $\alpha(g_i) = h_i$ and $\beta(h_i) = g_i$ $(1 \le i \le n)$. Clearly $\beta \circ \alpha$ fixes $\overline{g}$. Therefore α is an automorphism, showing that $\overline{g}, \overline{h}$ are in the same n–orbit. ♢

COROLLARY 2.5. *If $\overline{g}$ generates a subgroup of finite index of F_k, then $\overline{g}$ is $\exists$–determined.*

COROLLARY 2.6. *If $\alpha : F_k \mapsto F_k$ preserves $\exists$–formulas in both directions, then α is an automorphism.*

Proof. $\alpha(\overline{b})$ satisfies the same $\exists$–formulas as $\overline{b}$ does, so by Corollary 2.5, $\alpha(\overline{b})$ must be base. ♢

Let V be a variety. The definitions and the proofs of our results above, except (ii) of Lemma 2.2, carry over to the relatively free group $F_V(k)$ for the variety. Thus, a base, or more generally, a tuple generating a subgroup of finite index is $\exists$-determined in $F_V(k)$.

3. F_2 is $\exists$–homogeneous

A group G is called *$\exists$–homogeneous* if for all $n \geq 1$, all n –tuples in G are $\exists$–determined. To prove that F_2 is $\exists$–homogeneous, we need (ii) of the following result. The result was obtained by Turner [**10**] for 1-tuples, but the proof carries over to the general case. Recall that a *retract* of a group G is a subgroup $Im(\rho)$, where $\rho : G \mapsto G$ is an endomorphism such that $\rho \circ \rho = \rho$.

THEOREM 3.1 ([**10**]). (i) *$\overline{g}$ is a test tuple in F_k iff, for each proper retract R of F_k, there is an i such that $g_i \notin R$.*

(ii) *$\overline{g}$ is a test tuple for monomorphisms in F_k iff, for each proper free factor L of F_k, there is an i such that $g_i \notin L$.*

THEOREM 3.2. *F_2 is $\exists$–homogeneous.*

Proof. Suppose $\overline{g}, \overline{h}$ are n–tuples in F_2 and $\overline{g} \equiv_\exists \overline{h}$. Then $T^*(\overline{g}) = T^*(\overline{h})$. First assume $\overline{g}$ is a test tuple for monomorphisms. By (ii) of Lemma 2.2, we can choose monomorphisms α, β of F_2 such that $\alpha(g_i) = h_i$ and $\beta(h_i) = g_i$ $(1 \leq i \leq n)$. Then $\beta \circ \alpha$ fixes $\overline{g}$, so α is an automorphism.

Otherwise, by Theorem 3.1, the subgroup generated by $\overline{g}$ is contained in a proper free factor of F_2. Thus there is a primitive element c such that $g_i = c^{r_i}$ for some $r_i \in \mathbb{Z}$ $(1 \leq i \leq n)$. Now we apply the following fact.

LEMMA 3.3. *Suppose $c \in F_k$ is a primitive element and $\alpha(d) = c$ for a monomorphism α of F_k. Then d is primitive as well.*

Proof. Pick a base $c, b_2, \ldots b_k$ of $F(k)$. Let U be the range of α. We claim that c is primitive in U. Choose a Nielsen reduced base $u_1, \ldots, u_k$ of U (see Lyndon and Schupp [**6**]). Since $c \in U$, a non-cancelling product of m elements of this base or their inverses is a word of length $\geq m$ [**6**, Prop. 2.13]. Then necessarily $c = u_i$ or $c = u_i^{-1}$ for some i. Thus c is primitive in U, and hence $d = \alpha^{-1}(c)$ is primitive in F_k. ♢

Since $\overline{g} \equiv_\exists \overline{h}$, there is $d \in F_2$ such $h_i = d^{r_i}$ for each $i < n$. Since $T^*(g_1) = T^*(h_1)$ and the extraction of roots is unique in a free group, $T^*(c) = T^*(d)$. By (ii) of Lemma 2.2 there is a monomorphism α such that $\alpha(d) = c$. Then, by Lemma 3.3, d is primitive. Thus $\overline{g}, \overline{h}$ are in the same n–orbit. ♢

We do not know whether the free groups F_k, $k \geq 3$, are also $\exists$–homogeneous (or ω–homogeneous at all). By Proposition 2.4 and (i) of Theorem 3.1, it remains to be shown that $\overline{g}$ is first–order determined when $\overline{g}$ is contained in a proper retract R.

(The following might be useful here: by [**2**], retracts in f.g. free groups are closed under intersection, so that there is a unique least retract R containing $\overline{g}$.)

4. Finite homomorphic images

Recall that if $V \subseteq F(x_1, x_2, \ldots)$ and A is group, then $V(A)$ is the (fully invariant) subgroup of A generated by all elements $v(a_1, \ldots, a_r)$, where $v \in V \cap F(x_1, \ldots, x_r)$ and $a_1, \ldots, a_r \in A$. For instance, if $V = \{[x_1, \ldots, x_r]\}$, then $V(A)$ is the r-th member of the lower central series, also denoted by $\gamma_r A$. The following lemma is easily verified.

LEMMA 4.1. (i) *If A, B are groups and $f : A \to B$ is an epimorphism, then f maps $V(A)$ onto $V(B)$.*
(ii) *If $N \triangleleft A$, then $V(A/N) = (V(A)N)/N$.*

$\diamondsuit$

THEOREM 4.2. *F_k is, up to isomorphism, the only k–generated group having all finite nilpotent k–generated groups as homomorphic images.*

Proof. Let $F = F_k$. It suffices to show the following: if $a \in F_k - \{1\}$ generates the normal subgroup N, then there is a k–generated finite p-group Q which is not a quotient of F/N.
Since F is residually nilpotent [**4**, Thm 14.2.2], there is a least $c > 1$ such that $a \notin \gamma_c F$. Let $H = F/\gamma_c F$ be the free step-c nilpotent group of rank k, and let $\widetilde{N} = (N\gamma_c F)/\gamma_c F$. It suffices to find a finite step-c nilpotent p-group Q of rank $\leq k$ which is not a quotient of $F/(\gamma_c FN) = H/\widetilde{N}$. For suppose such a Q is a quotient of F/N, and choose an onto homomorphism $f : F/N \to Q$. By (i) of Lemma 4.1, the induced map $(F/N)/\gamma_c(F/N) \to Q/\gamma_c Q = Q$ is onto. But, by (ii) of the same Lemma, the group on the left equals $F/(N\gamma_c F)$.
Let $A = \gamma_{c-1}H$. Then $\widetilde{a} = (\gamma_c F)a \in A$, and A is contained in (and in fact, equals) the center of H. Clearly A is f.g., since f.g. nilpotent groups satisfy the maximum condition for subgroups. Because free nilpotent groups are torsion free [**7**, 31.62], A is a f.g. free abelian group. Choose a basis $b_1, \ldots, b_m$ of A in a way that $\widetilde{a} = b_1^q$ for some $q \in \mathbb{N}$, and pick r so that $p^r > q$.

CLAIM 4.3. *For each prime p and each $r \geq 1$, there is step-c nilpotent finite p-group Q of rank $\leq k$ such that $\gamma_{c-1}Q \cong \mathbb{Z}_{p^r}^m$.*

The group Q given by the Claim is as desired, namely Q is not a quotient of $H/\widetilde{N}$. For suppose so. Then, by (i) of Lemma 4.1, $A/\widetilde{N} = \gamma_{c-1}(H/\widetilde{N})$ maps onto $\gamma_{c-1}Q \cong \mathbb{Z}_{p^r}^m$ via an epimorphism F. But $A/\widetilde{N} \cong \mathbb{Z}_q \times \mathbb{Z}^{m-1}$. If p does not divide q this gives an immediate contradiction. Otherwise $q = p^s$ where $s < r$. Let c be a generator for $\mathbb{Z}_q$ in $\mathbb{Z}_q \times \mathbb{Z}^{m-1}$, and let $L = \ker F\langle c\rangle$. Then we obtain an epimorphism $(\mathbb{Z}_q \times \mathbb{Z}^{m-1})/L \to \mathbb{Z}_{p^r}^m/F(L)$. The group on the right has rank m, since $F(L)$ is contained in the subgroup of elements of order $\leq p^{r-1}$). Contradiction.
Proof of the Claim. We obtain Q as a quotient of H. As above, let $b_1, \ldots, b_m$ be a basis of the free abelian group $A = \gamma_{c-1}H$, and let $R = p^r A$. Since R is contained in the center, $R \triangleleft H$. Since H/R is a f.g. nilpotent group, it is residually finite (see [**4**, Ex. 17.2.8]). So we may choose M, $R \leq M \triangleleft H$ such that H/M is finite, and none of the finitely many nontrivial linear combinations of the Rb_i is in M/R. Let $Q_0 = H/M$. By (i) of Lemma 4.1, $\gamma_{c-1}Q_0 = (M\gamma_{c-1}H)/M$ is the image of $\gamma_{c-1}(H/R)$ under the natural map $\eta : H/R \to Q_0$ (whose kernel is M/R). Then,

since the elements $\eta(Rb_i)$ are linearly independent, $\gamma_{c-1}Q_0 \cong \mathbb{Z}_p^m$. Finally, Q_0 is the direct product of its q–Sylow subgroups. Let Q be the p–Sylow subgroup, then Q is a quotient of Q_0 and $\gamma_{c-1}Q = \gamma_{c-1}Q_0$. $\diamondsuit$

5. Prime models

Using Kharlampovich and Myasnikov [**5**], we observe that the theory of non-abelian free groups has no prime model. We use the following well-known fact.

PROPOSITION 5.1. *Let* $\mathbf{A}$ *be a countable structure. Then* $\mathbf{A}$ *is a prime model of its theory iff each* n*–orbit is first–order definable without parameters in* $\mathbf{A}$. $\diamondsuit$

In $F(a,b)$, the orbit of $[a,b]$ is the union of the conjugacy classes of $[a,b]$ and $[b,a]$, while in $F_k, k > 2$, each nontrivial orbit splits into infinitely many conjugacy classes (see [**6**, I.5.1]). This implies

LEMMA 5.2. *If the 1–orbit of* $[a,b]$ *is definable without parameters in* $F(a,b)$*, then* $F(a,b)$ *does not satisfy the same sentences as* F_k *for* $2 < k$.

Proof. If the orbit is definable by a formula $\varphi(x)$, the sentence saying that there are exactly two conjugacy classes of elements satisfying φ is true in $F(a,b)$, but in no group F_k for $k > 2$ (since the set of elements satisfying φ contains an orbit). $\diamondsuit$

THEOREM 5.3. $\mathrm{Th}(F_2)$ *has no prime model.*

Proof. If there is a prime model P, it can be elementarily embedded into F_2. Then P is a free group. If P has rank > 2, then for suitable words t_i in F_2, three generators g_0, g_1, g_2 of P satisfy the existential formula $\exists x \exists y \bigwedge_{i=0\ldots 2} g_i = t_i(x,y)$ in F_2, but not in P. Thus P is a free group $F(a,b)$ of rank 2. By Proposition 5.1, the orbit of $[a,b]$ is definable without parameters, whence $F(a,b)$ does not satisfy the same sentences as F_k for $2 < k$. Using Lemma 5.2, this contradicts the result of Kharlampovich and Myasnikov [**5**]. $\diamondsuit$

An important question which remains in the model theory of free groups is whether the theory of a non-abelian free group is stable. Stability would mean that there is no formula $\varphi(\overline{x}, \overline{y})$, where $\overline{x}, \overline{y}$ are tuples of variables of the same length, which determines linear orders on (not necessarily definable) arbitrarily large finite sets of tuples. It is known that the theory is not superstable (see [**3**, p. 694]).

References

[1] G. Baumslag, A. Myasnikov, and V. Shpilrain. Open problems in combinatorial group theory. *Contemporary Mathematics*, 250:1–27, 1999. Also see Magnus homepage at http://zebra.sci.ccny.cuny.edu/web.

[2] G. Bergman. Supports of derivations, free factorizations, and ranks of fixed subgroups in free groups. *Trans. Amer. Math. Soc.*, 351:1531–1550, 1999.

[3] Wilfrid Hodges. *Model Theory.* Enzyklopedia of Mathematics. Cambridge University Press, Cambridge, 1993.

[4] M. Kargapolov and J. Merzljakov. *Fundamentals of the theory of groups.* Springer–Verlag, 1979.

[5] O. Kharlampovich and A. Myasnikov. Tarski's problem about the elementary theory of free groups has a positive solution. *Electronic research announcements of the AMS*, 4:101–108, 1998.

[6] R.C. Lyndon and P.E. Schupp. *Combinatorial group theory.* sv, 1977.

[7] H. Neumann. *Varieties of groups*, volume 37 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* sv, 1967.

[8] A. Nies. Separating classes of groups by first–order formulas. To appear in Intern. J. Algebra Computation.

[9] D. Robinson. *A course in the theory of groups.* Springer–Verlag, 1988.

[10] E. Turner. Test words for automorphisms of free groups. *Bull. London Math. Soc.*, 28:255–263, 1996.

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