A Uniformity of Degree Structures*

André Nies

Abstract

We isolate a fact which holds for various degree structures arising from recursion theory and complexity theory and makes it possible to prove the undecidability of their theories in a more uniform way than in the original proofs, namely by interpreting the lattice of Σ^0_k–sets for some k.

Dedicated to Kena Prosper

1 Introduction

A reducibility $\leq_r$ gives a method to compare sets of natural numbers with respect to their computational complexity. Reducibilities are preorderings on sets; the r-degree of a set X, denoted by $\deg_r(X)$, is the equivalence class of all sets which have the same complexity as X. The r-degrees form an uppersemilattice (u.s.l.). Here we discuss the restricted u.s.l. $\mathbf{R}_r$ of r-degrees of recursively enumerable sets with respect to reducibilities arising from recursion theory (e.g. Turing- or many-one-reducibility), as well as the partial order of r-degrees of the recursive sets with respect to subrecursive reducibilities (e.g. polynomially bounded Turing- or many-one reducibility). Investigations of these degree structures usually show that they behave in a different way rather than leading to uniformities. For instance, some r.e. degree structures possess minimal degrees whereas others are dense. An elementary difference between any two r.e. degree structures (viewed as

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partial orders) exists already at the two-quantifier level, except for the case of r.e. *tt*- and *btt*-degrees. Undecidability of the theories was shown by widely varying coding schemes, relying on specific properties of the reducibility in question. Here we prove that a fact, called the Exact Degree Theorem, which only depends on a parameter $k \in \{2, 3, 4\}$, is shared by all degree structures discussed in this paper. This fact suggests a uniform coding scheme for proving undecidability of the given degree structure, namely interpreting with parameters the lattice $\mathcal{E}^k$ of Σ^0_k-sets under inclusion. The number k could be called the *arithmetical complexity* of the reducibility, namely k is the least number such that $\{\langle i, j\rangle : X_i \leq_r X_j\}$ is in Σ^0_k, if we assume an appropriate effective enumeration (X_i) of the sets in question. In the recursion theoretic case, we let $X_i = W_i$. In the complexity theoretic case, we restrict ourselves to some recursively presented ideal of the u.s.l. of recursive r-degrees: thus we assume a uniformly recursive listing (X_i) of sets of words in $\{0, 1\}^*$ such that $\{\deg_r(X_i) : i \in \omega\}$ forms an ideal. For instance, (X_i) could be an effective listing of the primitive recursive or of the elementary recursive sets. The arithmetical complexity is 2 for subrecursive reducibilities, 3 for many-one, *btt*, *tt* and *wtt*-reducibility, and 4 for Turing- and Q-reducibility.

The two notions involved in an Exact Degree Theorem are independent sequences and effectivity of sequences. A sequence $(\mathbf{a}_i)_{i\in\omega}$ of elements of a degree ordering is called *independent* if $\mathbf{a}_n \not\leq \sup_{i\in F} \mathbf{a}_i$ for every finite set $F \subseteq \omega$ and each $n \in \omega - F$, and *effective* if $\mathbf{a}_i = \deg_r(X_{f(i)})$ for some recursive function f. For an effective sequence $(\mathbf{a}_i)$ and an element $\mathbf{b}$ of the degree structure under consideration, the set $\{i : \mathbf{a}_i \leq \mathbf{b}\}$ is Σ^0_k. The Exact Degree Theorem states the converse, if the sequence is independent:

> *For independent sequences $(\mathbf{a}_i)$, each Σ^0_k-set S can be represented as $\{i : \mathbf{a}_i \leq \mathbf{b}\}$ for some degree $\mathbf{b}$.*

We call $\mathbf{b}$ an *exact degree* for $\{\mathbf{a}_i : i \in S\}$. The coding scheme to prove undecidability of the theory relies on the assumption that for some independent effective sequence $(\mathbf{a}_i)$, the set $\{\mathbf{a}_i : i \in \omega\}$ is definable from parameters, using a formula $\phi(x; \overline{\mathbf{p}})$ (where $\overline{\mathbf{p}}$ is the list of parameters). Represent the number i by the degree $\mathbf{a}_i$ and a Σ^0_k-set S of numbers by a degree $\mathbf{b}$, i.e. $S = \{i : \mathbf{a}_i \leq \mathbf{b}\}$. Then inclusion of sets represented by the degrees $\mathbf{b}, \mathbf{c}$ is definable from parameters: the formula

$$\phi(b, c; \overline{\mathbf{p}}) \equiv^{\text{def}} (\forall a)[\phi(a; \overline{\mathbf{p}}) \to (a \leq b \to a \leq c)]$$

expresses that $\{i : \mathbf{a}_i \le \mathbf{b}\} \subseteq \{i : \mathbf{a}_i \le \mathbf{c}\}$. So $\mathcal{E}^k$ is elementarily definable with parameters in the degree structure, and undecidability follows by model theoretic methods ([BM81], see also [ASNA92]) from the fact, proved in [Her84], that $\mathcal{E}^k$ has a hereditarily undecidable theory.

The existence of such a definable sequence can be shown for all recursion theoretic reducibilities and for the complexity theoretic reducibilities $\le_m^p, \le_T^p$ if the recursively presentable ideal is large enough. To prove the existence, specific properties of the reducibility are exploited, whereas the proof of the Exact Degree Theorems can be given for each k uniformly. See [ASNA92] for *wtt*-, [NS95] for *tt*-, [Nie92] for *btt*-, [Nie93] for *m*- and [NSS] for *T*-reducibility. Also see [ASN92] and [SS91] for $\le_m^p, \le_T^p$, respectively.

The results are also interesting from the point of view of definability: $\mathcal{E}^k$ is elementarily definable in a very direct way, since each degree represents a Σ_k^0- set. An extended version of the Exact Degree Theorems can be proved which shows that also the inclusion relation on Σ_k^0-sets is given directly by the degree ordering: if $S \subseteq T$ are Σ_k^0-sets, then $S = \{i : \mathbf{a}_i \le \mathbf{b}\}$ and $T = \{i : \mathbf{a}_i \le \mathbf{c}\}$ for some degrees $\mathbf{b}, \mathbf{c}$ such that $\mathbf{b} \le \mathbf{c}$.

The idea of obtaining undecidability by defining $\mathcal{E}^k$ was first applied in [ASNA92] for the r.e. *wtt*-degrees and in [ASN92] for the polynomial time *m*-degrees of recursive sets. There, exact pairs for ideals instead of exact degrees were used to represent the Σ_k^0-sets, so the coding is a little less direct. For definitions and basic facts about reducibilities in recursion theory, see [Odi89], and for reducibilities in complexity theory see [BDG88].

The degree structures considered here fall into three groups, determined by the arithmetical complexity k. For each group, there is one version of the Exact Degree Theorem. The general scheme for the proofs is that, for each i, coding requirements try to code A_i into B, while, for every r-reduction procedure $[n]$, diagonalization requirements try to cause $A_i \neq [n]^B$. If $i \in S$, then some coding requirement succeeds. Otherwise, for each n some diagonalization requirement associated with $[n]$ succeeds. Using the fact that S is a Σ_k^0-set we can choose a uniformly recursively enumerable (u.r.e.) sequence of sets (X_k) so that the value of $S(i)$ depends in an appropriate way on the subsequence $(X_{\langle i,n\rangle})_{n\in\omega}$. The actions of the Requirements concerned with A_i are controlled by this subsequence. The groups considered are

1. $k = 2$: recursively presented degree structures given by subrecursive reducibilities, e.g. polynomially time bounded reducibilities and LOGSPACE reducibility;

2. $k = 3$: r.e. m-, btt-, tt- and wtt-degrees, as well as r.e. degrees with respect to other tt-like reducibilities (see [Odi89]);

3. $k = 4$: r.e. Turing- and Q-degrees.

In Sections 2, 3 and 4, we give a proof of the Exact Degree Theorem for each group. We also briefly consider the enumeration degrees of Σ^0_2-sets, which possess arithmetic complexity 4 and satisfy the Exact Degree Theorem for $k = 4$. However, they will be treated like the members of group 2, using a similar construction relativized to $\emptyset'$.

It is possible to handle the reducibilities in the groups 1 and 2 on the basis of very weak axioms. First we introduce some notation:

Notation. We fix an effective pairing function $\langle , \rangle$ on natural numbers, with effective projections, which is monotonic in its arguments. $X^{[i]}$ denotes the set $\{x : \langle x, i\rangle \in X\}$, and $\bigoplus_{i\in F} C_i$ stands for $\{\langle x, i\rangle : x \in C_i \wedge i \in F\}$. In the complexity theoretic case, we fix $\{0, 1\}$ as the alphabet. Numbers are written in binary, $\langle , \rangle$ is a polynomial pairing function, monotonic in its arguments with polynomial time computable projections such that $|\langle x, y\rangle| \geq |x|, |y|$.

For all reducibilities we require the following 4 axioms:

(A1) The reducibility is a preordering on sets of numbers (words, in the complexity theoretic case).

(A2) $X \leq_r Y$ does not depend on finite variants of X and Y.

(A3) The degree structure induced by the preordering on all sets is an u.s.l., the supremum is given by

$$\sup_{1\leq i\leq n} \deg_r(A_i) = \deg_r(\oplus_{1\leq i\leq n} A_i).$$

(A4) As a fourth axiom, we require in the recursion theoretic case that

$$X \leq_m Y \Rightarrow X \leq_r Y.$$

In the complexity theoretic case, in order to include honest reducibilities and LOGSPACE reducibility, we only require that, for each word c, an m-reduction via the map $x \mapsto \langle x, c\rangle$ is also an r-reduction.

The fifth axiom is also different for group 1 and group 2. Let (M_e) be a listing of oracle Turing machines associated with the reducibility, and let $M_{e,s}(Z)(x)$ be the output of the computation of M_e with oracle Z if this computation converges in s steps, and 0 otherwise. The axiom required for group 1 is an abstraction from a property of polynomially bounded oracle computations:

(A5) $\{X : X \leq_r Y\} = \{M_{e,g_e(|x|)}(Y)(x) : x \in \omega\}$, for an effective sequence (g_e) of total recursive time bounds.

It is essential that g_e be total for each e; thus only restricted classes of time bounds like polynomials or primitive recursive functions are acceptable, but not the class of all total recursive functions. The axiom (A5) holds for any polynomial bounded reducibility extending $\leq_m^p$, but also for LOGSPACE reducibility. The axiom implies that the arithmetical complexity is bounded by 2, since

$$X_i \leq_r X_j \Leftrightarrow (\exists e)(\forall x)[X_i(x) = M_{e,g_e(|x|)}(X_j)(x)].$$

For group 2, we use the following notation: given a partially recursive function f, let $M_{e,s}(Z,f)(x)$ be $M_{e,s}(Z)(x)$, if both $f(x)$ and the computation converge in s steps and the computation uses only oracle questions $< f(x)$, and let $M_{e,s}(Z,f)(x)$ be undefined otherwise. Moreover, let

$$M_e(Z,f)(x) = \lim_s M_{e,s}(Z,f)(x)$$

(i.e. $M_e(Z,f)(x)$ is defined iff $M_{e,s}(Z,f)(x)$ is defined from some stage on). Instead of (A5), we require that

(A5*) $\{X : X \leq_r Y\} = \{M_e(Y, f_e) : e \in \omega \wedge M_e(Y, f_e) \text{ total}\}$ for an effective sequence (M_e, f_e) of oracle Turing machines and recursive bounds on the maximal oracle question asked.

It is easy to verify that all reducibilities in group 2 satisfy the axioms. The axiom (A5*) implies that the arithmetical complexity is bounded by 3, since an r.e. oracle can only change $f_e(n)+1$ times on the interval $[0, f_e(n))$:

$$W_i \leq_r W_j \Leftrightarrow (\exists e)(\forall x)(\forall t)(\exists s \geq t)[W_i(x) = M_{e,s}(W_{j,s})(x)]. \tag{1}$$

Finally, for the r.e. T-degrees, the arithmetical complexity is 4 (see [Soa87]), as well as for the r.e. Q-degrees: recall that, if C, D are r.e. sets, then $C \leq_Q D$ if there exists a recursive function g such that

$$x \in C \Leftrightarrow W_{g(x)} \subseteq D \tag{2}$$

(see [Odi89]). Then $W_i \leq_Q W_j$ is Σ^0_4 because $\{x : W_{g(x)} \subseteq D\}$ is Π^0_2. It is also easy to verify that $k = 4$ for the structure of Σ^0_2 enumeration-degrees.

The axiom systems have trivial models, so the bound computed on the arithmetical complexity may not be optimal. But for reducibilities in the groups above, independent effective sequences exist. So an application of the respective Exact Degree Theorems to a Σ^0_k-complete set shows that the bounds (with respect to the particular representations of the degree structures) are optimal.

2 The Exact Degree Theorem for reducibilities of arithmetical complexity 3

We begin with the reducibilities satisfying the axioms for group 2.

Theorem 2.1 (Exact Degree Theorem for arithmetical complexity 3) *Let $\leq_r$ be a reducibility satisfying (A5*) (e.g. m-, btt-, tt- or wtt-reducibility) and suppose that $(A_i)_{i\in\omega}$ is a u.r.e. sequence of sets such that $\deg_r(A_i)_{i\in\omega}$ is independent. Then, for each Σ^0_3-set S, there exists an r.e. set B such that, for all i,*

$$i \in S \Leftrightarrow A_i \leq_r B.$$

Notation. We assume that a sequence (M_e, f_e) is given so that (A5*) is satisfied, and write $[e]^Z(x)$ instead of $M_e(Z, f_e)(x)$. Also we write $A^{[F]}$ for $\oplus_{i\in F} A^{[i]}$. The proof contains an elementary tree construction on the binary tree $T = 2^\omega$. We use standard notation from [Soa87, p. 301]. We fix some numbering $\gamma \mapsto n(\gamma)$ of strings on T and write $\langle x, \gamma\rangle$ and $X^{[\gamma]}$ instead of $\langle x, n(\gamma)\rangle$and $X^{[n(\gamma)]}$. We also use the notation $X^{[<\gamma]}$ etc. in the obvious sense (referring to the lexicographical ordering of strings) and sometimes write $X =^* Y^{[n]}$ instead of (a.e. z)$[z \in X \Leftrightarrow \langle z, n\rangle \in Y]$ (the symbol "*a.e.*" stands for "for almost every"). The notation $X =^* Y^{[\gamma]}$ is used in a similar way.

Proof. By [Soa87, p. 66], for any Π^0_2-set P there exists a u.r.e. sequence $(X_k)_{k\in\omega}$ such that each set X_k is an initial segment of ω and, for each k,

$$k \in P \Leftrightarrow X_k = \omega.$$

Let $S \in \Sigma^0_3$. Then there exists a u.r.e. sequence (X_k) of initial segments such that, for each i,

$$i \in S \Rightarrow (\exists n)[X_{\langle i,n\rangle} = \omega] \tag{1}$$
$$i \notin S \Rightarrow (\forall n)[X_{\langle i,n\rangle} \text{ is finite}]. \tag{2}$$

We will have to guess at "$X_k = \omega$?" on a tree. To explain the strategies, we first ignore the necessity of such a guessing procedure. There are infinitely many coding requirements $C_{\langle i,n\rangle}$ which try to achieve $A_i =^* B^{[\langle i,n\rangle]}$. For each n, there is a diagonalization requirement $Q_{\langle i,n\rangle}$ associated with $[n]$, which tries to cause $A_i \neq [n]^B$. We incorporate assumptions on the sequence (X_k) into the requirements as follows:

$$\begin{array}{ll} C_{\langle i,n\rangle}: & X_{\langle i,n\rangle} = \omega \Rightarrow A_i =^* B^{[\langle i,n\rangle]} \\ Q_{\langle i,n\rangle}: & (\forall n' \leq n)[X_{\langle i,n'\rangle} \text{ is finite}] \Rightarrow A_i \neq [n]^B. \end{array}$$

The coding for $C_{\langle i,n\rangle}$ is carried out whenever $|X_{\langle i,n\rangle}|$ increases. The strategy for $Q_{\langle i,n\rangle}$ simply consists in filling up $B^{[>\langle i,n\rangle]}$: let

$$l(\langle i,n\rangle, s) = \max\{x : (\forall y < x)[A_i(y) = [n]^B(y)[s]]\}.$$

Whenever, at stage s, $l(\langle i,n\rangle, s)$ is greater than all previous values, then enumerate $[0,s)^{[>\langle i,n\rangle]}$ into B.

We sketch how to show that $Q_{\langle i,n\rangle}$ is met. Assume for a contradiction that $i \notin S$ and that $A_i = [n]^B$. By axiom (A5*), the last condition is equivalent to $\sup_s l(\langle i,n\rangle, s) = \infty$, using the same argument as for (1). By (2), the coding of the requirements $C_{\langle i,n'\rangle}, n' \leq n$, ceases at some stage. Therefore

$$B =^* B^{[\{\langle j,m\rangle : \langle j,m\rangle < \langle i,n\rangle \wedge j \neq i\}]}.$$

Inductively suppose that for each $\langle j,m\rangle < \langle i,n\rangle$, $j \neq i$, either $A_j =^* B^{[\langle j,m\rangle]}$ or $B^{[\langle j,m\rangle]}$ is finite. Then $B \leq_m \oplus_{i\in J} A_i$, where J is the finite set of first components of such pairs $\langle j,m\rangle$. Since $i \notin J$ and $A_i = [n]^B$, this contradicts the independence of the sequence (A_j).

This argument also shows that $Q_{\langle i,n\rangle}$ is finitary if $i \notin S$. However, if $i \in S$, it might be the case that $X_{\langle i,n'\rangle} = \omega$ for some $n' \leq n$, whence the coding of $C_{\langle i,n'\rangle}$ is infinitary and the argument fails. Therefore we have to equip the requirement $Q_{\langle i,n\rangle}$ with a guess at whether its hypothesis is true.

Our guessing procedure at "$X_k = \omega$?" is standard. Let $\delta_0 = \lambda$. If $s > 0$, then by induction on k, with $1 \leq k \leq s$, define $\delta_s \upharpoonright k$: if $\nu = \delta_s \upharpoonright\ k-1$

and $t < s$ is the greatest stage number such that $t = 0$ or $\nu \subseteq \delta_t$, then let $\delta_s(k-1) = p$, where $p = 0$ if $|X_{k,t}| < |X_{k,s}|$ and $p = 1$ otherwise.

Since T is finitely branching, there exists a *true path*, namely a path f through T such that, for each e, letting $\alpha = f \restriction e$, we have that (a.e. s)$[\alpha \leq \delta_s]$ and $(\exists^\infty s)[\alpha \subseteq \delta_s]$. Then

$$X_k = \omega \Leftrightarrow f(k) = 0. \tag{3}$$

For each α, $|\alpha| = \langle i, n\rangle + 1$, we introduce versions C_α, Q_α of the requirements $C_{\langle i,n\rangle}$ and $Q_{\langle i,n\rangle}$. Since at stage s, the current approximation of the true path is δ_s, by (3), the hypothesis of a requirement $C_{\langle i,n\rangle}$ or $Q_{\langle i,n\rangle}$ translates into a combinatorial condition on the guess of a version C_α, or Q_α. Hence the version can directly obtain from α the information whether, according to its guess, the hypothesis is true. If not, the version does not act at all. This eliminates the problem that occurred above: for $\alpha = f \restriction \langle i, n\rangle + 1$, if $X_{\langle i,n'\rangle} = \omega$ for some $n' \leq n$, then $\alpha(\langle i, n'\rangle) = 0$, whence Q_α never acts.

The strategies are adapted to the tree construction: C_α now tries to code A_i into $B^{[\alpha]}$, and $Q^{[\alpha]}$ tries to fill up $B^{[>\alpha]}$. In this way, a requirement Q_α, where α is to the right of the true path, can do no harm to a coding requirement C_α such that α is on the true path.

Construction of B.

Stage 0. Let $B_0 = \emptyset$.

Stage $s, s > 0$. For each $k < s$, $k = \langle i, n\rangle$, carry out the following. Let $\alpha = \delta_s \restriction k + 1$.

Case 1. $(\forall n' \leq n)[\alpha(\langle i, n'\rangle) = 1]$. Then let $t < s$ be the greatest stage number such that $t = 0$ or $\alpha \subseteq \delta_t$. If $l(\langle i, n\rangle, t) < l(\langle i, n\rangle, s)$, then enumerate the set $[0, s)^{[>\alpha]}$ into B. We say that the requirement Q_α *acts*.

Case 2. $\alpha(\langle i, n\rangle) = 0$. Then, for each $x < s$, if $x \in A_{i,s}$, enumerate $\langle x, \alpha\rangle$ into B. (Note that $\langle x, \alpha\rangle$ may be in B already due to action of a Q-requirement.) We say that the requirement C_α *acts*.

If none of these cases applies, do nothing.

Verification.

Lemma 2.1 *Let $k = \langle i, n\rangle$ and $\alpha = f \restriction k + 1$. Then the following hold.*

(i) *If* $X_k = \omega$, *then* $A_i =^* B^{[\alpha]}$.

(ii) *The requirement* Q_α *acts only finitely often.*

Proof. Inductively assume that the lemma is true for all $k' < k$. Let s_0 be a stage number such that, for each $s \geq s_0$

$$\alpha \leq \delta_s \wedge (\forall \beta \subset \alpha)[Q_\beta \text{ does not act at stage } s].$$

(i) By (3), $\alpha(k) = 0$. Moreover, by the choice of s_0, no element of $\omega^{[\alpha]}$ is enumerated into B by an action of a Q-requirement at any stage $s \geq s_0$. Therefore $A_i =^* B^{[\alpha]}$.

(ii) Assume for a contradiction that Q_α acts infinitely often. Then $\alpha(\langle i, n' \rangle) = 1$ for each $n' \leq n$. Moreover $\sup_s l(\langle i, n \rangle, s) = \infty$, i.e. $A_i = [n]^B$. Let

$$J = \{j : (\exists m)[\langle j, m \rangle \leq k \wedge X_{\langle j,m \rangle} = \omega]\}$$

and let $A^{[J]} = \oplus_{j \in J} A_j$. Note that $i \notin J$. We show $B \leq_m A^{[J]}$, contrary to the independence of the sequence $(\deg_r(A_i))$.

Let $\gamma \in T$ be arbitrary. If $\alpha < \gamma$, then $B^{[\gamma]} = \omega$. If $\gamma <_L \alpha$ (i.e. γ precedes α but is not an initial segment of α), then $B^{[\gamma]}$ is finite by the inductive hypothesis in (ii).

Finally, suppose that $\gamma \subseteq \alpha$, and let $|\gamma| = \langle j, m \rangle$. If $X_{\langle j,m \rangle} = \omega$, then $j \in J$ and, by (i), $A_j =^* B^{[\gamma]}$. If $X_{\langle j,m \rangle}$ is finite, then $f(\langle j, m \rangle) = 1$, and hence no number is enumerated into $B^{[\gamma]}$ for the sake of coding via Case 2. However, if a requirement Q_β acts and enumerates an element into $B^{[\gamma]}$, then$\beta < \gamma$. Therefore, by the inductive hypothesis in (ii), $B^{[\gamma]}$ is finite. This shows that $B \leq_m A^{[J]}$, a contradiction.

Lemma 2.2 $i \in S \Leftrightarrow A_i \leq_r B$.

Proof. For one direction suppose that $i \notin S$, and assume for a contradiction that $A_i = [n]^B$. Let $\alpha = f \upharpoonright \langle i, n \rangle + 1$. Since $i \notin S$, it follows that $(\forall n' \leq n)[\alpha(\langle i, n' \rangle) = 1]$. Hence Q_α acts infinitely often, contrary to the first Lemma. Now suppose that $i \in S$, and choose n such that $X_{\langle i,n \rangle} = \omega$. Then, by the first Lemma and the axioms (A2) and (A4), $A_i \leq_r B$. $\square$

Theorem 2.2 (Exact Degree Theorem for Σ^0_2 e-degrees) *Suppose that $(A_i)_{i\in\omega}$ is a uniformly Σ^0_2-sequence of sets such that $\deg_e(A_i)_{i\in\omega}$ is independent. Then, for each Σ^0_4-set S, there exists a Σ^0_2-set B such that $i \in S \Leftrightarrow A_i \leq_e B$.*

Proof. It is enough to relativize the previous construction to $\emptyset'$. Thus let $[n]$ denote the n-th enumeration reduction, i.e.

$$[n]^Y(z) = 1 \Leftrightarrow (\exists u)[D_u \subseteq Y \wedge \langle z, u\rangle \in W_n].$$

If Y is Σ^0_2, there is a $\emptyset'$ sequence (Y_s) of strong indices for finite sets such that $Y = \bigcup Y_s$. Thus we obtain an approximation $[n]^Y(z)[s]$, replacing Y and W by Y_s and W_s respectively, and we can adapt the definition of $l(n,s)$. Since S is $\Sigma^0_3(\emptyset')$, we obtain $X_{\langle i,n\rangle}$ approximating S, as well as (δ_s), both now recursive in $\emptyset'$. Running the construction produces a set B which is r.e. in $\emptyset'$, i.e. Σ^0_2. The verification goes through without changes. □

3 The Exact Degree Theorem for subrecursive reducibilities

We now prove the Exact Degree Theorem for reducibilities satisfying axiom (A5). The Theorem can be stated without reference to recursively presented ideals: as an effectivity condition on the sequence $(\deg_r(A_i))$ it suffices to require that $A = \oplus_i A_i$ be recursive. We prove that each Σ^0_2-set S has an exact degree $\deg_r(B)$ such that $B \leq^p_m \oplus_i A_i$ (actually $B = A \cap Q$ for some set Q in P). We first discuss the relevance of this bound on B. Suppose that $\leq_r$ is a reducibility which extends polynomial time m-reducibility and that C is a recursively presented ideal in the u.s.l. of r-degrees of recursive sets (as an example, consider the polynomial T-degrees of the primitive recursive sets or of the sets in EXPTIME). Call a sequence $(\mathbf{a}_i)$ of r-degrees *C-uniform* if $\mathbf{a}_i = \deg_r(A_i)$ for some sequence of sets such that $\deg_r(\oplus_i A_i) \in C$. Then, by the result above, if an independent sequence $(\mathbf{a}_i)$ is C-uniform, every Σ^0_2-set S can be represented by a degree in C. This gives the following method to prove undecidability of $Th(C, \leq)$):

> *If, for some C-uniform independent sequence $(\mathbf{a}_i)_{i\in\omega}$, the set $\{\mathbf{a}_i : i \in \omega\}$ is definable in $(C, \leq)$ from parameters, then $\mathcal{E}^2$ is elementarily definable with parameters in C. In particular, in that case $Th(C, \leq)$ is undecidable.*

The method can be applied to the polynomial time m-degrees of primitive recursive sets, using a result from [ASN92] and observing that the corresponding construction can be carried out within the primitive recursive sets. It can also be applied to the polynomially honest m-degrees of elementary recursive sets. Note that the speed-up technique introduced by Ambos-Spies (see e.g. [ASN92]) produces sets which are not elementary recursive, however if the construction is adapted to honest m-reducibility, they will be. For more restricted degree structures, like the polynomial time T-degrees of sets in EXPTIME, it is not known how to obtain such a definable sequence (or in fact whether the theory is undecidable).

Theorem 3.1 (Exact Degree Theorem for subrecursive reducibilities) *Suppose that $\leq_r$ satisfies the axioms* (A1)-(A5). *Let (A_i) be any sequence of subsets of $\{0,1\}^*$ such that $A = \oplus_i A_i$ is recursive and the sequence $\deg_r(A_i)_{i\in\omega}$ is independent. Then, for each Σ^0_2-set S, there exists a set B such that $i \in S \Leftrightarrow A_i \leq_r B$, for every i. Moreover, $B = A \cap Q$ for some set Q in P.*

Notation. We assume that a sequence (M_e, g_e) is given so that (A5) is satisfied, and write $\langle e\rangle^Z(x)$ instead of $M_{e,g_e(|x|)}(Z)(x)$. Note that, if Z is recursive, then the ternary relation$\langle e\rangle^Z(x) = y$ is recursive. By $[u,v)$ we denote the set of words $\{x : u \leq |x| < v\}$. Words are ordered by

$$x < y \Leftrightarrow |x| < |y| \vee (|x| = |y| \wedge x \text{ precedes } y \text{ lexicographically}).$$

Proof. Since S is a Σ^0_2-set there exists a recursive function $g(i,n)$ which is monotonic in the second argument such that $i \in S \Leftrightarrow \lim_n g(i,n) < \infty$. Let $g(i) = \lim_n g(i,n)$. For each pair $\langle i,e\rangle$, we meet the diagonalization requirement

$$Q_{\langle i,e\rangle} : e < g(i) \Rightarrow A_i \neq \langle e\rangle^B.$$

Then $i \notin S \Rightarrow A_i \not\leq_r B$.

For the converse, we ensure that

$$i \in S \Rightarrow A_i =^* B^{[i]}. \qquad (1)$$

We will effectively determine a strictly increasing time constructible function f (see [BDG88]) and a sequence of numbers $i(n)$ (with $n \in \omega$). The segment

$[f(n), f(n+1))$ is devoted to some diagonalization requirement $Q_{\langle i(n),e\rangle}$. We define B in this segment by omitting the row $i(n)$ from $\oplus_{i\leq n} A_i$, i.e.

$$B \cap [f(n), f(n+1)) = \oplus_{i\leq n\wedge i\neq i(n)} A_i \cap [f(n), f(n+1)). \tag{2}$$

It is possible to diagonalize for $Q_{\langle i(n),e\rangle}$ by choosing $f(n+1)$ sufficiently large: let

$$Z = B \cap [0, f(n)) \cup (\oplus_{i\leq n\wedge i\neq i(n)} A_i \cap \{z : |z| \geq f(n)\}).$$

Since $Z =^* \oplus_{i\leq n\wedge i\neq i(n)} A_i$, we have that $A_{i(n)} \not\leq_r Z$ by (A2),(A3) and the independence of the sequence $(\mathbf{a}_i)$. Hence there exists a word y which can be computed effectively such that $A_{i(n)}(y) \neq \langle e\rangle^Z(y)$. Now, if we make sure that $f(n+1)$ exceeds the length of any oracle query asked in the computation $\langle e\rangle^Z(y)$, then we diagonalize, since $B \cap [0, f(n+1)) = Z \cap [0, f(n+1))$ by the definition of B.

Actually we will define f so large that all the possible values of $e \leq n, B \restriction f(n)$ and $i(n)$ are covered and also that $i(n)$ can be computed in time polynomial in $f(n)$. First let

$$\begin{aligned} h_0(k) = \min\{r : r \geq 2k \wedge (\forall B_0 \subseteq [0,k))(\forall i \leq k)(\forall e \leq k)(\exists y) \\ [|y| \leq k \wedge A_i(y) \neq \langle e\rangle^{B_0 \cup (\oplus_{j\leq n\wedge j\neq i} A_i \cap \{z:|z|\geq k\})}(y) \wedge \text{ no oracle} \\ \text{query of length} \geq r \text{ is asked in the computation of } M_e \text{ on input } y]\}. \end{aligned}$$

The function h_0 is recursive since each oracle query asked in the computation of M_e on input y is bounded by $g_e(|y|)$. Let h be a time constructible function dominating h_0, and let T_g be a monotonic time constructible function bounding the time needed to compute $g(i,n)$ for any i such that $|i| < n$. Define f by

$$f(0) = T_g(0) + 1, \quad \text{and } f(n+1) = \max(h(f(n)), T_g(n)).$$

Note that f is time constructible and $f(n) \geq 2^n$. We think of the definition of B as a construction in stages. At stage n, at most one diagonalization requirement $Q_{\langle i,e\rangle}$ is called *active*: $\langle i, e\rangle$ is the least pair (with respect to to the ordering of words) such that $e < g(i,n)$, $2^{|\langle i,e\rangle|} < n$ and $Q_{\langle i,e\rangle}$ has not been active before. We let $i(n) = i$ if a requirement $Q_{\langle i,e\rangle}$ is active at stage n and $i(n) = n+1$ else. Now the maps f and i define B via (2). We verify that B has the desired properties. Clearly the requirement $Q_{\langle i,e\rangle}$ is met, since $Q_{\langle i,e\rangle}$ is active at some stage if $e < g(i)$. For (1), note that A_i is coded into

$B^{[i]}$ whenever $i \leq n$ and $i \neq i(n)$. Every diagonalization requirement $Q_{\langle i,e\rangle}$ is active at most once, and only if $e < g(i)$. Hence, if $g(i)$ is finite, then $i \neq i(n)$ for almost every n and hence $A_i =^* B^{[i]}$. So the axioms (A2) and (A4) imply $A_i \leq_r B$. To complete the proof, let

$$Q = \{\langle x,j\rangle : (\exists n)[f(n) \leq |\langle x,j\rangle| < f(n+1) \wedge j \leq n \wedge j \neq i(n)]\}.$$

Then $B = A \cap Q$. To show that $Q \in P$, given a word z, first compute j such that $z = \langle x,j\rangle$ for some x, and compute n such that $f(n) \leq |z| \leq f(n+1)$. Since $f(m)$ can be computed in time $Cf(m)$ for some constant C, this can be done in time $O(|z|^2)$ by successively computing $f(0), f(1), \ldots$ until a number n is found such that the computation of $f(n+1)$ does not terminate in time $C|z|$. We claim that $i(n)$ can be computed in time polynomial in $f(n) \leq |z|$. This shows that $j \neq i(n)$ can be checked in time polynomial in $|z|$, so $Q \in P$.

The algorithm to compute $i(n)$ goes through the steps $0,\ldots,n$. At each step m it computes which requirement $Q_{\langle i,e\rangle}$ is active at stage m of the construction of B and maintains a list of requirements $Q_{\langle i',e'\rangle}$ which have been active up to stage m. Since $|\langle i',e'\rangle| < m$ for such a requirement, this list has length $O(m^2)$. To compute $\langle i,e\rangle$, it finds the minimal pair $\langle i,e\rangle$ of length $< m$ not yet in the list such that $e < g(i,m)$. Since $n < \log(|z|)$ there are only polynomially many numbers to be considered. Any possible candidate $\langle i,e\rangle$ satisfies $|i| < m$. Since $T_g(m) \leq f(n)$, it is clear that $g(i,m)$ can be computed in time $f(n)$. So finding the least pair $\langle i,e\rangle$ takes time polynomial in $f(n)$. If $\langle i,e\rangle$ exists, add this pair to the list. The output is i if $\langle i,e\rangle$ was computed at step n, and $n+1$ else. The whole computation takes time polynomial in $f(n)$. □

4 The Exact Degree Theorem for Turing- and Q-reducibility

We first prove the theorem for T-reducibility and then note how to adapt the proof for Q-reducibility.

Theorem 4.1 (Exact Degree Theorem for T-reducibility) *Let $(A_i)_{i\in\omega}$ be a u.r.e. sequence of sets such that $\deg_r(A_i)_{i\in\omega}$ is independent. Then, for each Σ^0_4-set S, there exists an r.e. set B such that $i \in S \Leftrightarrow A_i \leq_T B$, for every i.*

Notation. In the proof of the Theorem, we make use of "hat computations" (see [Soa87, p. 131]). The crucial property of hat computations is

Remark 4.1 *Each hat computation which exists at a nondeficiency stage of the oracle set is final.*

By $\langle i, n, m\rangle$ we denote the number $\langle i, \langle n, m\rangle\rangle$. Before proving the theorem, we construct an appropriate representation of the set S.

Lemma 4.1 *There is a u.r.e. sequence $X_{\langle i,n,m\rangle}$ of initial segments of ω such that*

$$i \in S \Rightarrow \ (a.e.\ n, m)[X_{\langle i,n,m\rangle} \ \ \mathit{finite}] \tag{1}$$

$$i \notin S \Rightarrow (\forall n)(\exists m)[X_{\langle i,n,m\rangle} = \omega]. \tag{2}$$

Proof of the Lemma. Since S is Σ^0_4, there is a u.r.e. sequence (Y_k) of initial segments of ω such that

$$i \in S \Leftrightarrow (\exists n)(\forall m)[Y_{\langle i,n,m\rangle} \ \mathit{finite}].$$

We now modify (Y_k) to satisfy (1) by reducing the number of sets which equal ω. By the row $\langle i, n\rangle$ we mean the collection of sets $(Y_{\langle i,n,m\rangle})_{m\in\omega}$. First we replace $Y_{\langle i,n,m\rangle}$ by

$$Y_{\langle i,0,\tau_0\rangle} \cap \ldots \cap Y_{\langle i,n,\tau_n\rangle}$$

(where $m = \langle \tau_0, \ldots, \tau_n\rangle$). Then, if the row $\langle i, n\rangle$ only contains finite sets, so do all the rows $\langle i, n'\rangle$, $n' \geq n$.

Next, we modify (Y_k) to obtain a u.r.e sequence (X_k) such that each row has at most one infinite member. The following process is applied uniformly to each row $\langle i, n\rangle$ to obtain (X_k). Let $C_m = Y_{\langle i,n,m\rangle}$. We replace this row by a row $D_{\langle m,g\rangle}$ where g is thought of as a guess about $|\bigcup_{m'<m} C_{m'}|$. As long as the guess is correct, $D_{\langle m,g\rangle}$ is allowed to copy C_m. Formally, let $D_{p,0} = \emptyset$ and, for $s > 0$ and $p = \langle m, g\rangle$, if $g = |\bigcup_{m'<m} C_{m',s}|$ then let $D_{p,s} = C_{m,s}$, else let $D_{p,s} = D_{p,s-1}$. Clearly, there is at most one $\langle m, g\rangle$ such that $D_{\langle m,g\rangle} = \omega$.

Proof of the Theorem. We build an r.e. set B such that

$$i \in S \Rightarrow A_i \leq_T B \tag{3}$$

$$i \notin S \Rightarrow (\forall n)[A_i \neq \{\widehat{n}\}^B]. \tag{4}$$

Intuitively speaking, the number n in (2) is interpreted as an index of a Turing reduction. For each number m, a requirement $Q_k = Q_{\langle i,n,m\rangle}$ tries to cause $A_i \neq \{\widehat{n}\}^B$. If $i \notin S$, then some requirement $Q_k = Q_{\langle i,n,m\rangle}$, such that $X_k = \omega$, will succeed. In this way we will satisfy (4). If X_k is finite, then Q_k will only exert a finite influence on the construction. We meet the requirements

$$Q_{\langle i,n,m\rangle} : X_{\langle i,n,m\rangle} = \omega \Rightarrow A_i \neq \{\widehat{n}\}^B.$$

We write $i_Q(k) = i$ if Q_k works on (4) for i, i.e. if $k = \langle i,n,m\rangle$ for some n, m.

For (3) we meet the coding requirements C_k. Each coding requirement attempts to code a set $A_{i_C(k)}$ into $B^{[k]}$; for each i there are infinitely many numbers k such that C_k works on (3) for i. For definiteness, let $i_C(k) = i$ if $k = \langle i,n\rangle$ for some n. The priority ordering of the requirements is

$$C_0 < Q_0 < C_1 < Q_1 < \ldots$$

A requirement C_k may be affected by the *lower* priority requirements Q_p such that $i_Q(p) = i_C(k)$. C_k relies on a hypothesis which implies that each single requirement Q_p affects C_k only finitely often:

$$C_k : (\forall p \geq k)[i_Q(p) = i \Rightarrow X_p \text{ is finite}] \Rightarrow A_i = \Delta_k(B^{[k]})$$

where $i = i_C(k)$, and Δ_k is a Turing reduction we build during the construction. Given $i \in S$, since infinitely many coding requirements work on (3), by (1) we can choose k such that $i_C(k) = i$ and the hypothesis of the coding requirement C_k is correct. Therefore $A_i \leq_T B$. We emphasize that the part of the construction concerned with the set A_i goes two completely different ways depending on whether $i \in S$ or not: if $i \in S$, then for some k such that $i = i_C(k)$, C_k succeeds. Moreover, by (1), almost all the requirements Q_p such that $i_Q(p) = i$ are finitary. If $i \notin S$, then for each n there exists m such that $Q_{\langle i,n,m\rangle}$ succeeds. Moreover, each requirement C_k, $i_C(k) = i$, is affected infinitely often by some single requirement Q_p where $p \geq k$ and $i_C(p) = i$, whence the coding of C_k fails.

The strategy for C_k

If the set S is not Σ^0_3, we cannot have a *wtt*-reduction in (3). Therefore we have to exploit the full power of Turing reductions. The requirements C_k use coding via markers. With each C_k we associate a sequence of markers $(\gamma_k(x))_{x\in\omega}$. For almost every s, the marker $\gamma_k(x)$ possesses a value $\gamma_{k,s}(x) \in$

$\omega^{[k]}$ at the end of stage s, unless $\gamma_k(x)$ gets cancelled (because x entered A_i). The sequence of markers $\gamma_k(x)$ will satisfy the following two conditions:

$$\gamma_{k,s-1}(x) \neq \gamma_{k,s}(x) \Rightarrow B_{s-1} \upharpoonright \gamma_{k,s-1}(x)+1 \neq B_s \upharpoonright \gamma_{k,s-1}(x)+1 \qquad (5)$$

$$\begin{aligned}&(\text{a.e. } x)[x \in A_i \Rightarrow \text{ at some stage } t+1,\\ &\qquad \gamma_{k,t}(x) \text{ is enumerated into B and } \gamma_k(x) \text{ gets cancelled forever }]. \qquad (6)\end{aligned}$$

Moreover, if the hypothesis of requirement C_k is correct, then, for each x

$$\text{if } \gamma_k(x) \text{ does not get cancelled, then} \lim_s \gamma_{k,s}(x) \text{ exists.} \qquad (7)$$

It is now clear how to build a T-reduction from A_i to $B^{[k]}$: given input y, compute a stage t such that $\gamma_k(y)$ has been cancelled or $B_t \upharpoonright \gamma_{k,t}(y)+1 = B \upharpoonright \gamma_{k,t}(y)+1$. Then $A_i(y) = A_{i,t}(y)$. In this way we satisfy C_k. As usual, the values $\gamma_{k,s}(x)$ will be monotonic in x and in s.

Notation. We write $\gamma_k(x) \downarrow\downarrow [s]$ if either $\gamma_k(x)$ has been cancelled at the end of stage s or this marker remains uncanceled and $\gamma_{k,s}(x) = \lim_t \gamma_{k,t}(x)$. Let

$$P = \{p : (\forall x)(\exists s)][\gamma_p(x) \downarrow\downarrow [s]]\}.$$

If $k \notin P$, then let $x(k)$ be the minimal number x such that $x \notin A_{i_C(k)}$ and $\lim_s \gamma_k(x)$ does not exist (there is such a number x since $A_{i_C(k)}$ is coinfinite.)

The strategy for Q_k

The length of agreement associated with Q_k is

$$l(k,t) = \max\{x : (\forall y < x)[A_i(y)[t] = \{\widehat{n}\}^B(y)[t]]\}$$

(where $i = i_Q(k)$).

Let $J_k = \{i_C(p) : p \leq k \wedge p \in P\}$. If $X_k = \omega$, then the strategy for Q_k consists of the following:

a) we cause $i \notin J_k$

b) roughly speaking, at stage s we restrain the lower priority requirements in order to protect computations $\{\widehat{n}\}^B(y)$, where $y < l(k,s)$.

From this, it will be possible to argue that $A_i \neq \{\widehat{n}\}^B$: otherwise $A_i \leq_T \oplus_{j\in J_k} A_j$, contrary to the independence of the sequence $(\deg_r(A_i))$.

To cause $i \notin J_k$, Q_k has to detach some marker $\gamma_p(x)$ for each $p \leq k$ such that $i_C(p) = i_Q(k)$. To make the influence of Q_k on C_p finite if the set X_k is finite, we only allow this if $|X_k|$ has increased since the last time when $\gamma_p(x)$ was detached.

Let

$$\widehat{u}(k,s) = \max\{\widehat{u}(B_s; n, y, s) : y < l(k,s)\}.$$

The restraint imposed by Q_k on the lower priority requirements is defined as follows: $r(k,0) = 0$, and for $s > 0, r(k,s) = r(k,s-1)$ if $X_{k,s} \neq Xk, s-1$, and $r(k,s) = \min(r(k,s-1), \widehat{u}(k,s-1))$, otherwise. By this definition, we let $r(k,s)$ drop back at any stage, but we allow an increase of $r(k,s)$ only at stages where $|X_k|$ has increased. Therefore, if X_k is finite, then $\lim_s r(k,s)$ exists and is finite.

We now sketch the argument for $A_i \neq \{\widehat{n}\}^B$ (supposing that $X_k = \omega$). Assume for a contradiction that $A_i = \{\widehat{n}\}^B$. Then, for each y, there exists a stage number s^* such that $X_{k,s^*} \neq X_{k,s^*-1}$ and all the computations $\{\widehat{n}\}^B(y'), y' \leq y$, are stable from stage s^* on, i.e.

$$l(k,s^*) > y \wedge B_{s^*} \restriction u = B \restriction u \tag{8}$$

where $u = \widehat{u}(B_s; n, y, s^*)$. By (5) and (6), $\gamma_q(x) \downarrow\downarrow [s^*]$ for each marker $\gamma_q(x)$ such that $\gamma_{q,s}(x) < u$. In particular, if $p \leq k$ and $p \notin P$, then

$$\gamma_{p,s}(x(p)) \geq u. \tag{9}$$

Let $A^{[J]} = \oplus_{j\in J} A_j$ (where $J = J_k$). We show how to compute A_i from $A^{[J]}$. Given an input y, compute a stage number s such that $X_{k,s} \neq X_{k,s-1}, l(k,s) > y$, (9) holds for those $p \leq k$ such that $p \notin P$ and, if $p < k$ and $p \in P$ then $x \notin A_j$ for each marker $\gamma_p(x)$ which has value $< u$. (By the remarks above, s^* is such a stage.) Since $|X_{k,s}|$ has increased, Q_k can "update" its restraint. This implies that (8) holds for s instead of s^*. Hence $A_{i,s}(y) = \{\widehat{n}\}_s^{B_s}(y) = \{\widehat{n}\}^B(y)$. In this way we determine $A_i(y)$.

Finally, we consider the behavior of the restraint function $r(k,s)$ if $X_k = \omega$. Since $A_i \neq \{\widehat{n}\}^B$, by a standard argument involving Remark 4.1 (see the "Window Lemma" in [Soa87, p. 134]) together with the fact that we always allow the restraint to drop back, we conclude

$$\liminf_s r(k,s) = \lim_{t\in T} r(k,t) \text{ is finite.}$$

Then $\liminf_s r(<j,s)$ is finite as well for each j, where

$$\liminf_s r(<j,s) = \max\{r(k,s) : k < j\}.$$

Thus, the requirement Q_k leaves enough room for the lower priority requirements.

Construction of B.
Stage 0. Let $B_0 = \emptyset$. Each marker $\gamma_k(x)$ is declared undefined.
Stage $s, s > 0$.

Phase 1. (The Q-requirements detach markers) For each k, $0 \leq k < s$ and for each $p \leq k$ such that $i_C(p) = i_Q(k)$ do the following: if a number $x < s-1$ exists such that $\gamma_p(x)$ is defined, $k \leq x$, $r(<k,s) \leq \gamma_{p,s-1}(x)$, and $X_{k,t} \neq X_{k,s}$, where $t < s$ is the greatest stage number such that $\gamma_p(x)$ was detached by Q_k or $t = x$, then let x be a minimal such number. Enumerate $\gamma_{p,s-1}(x)$ into B. For $x \leq x' < s-1$ we say that Q_k *detaches* $\gamma_p(x')$ and that $\gamma_p(x')$ is now undefined.

Phase 2. (Giving values to markers and coding) For each $k < s$ do the following:

Step 1. Let $x < s$ be minimal such that $\gamma_k(x)$ is undefined. Give values $\gamma_{k,s}(x') \in \omega^{[k]}$ to the markers $\gamma_k(x')$, $x \leq x' < s$ which have not been cancelled. Choose these values increasing in x', greater than s and greater than any former value.

Step 2. Let $i = i_C(k)$. For each $y < s$, if $y \in A_{i,s}$, $\gamma_k(y)$ has not been cancelled and $\gamma_{k,s}(y) \geq r(<k,s)$, then enumerate $\gamma_{k,s}(y)$ into B and cancel the marker $\gamma_k(y)$.

The Verification.

Lemma 4.2 *Let $k = \langle i,n,m \rangle$.*

(i) $r(<k) = \lim_{t\in T} r(<k,t+1)$ *exists and is finite.*

(ii) *If $X_k = \omega$, then the following hold:*

(ii.1) *If $p \leq k$ and $i_C(p) = i$, then $p \notin P$. In particular, $i \notin J_k$.*

(ii.2) $A_i \neq \{\widehat{n}\}^B$.

(iii) $\lim_{t\in T} r(k, t+1)$ *exists and is finite.*

Proof. Inductively assume that the lemma is true for all $k' < k$. Then (i) is immediate.

(ii.1) Suppose that $p \leq k$ and $i_C(p) = i$. We show that, for some x,

$$\gamma_p(x) \text{ is detached infinitely often by } Q_k. \tag{10}$$

Let $x \geq k$ be a number such that $x \notin A_i$ and $x \geq r(< k)$ (x exists because A_i is coinfinite). Suppose that t_0 is a stage number such that $\gamma_p(x)$ is detached by Q_k at stage t_0 or $t_0 = x$, and let $t > t_0$ be the smallest stage number such that $t-1 \in T$, $X_{k,t_0} \neq X_{k,t}$, $t > x+1$, $t > k$ and $r(< k, t) \leq r(< k)$. Then at some stage $t', t_0 < t' \leq t$, the marker $\gamma_p(x)$ is detached by Q_k. This shows (10).

(ii.2) One carries out in detail the argument sketched above. See [Nie92].

(iii) If X_k is finite then $\lim_s r(k, s)$ exists and is finite. If $X_k = \omega$, then (iii) can be inferred from the fact that $A_i \neq \{\widehat{n}\}^B$.

Lemma 4.3 $i \in S \Leftrightarrow A_i \leq_T B$.

Proof. For one direction suppose that $i \notin S$. Let n be arbitrary; we show that $A_i \neq \{\widehat{n}\}^B$. Since $i \notin S$, we can choose a number m such that $X_{\langle i,n,m\rangle} = \omega$. Then, by (ii.2) of the previous Lemma, $A_i \neq \{\widehat{n}\}^B$.

For the converse direction, suppose that $i \in S$. By (1), there exists a p such that the hypothesis of the coding requirement C_p is correct. We verify (5)-(7). The condition (5) is immediate, and (6) holds by Phase 2, Step 2 of the construction, since $\lim_s r(< p, s)$ is finite. For (7), let x be arbitrary and, by induction, choose s such that

$$(\forall x' < x)[\gamma_p(x') \downarrow\downarrow [s]].$$

After stage s, $\gamma_p(x)$ can only be detached by one of the finitely many requirements Q_k, where $k \leq x$, $p \leq k$ and $i_C(p) = i_Q(k)$, and only as often as $|X_k|$ increases. Therefore (4.1) holds for x. □

To obtain the result for Q-reducibility, we slightly modify the construction in the proof: whenever a marker is detached in Phase 1, then its value is enumerated into B. Then, if C_p is a coding requirement with correct hypothesis and $i_C(p) = i$, the coding of C_p in Phase 2 gives $A_i \leq_Q B$. To see this, in (2) let $W_{g(x)}$ be the r.e. set of all values of the marker $\gamma_p(x)$. Then $A_i \leq_Q B$ via g. $\square$

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Department of Mathematics, University of Chicago, Chicago, IL 60637-1514, USA
e-mail address: `nies@math.uchicago.edu`