Reals which compute little

André Nies

Abstract. We investigate combinatorial lowness properties of sets of natural numbers (reals). The real A is super-low if $A' \leq_{tt} \emptyset'$, and A is jump-traceable if the values of $\{e\}^A(e)$ can be effectively approximated in a sense to be specified. We investigate those properties, in particular showing that super-lowness and jump-traceability coincide within the r.e. sets but none of the properties implies the other within the ω-r.e. sets. Finally we prove that, for any low r.e. set B, there is is a K-trivial set $A \not\leq_T B$.

1. Introduction

In computability theory, one measures and compares the computational complexity of sets of natural numbers (also called *reals*). The first question one is interested in is whether the real is computable. Reals which come close to being computable are therefore of particular interest. A *lowness properties* of a real A says that, in some sense, A has low computational power when used as an oracle (and therefore A is close to being computable). To qualify as a lowness property, we require that the property be downward closed under Turing reducibility $\leq_T$, and that each real A with that property is generalized low, namely $A' \leq_T A \oplus \emptyset'$. In this paper we study and compare two lowness properties, being super-low and being jump-traceable.

Superlow reals. Recall that a real A is *low* if its jump A' is Turing-below the halting problem $\emptyset'$, or, equivalently, $A'(e) = \lim_s g(e, s)$ for a computable $0, 1$-valued g. The following concept is more restrictive.

Definition 1.1. *The real A is* super-low *if $A' \leq_{tt} \emptyset'$. Equivalently, $A'(e) = lim_s g(e, s)$ for a computable $0, 1$-valued g such that $g(e, s)$ changes at most $b(e)$ times, for a computable function b.*

This notion goes back to work of Mohrherr [**8**], and an unpublished manuscript of Bickford and Mills [**1**] (where only super-low r.e. sets are studied, called "abject" there). The canonical construction of a low simple set (see [**10**, Thm VII.1.1]) produces in fact a super-low set: one satisfies lowness requirements

$$L_e : \exists^\infty s\ \{e\}^A(e) \downarrow [s\text{-}1] \ \Rightarrow\ \{e\}^A(e) \downarrow.$$

L_e is injured at most e times by requirements

$$P_i : |W_i| = \infty \ \Rightarrow\ W_i \cap A \neq \emptyset,$$

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$i < e$, which enumerate a number $x \geq 2i$ such that $x \in W_{i,s}$ into A at a stage s if $W_{i,s} \cap A_{s-1} = \emptyset$. Then $\{e\}^A(e)$ can become undefined at most e times. Thus, if we let $g(e,s) = 1$ when $\{e\}^A(e)$ converges at stage s and $g(e,s) = 0$ otherwise, then g is an approximation as in Definition 1.1, where $b(e) = e$.
The low basis theorem of Jockusch and Soare [**6**] can also be strengthened to "super-low": each non-empty Π^0_1 class has a super-low member (Proposition 3.1 below).
Jump-traceable reals. We write $J^A(e)$ for $\{e\}^A(e)$, the jump at argument e. While lowness and super-lowness restrict the *domain* $A' = \{e : J^A(e)\downarrow\}$ of J^A, jump traceability expresses that $J^A(e)$ has few possible *values*. Given $T \subseteq \mathbb{N}$, let $T^{[x]} = \{y : \langle y, x\rangle \in T\}$.

DEFINITION 1.2. (i) *An r.e. set $T \subseteq \mathbb{N}$ is a* TRACE *if for some computable h, $\forall n |T^{[n]}| \leq h(n)$. We say that h is a* BOUND *for T.*
(ii) *The real A is* JUMP-TRACEABLE *if there is a trace T such that*

$$\forall e\ J^A(e)\downarrow \Rightarrow\ J^A(e) \in T^{[e]}.$$

This modifies the property of being recursively traceable, used in [**11**] to give a characterization of the reals that are low for Schnorr tests. We will see below that, because of the universality of the jump, jump traceability of A actually restricts the possible values of any partial A-recursive function via a trace.
Both super-lowness and jump traceability are closed downward under $\leq_T$ and imply GL_1. Thus they satisfy our criteria for being lowness properties. Super-low reals A are ω-r.e., that is, $A \leq_{tt} \emptyset'$. On the other hand, we will see that there is a perfect Π^0_1-class of jump-traceable reals. *Among our main results are:*

- *super-lowness and jump-traceability coincide within the r.e. sets*
- *none of the properties implies the other within the ω-r.e. sets.*

We also prove that jump traceability is Σ^0_3 on the ω-r.e. sets, namely, if $(\Theta_e)_{e\in\mathbb{N}}$ is an effective listing of all tt-reduction procedures defined on an initial segment of $\mathbb{N}$, then $\{e : \Theta_e(\emptyset')\ \textit{jump-traceable}\}$ is Σ^0_3. The same result follows for the r.e. sets. Recall that $\{e : W_e \text{ low}\}$ is Σ^0_4-complete [**10**, Cor. XII 4.7]. Since our two properties coincide for r.e. sets, super-lowness is strictly stronger than lowness even for the r.e. sets.
Our "combinatorial" lowness properties can be used to study very interesting lowness properties related to randomness and prefix Kolmogorov complexity. We first recall some definitions. For each real A, we want to define $K^A(y)$, the length of a shortest prefix description of y using oracle A. An *oracle machine* is a partial recursive functional $M : 2^\omega \times 2^{<\omega} \mapsto 2^{<\omega}$. We write $M^A(x)$ for $M(A,x)$. M is an *oracle prefix machine* if the domain of M^A is an antichain under inclusion of strings, for each A. Let $(M_d)_{d\in\mathbb{N}^+}$ be an effective listing of all oracle prefix machines. The universal oracle prefix machine U is given by

$$U^A(0^d 1\sigma) = M^A_d(\sigma).$$

Let $K^A(y) = \min\{|\sigma| : U^A(\sigma) = y\}$. If $A = \emptyset$, we simply write $U(\sigma)$ and $K(y)$. $U_s(\sigma) = y$ indicates that $U(\sigma) = y$ and the computation takes at most s steps.
The real A is *K-trivial* if the K-complexity of its initial segments is as low as possible, up to a constant c, namely $\forall n\ K(X \upharpoonright n) \leq K(n) + c$. Let $\mathcal{K}$ denote this class of reals. $\mathcal{K}$ contains nonrecursive r.e. sets and is closed under $\oplus$ (see [**3**] for proofs and more references). Here we show that,

- *for each low r.e. B, there is an r.e. $A \in \mathcal{K}$ such that $A \not\leq_T B$.*

In [**9**] I prove that $\mathcal{K}$ is closed downward under Turing reducibility, and each $A \in \mathcal{K}$ is truth table-below some r.e. $D \in \mathcal{K}$. Thus $\mathcal{K}$ is an example of a lowness property which is an ideal in the ω-r.e. reals, generated by its r.e. members. In contrast, the super-low r.e. sets do not form an ideal, since there are super-low r.e. sets A_0, A_1 such that $A_0 \oplus A_1$ is Turing complete (see [**1**] or Theorem below).
We also show in Nies [**9**] that each $A \in \mathcal{K}$ is superlow. Since the construction in [**3**] produces a noncappable $A \in \mathcal{K}$, the class of super-low r.e. degrees is downward dense in the nonrecursive r.e. degrees.
The notion of jump traceability can be used to characterize reals which are computationally weak in the following sense.

Definition 1.3. *Let $p : \mathbb{N} \mapsto \mathbb{N}$ be a non-decreasing computable function such that $lim_n p(n) - n = \infty$. A real A is p-low if $\forall y\ K(y) \le p(K^A(y) + c_0) + c_1$ for some constants $c_0, c_1 \in \mathbb{N}$.*

Thus, for such A, $K^A(y)$ is not much smaller than $K(y)$. Let $\mathcal{M}[p]$ denote this class of reals. In the last section of [**4**] we show that

$$A \textit{ jump traceable } \Leftrightarrow \exists\, p \textit{ computable } \ A \in \mathcal{M}[p].$$

Preliminaries. If $f : \mathbb{N} \to \mathbb{N}$, then we say f is ω-r.e. if $f \le_{wtt} \emptyset'$, that is f can be computed from $\emptyset'$ with recursively bounded use. This is easily seen to be equivalent to $f(e) = \lim_s g(e, s)$, where g is a computable function such that $g(e, s)$ changes at most $b(e)$ times, for a computable function b. For instance, if T is a trace in the sense of Definition 1.2, then $f(n) = \max T^{[n]}$ is ω-r.e., via $g(n, s) = \max T^{[n]}_s$ and $b(n) = h(n)$.
The following notation is also useful. A Δ^0_2–*approximation* $(A_r)_{r\in\mathbb{N}}$ of a real A is an effective sequence of finite sets such that $A(x) = \lim_r A_r(x)$. Then $A \le_{tt} \emptyset'$ iff $A \le_{wtt} \emptyset'$ iff the number of changes in such an approximation is recursively bounded.
Recall that we write $J^A(e)$ for $\{e\}^A(e)$. If A is given by a Δ^0_2–approximation, we write $J^A(e)[s]$ for $\{e\}^{A_s}_s(e)$. The use of the computation $J^A(e)$ is denoted $j(A, e)$, and the use of $J^A(e)[s]$ is denoted $j(A, e)[s]$.
Recall that a *partial recursive functional* is an r.e. set Ψ of "axioms" $\langle \sigma, e, v\rangle$, $\sigma \in 2^{<\omega}$ such that if $\langle \sigma, e, v\rangle, \langle \sigma', e, v'\rangle \in \Psi$ and σ, σ' are compatible, then $v = v'$. Given Δ^0_2–approximation (A_s), to *define* $\Psi^A(e) = v$ *with use* u *at stage* s means to put the axiom $\langle A_s \restriction u, e, v\rangle$ into Ψ.
While the proof of the following fact is not hard, it depends on the particular implementation of the universal machine.

Fact 1.4. *From a partial recursive functional Ψ, one can effectively obtain a primitive recursive function α, called a* reduction function *for Ψ, such that*

$$\forall X\ \forall e\ \Psi^X(e) \simeq J^X(\alpha(e)).$$

2. Jump-traceability

In this Section we collect some basic facts on jump-traceability, prove existence of a perfect class of jump-traceable reals, and place this notion in context.

Jump traceable reals at large.

Fact 2.1. *If A is jump-traceable via T, then there is a trace S such that, for each partial recursive functional Ψ,*

$$a.e.m[\Psi^A(m)\downarrow \Rightarrow \ \Psi^A(m)\in S^{[m]}].$$

For, define S by $S^{[m]} = \bigcup_{i\leq m} T^{[\alpha_i(m)]})$, where (α_i) is a listing of all primitive recursive unary functions. Then S is a trace which is as required by Fact 1.4.

Proposition 2.2. *Let A be any jump-traceable real. Then A is generalized low$_1$, namely $A' \leq_T A\oplus\emptyset'$. The reduction procedure can be obtained effectively from an r.e. index for the trace T. In this reduction, the use on $\emptyset'$ is recursively bounded, and the use on A is ω-r.e.*

Proof. Consider the partial A–recursive functional

$$\Psi^A(e) = \mu s\ [J^A(e)\downarrow \text{ in } s \text{ steps}].$$

Choose a reduction function α by Fact 1.4. To see if $e\in A'$, first compute $t = \max T^{[\alpha(e)]}$, using $\emptyset'$ as an oracle. Then, using $A\upharpoonright t$, check whether $J^A(e)\downarrow$ in $\leq t$ steps. If so, answer Yes, otherwise No.
Since T is a trace, the use on $\emptyset'$ to compute t is recursively bounded. □

Recall that a Π^0_1-class P is a subset of 2^ω given as the set of paths $[B]$ through a recursive subtree of B of $2^{<\omega}$.

Theorem 2.3. *There is a perfect Π^0_1-class of reals which are jump-traceable via a fixed trace T, whose bound is $h(e) = 2\cdot 4^e$.*

Proof. We will define an effective sequence of 1-1-maps (F_s) such that $F_s : 2^{<\omega}\mapsto 2^{<\omega}$ preserving the ordering and compatibility relations, and for each $\alpha\in 2^{<\omega}$, $F_s(\alpha)\subseteq F_{s+1}(\alpha)$ and $\lim_s F_s(\alpha)$ exists. If we let $B = \{\rho : \forall s\ \exists\alpha[|\alpha|\leq|\rho|\ \&\ \rho\subseteq F_s(\alpha)]$, then $Q = [B]$ is a perfect Π^0_1–class. We define $F_s(\alpha)$, $|\alpha| = e$ in a way to minimize the number of values $J^{F(\alpha)}(e)$. Any such value we see at some stage needs to be enumerated into $T^{[e]}$.
Construction. Let $F_0(\alpha) = \alpha$ for each α. At stage $s+1$ look for the length-lexicographically least α such that, where $|\alpha| = e$, there is $\beta\succeq\alpha$ such that $y = J^{F(\beta)}(e)\downarrow$ and $y\notin T^{[e]}$. If there is such an α, enumerate y into $T^{[e]}$. Define $F_{s+1}(\alpha) = F_s(\beta)$. Moreover, for all $\rho\neq\lambda$ define $F_{s+1}(\alpha\rho) = F_s(\beta\rho)$.
This ends the construction. A value $F_s(\alpha)$, $|\alpha| = e$, changes at most $2^{e+1}-1$ times, and causes the enumeration of at most that many elements into $T^{[e]}$. Thus $|T^{[e]}|\leq 2\cdot 4^e$. □

This construction could be massaged a bit to obtain a bound close to 2^e. However, it is unknown if there is still a perfect class for much smaller bounds. Below we show that there is a fixed C such that, if A is low for K via b (i.e., $\forall y\ K^A(y)\geq K(y)-b$), then A is jump traceable via the bound $C2^b i\log i$ (see Prop. 5.9 below). However, each low for K set is Δ^0_2. In [**4**] we construct a non-computable r.e. A which is *strongly jump traceable*, namely, jump traceable via each unbounded montonic computable h.

Jump traceable ω-r.e. sets. Next we determine the index set complexity of jump traceability on the ω-r.e. sets. Recall that $(\Theta_e)_{e\in\mathbb{N}}$ is an effective listing of all (possibly partial) tt-reduction procedures defined on an initial segment of $\mathbb{N}$. Thus $\Theta_e(x)$ can be viewed as a truth table. Then we obtain an effective listing $(V_e)_{e\in\mathbb{N}}$ of the ω–r.e. sets by letting $V_{e,s}(x) = \Theta_e(\emptyset';x)[s]$, which is interpreted as 0 if $\Theta_e(x)[s]$ is undefined. Now let $V_e(x) = \lim_s V_{e,s}(x)$.

PROPOSITION 2.4. *$\{e : V_e$ jump-traceable $\}$ is Σ^0_3-complete. Similarly, $\{e : W_e$ jump-traceable $\}$ is Σ^0_3-complete.*

PROOF. V_e is jump-traceable iff $\exists T \subseteq \mathbb{N}\ \exists h$ total

$$(\forall n |T^{[n]}| \leq h(n)\ \&\ \forall x \forall s \exists t \geq s\, [J^{V_e}(x)[t] \uparrow\ \vee\ J^{V_e}(x)[t] \downarrow \in T^{[x]}_t]).$$

The direction "$\Rightarrow$" is clear. For the other direction, note that if $J^{V_e}(x)\downarrow$ then the condition implies $J^{V_e}(x) \in T^{[x]}$.
For each $e \in \mathbb{N}$ we can effectively obtain $\widehat{e}$ such that $W_e = V_{\widehat{e}}$. This proves the second index set is Σ^0_3. For Σ^0_3-hardness it suffices to consider the r.e. case. But it is easy to show that *any* nontrivial Σ^0_3-class of r.e. sets which is closed under finite differences and contains the computable sets has a Σ^0_3-complete index set. ☐

Digression: r.e. traceable reals. A real A is *r.e. traceable* if there is a trace S such that $\forall \Gamma(\Gamma^A$ total $\Rightarrow$ a.e. $m\ \Gamma^A(m) \in S^{[m]})$. This was studied in (Ishmukhametov [**5**]), who used the term *weakly recursive.* By Fact 2.1 each jump traceable real is r.e. traceable. Since the function $g(x) = \max T^{[x]}$ is ω-r.e., a weakly recursive real is array recursive [**2**], which means that there is an ω-r.e. function which eventually dominates any A–computable function. For r.e. sets A, the converse implication holds by [**5**], a fact which can be proved in the same style as the proof of Theorem 4.1 below.
The r.e. traceable Δ^0_2 reals have an interesting uniformity property. Recall that a real A is Low_2 if $\mathrm{Tot}^A = \{e : \{e\}^A \text{ total}\} \in \Sigma^0_3$.

PROPOSITION 2.5. *The r.e. traceable Δ^0_2-reals (and hence the jump traceable Δ^0_2-reals) are uniformly low$_2$. Thus, from a Δ^0_2 approximation (A_s) to A one can effectively obtain a Σ^0_3-index for Tot^A.*

The point is that the Δ^0_2 approximation alone suffices, in case it actually approximates a r.e. traceable real, to obtain the Σ^0_3-index.

PROOF. Using the same argument as in [**11**, Fact 1], if a real A is weakly recursive, then it is irrelevant what the actual bound h for the trace is, as long as $\lim_n h(n) = \infty$. Thus there is a trace S such that $|S^{[m]}| \leq m$. Let S_i be a u.r.e. list of all traces with bound $h(m) = m$, and let $V^{[e]} = \bigcup_{i \leq e} S^{[e]}_i$, so that V is a trace which works for all r.e. traceable reals. Let $g(m, s) = \max V^{[m]}_s$. Then $\{e\}^A$ is total iff

$$\exists x \exists s\ \forall t \geq s\ \forall z < x\ e^A(z) \downarrow [t]\ \&\ \forall z \geq x \exists v \geq t\ u(A; e, z)[v] \leq g(z, v).$$

The direction from left to right holds since $u(A; e, z)$, the use of $\{e\}^A(z)$ is an A–recursive function. The converse direction holds because, for each z, there are only finitely many possibilities for $\{e\}^A(z)[v]$.
The right hand side gives a Σ^0_3 index for Tot^A, which was obtained uniformly in the Δ^0_2-approximation to A. ☐

3. Super-low reals

Jockusch and Soare [**6**] proved that each non-empty Π^0_1 class has a low member. An analisis of their proof yields

PROPOSITION 3.1. *Each non-empty Π^0_1 class has a super-low member.*

Proof. Suppose $P = [B]$, where B is an infinite recursive subtree of $2^{<\omega}$. For each finite set F, let $B_F = \{\sigma \in B : \ \forall e \in F \ J^\sigma(e) \uparrow\}$. Since being finite is a Σ^0_1-property of recursive trees, there is a computable g defined on (strong indices for) finite subsets of $\mathbb{N}$ such that

$$B_F \text{ finite } \Leftrightarrow \ g(F) \in \emptyset'.$$

As in [**6**], let $B = B_{F_0} \supseteq B_{F_1} \supseteq \ldots$ be a sequence of recursive trees defined as follows: let $F_0 = \emptyset$, and $F_{i+1} = F_i$ if $B_{F_i \cup \{i\}}$ is finite, and $F_{i+1} = F_i \cup \{i\}$ else. Then one can compute F_i from $\emptyset'$, where the use is bounded by the computable function $\max\{g(F)+1 : F \subseteq \{0,\ldots,i\}\ \}$.
By compactness, there is a (unique) path $A \in \bigcap_i [B_{F_i}]$. This path satisfies $J^A(e) \uparrow \Leftrightarrow \ e \in F_{e+1}$. Thus $A' \leq_{wtt} \emptyset'$ and hence $A' \leq_{tt} \emptyset'$. $\square$

Corollary 3.2. *There is a Martin-Löf random super-low real.*

Proof. This follows since the set of random reals forms a union of Π^0_1-classes (given by a universal Martin-Löf test). $\square$

In contrast, a Martin-Löf random real R is not of n-r.e. degree unless $R \equiv_T \emptyset'$. This is because there is a fixed point free $f \leq_T R$ (i.e., $\forall e \ W_e \neq W_{f(e)}$), and the Arslanov completeness criterion applies to n-r.e. sets (see [**10**, p. 277]).
Recall that, by the Sacks Splitting Theorem, there are low r.e. sets A_0, A_1 such that $\emptyset' \leq_T A_0 \oplus A_1$. Again, we strengthen this to super-low. This was first proved in [**1**].

Theorem 3.3. [**1**] *There are super-low r.e. sets A_0, A_1 such that $\emptyset' \leq_T A_0 \oplus A_1$.*

Proof. We enumerate A_0, A_1 and also build a Turing functional Γ such that $\emptyset' = \Gamma(A_0 \oplus A_1)$. The use of $\Gamma(A_0 \oplus A_1; p)$ is denoted $\gamma(A_0 \oplus A_1; p)$ (and pictured as a movable marker). For the duration of this proof, k, l denote numbers in $\{0,1\}$, p, q denote numbers in $\mathbb{N}$ and $[p,k]$ stands for $2p+k$.
To avoid that $J^{A_k}(p)$ change too often, we ensure that at each stage s, for each p, k such that $[p,k] \leq s$,

$$J^{A_k}(p)[s] \downarrow \Rightarrow \ \gamma(A_0 \oplus A_1)([p,k]) > j(A_k, p)[s] \tag{1}$$

Construction. At stage s, define $\Gamma(A_0 \oplus A_1; s)$ with large use, and do the following.

a) If there is $[p,k]$ such that $J^{A_k}(p)[s-1] \uparrow$ and $J^{A_k}(p) \downarrow$ at the beginning of stage s, choose $[p,k]$ minimal such. Put $\gamma(A_0 \oplus A_1; [p,k])$ into A_{1-k} and redefine $\Gamma(A_0 \oplus A_1; q)$, $s \geq q \geq [p,k]$, with the correct value and large use.
b) If $n \in \emptyset'_s - \emptyset'_{s-1}$, then put $\gamma(A_0 \oplus A_1; n)$ into A_{1-k}.

A typical set-up looks like this:

(Here, $J^{A_1}(p)$ converged after $J^{A_0}(p)$.)

Verification We first check (1) by induction on s. The condition holds for $s = 0$. If $s > 0$, we may suppose there is $[p,k]$ minimal such that $J^{A_k}([p,k])[s-1] \uparrow$ and $J^{A_k}(p)[s] \downarrow$ (else there is nothing to prove), in which case we put $v = \gamma(A_0 \oplus A_1; [p,k])$ into A_{1-k}.

- If $[q,l] < [p,k]$, then $v > \gamma(A_0 \oplus A_1; [q,l]) \geq j(A_l, q)[s]$ by inductive hypothesis, so that (1) remains true for $J^{A_l}(q)[s]$.
- Since we enumerate v into the "other" side, $J^{A_k}(p)[s]$ remains convergent, so we ensure $\gamma(A_0 \oplus A_1; [p,k]) > j(A_k, p)[s]$.
- For $[q,l] > [p,k]$, computations $J^{A_l}(q)[s]$ have their use below the new value of $\gamma([q,l])$.

Next we show that both A_0 and A_1 are super-low. As in the standard construction of a (super-)low simple set, let $g_k(p,s) = 1$ if $J^{A_k}(p)[s] \downarrow$ and let $g_k(e,s) = 0$ otherwise. We define a computable function c such that $c([p,k])$ is a bound on the number of times $J^{A_k}(p)$ can become defined. Then $b_k(p) = 2c([p,k]) + 1$ bounds how often $g_k(p,s)$ changes.
By (1), $J^{A_0}(p)$ becomes undefined at most $2p$ times due to change of $\emptyset' \upharpoonright 2p$. Otherwise, $J^{A_k}(p)$ becomes undefined only when some computation $J^{A_{1-k}}([q,1-k])$ becomes defined, where $[q,1-k] < [p,k]$. Thus the recursive function c given by $c(0) = 1$, $c([p,k]) = 2p + \sum\{c([q,1-k]) : \ [q,1-k] < [p,k]\}$ is as desired.
As a consequence, each marker $\gamma(A_0 \oplus A_1; m)$ reaches a limit. Thus $\emptyset' = \Gamma(A_0 \oplus A_1)$. $\square$

In contrast to the case of the Sacks Splitting theorem, we cannot achieve that Γ above is a *wtt*-reduction. Bickford and Mills [**1**, Thm. 4.1] show that, in fact, no super-low r.e. set is cuppable in the r.e. *wtt*-degrees.

4. Traceability versus super lowness

THEOREM 4.1. *Let A be r.e. Then the following are equivalent.*

(i) *A is jump traceable*
(ii) *A is super-low.*

Both directions are effective.

PROOF. (i) $\Rightarrow$ (ii). Suppose A is jump traceable via a trace T with bound h. By convention, for each s, $T_s \subseteq [0,s)$. Consider the following partial A–recursive function:

$$q(e) = \mu s (J^A(e)[s] \downarrow \ \& \ A_s \upharpoonright j(A_s, e, s) = A \upharpoonright j(A_s, e, s)).$$

By Fact 1.4 there is a total computable α such that, for all e, $q(e) \simeq J^A(\alpha(e))$. Then, for each s,

$$(J^A(e)[s] \uparrow \ \& \ J^A(e) \downarrow) \ \Rightarrow \ J^A(\alpha(e)) \geq s, \tag{2}$$

since $J^A(\alpha(e)) < s$ implies that $J^A(e)$ has reached a final value by stage s.
We define computable functions $g(e,s)$, $b(e)$ as in Definition 1.1 witnessing that A is super-low. Let $g(e,0) = 0$. For $t > 0$, if $J^A(e)[t] \uparrow$ then let $g(e,t) = 0$. Now suppose $J^A(e)[t] \downarrow$. If $g(e,t-1) = 1$ then let $g(e,t) = 1$. If $g(e,t-1) = 0$, then we first test the stability of the computation $J^A(e)[t]$ before allowing $g(e,t) = 1$: let

$s < t$ be the greatest stage such that $J^A(e)[s] \uparrow$. If $v = J^A(\alpha(e))[t] \downarrow$, $s \leq v$ and $v \in T_t^{[e]}$ then let $g(e,t) = 1$, otherwise $g(e,t) = 0$.
We claim that $g(e,t)$ changes at most $2h(\alpha(e)) + 2$ times. It suffices to show that $g(e,t)$ changes from 1 to 0 and back to 1 at most $h(\alpha(e))$ times. Thus, suppose $s > 0$, $g(e,s-1) = 1$, $g(e,s) = 0$ (so that $J^A(e)[s] \uparrow$) and $t > s$ is least such that $g(e,t) = 1$. Then $v = J^A(\alpha(e))[t] \downarrow$ and $s \leq v$. Since $T_s \subseteq [0,s)$ and $v \in T_t^{[e]}$, $T_t^{[e]} - T_s^{[e]} \neq \emptyset$. This can happen at most $h(\alpha(e))$ times.

$J^A(e) \uparrow$ $\qquad\qquad\qquad\qquad$ $v \in T_t^{[e]}$

s $\qquad\qquad$ $v = J^A(\alpha(e))$ $\qquad\qquad$ t

$g(e,s-1) = 1,\ g(e,s) = 0$ $\qquad\qquad\qquad$ $g(e,t) = 1$

It remains to be shown that $A'(e) = \lim_s g(e,s)$. If $J^A(e) \uparrow$, then $g(e,s) = 0$ for infinitely many s, so $\lim_s g(e,s) = 0$. Now suppose $J^A(e) \downarrow$. Let s be greatest such that $J^A(e)[s] \uparrow$. Since $J^A(\alpha(e)) \downarrow$, there is a $t \geq s$ such that the computation $v = J^A(\alpha(e))$ is stable and $v \in T_t^{[e]}$. Then $s \leq v$ by (2). So we define $g(e,t') = 1$ for each $t' \geq t$.
Note that we have obtained g and b effectively in the trace T and its bound.
(ii) $\Rightarrow$ (i). Suppose A is super-low. Thus A' is ω-r.e. via some functions g, b. We enumerate a trace T to show A is jump traceable, and also define an auxiliary partial recursive functional Ψ, which copies computation of the jump J with some delay. We assume a partial recursive functional $\widetilde{\Psi}$ is given, and let α be the reduction function for $\widetilde{\Psi}$ according to Fact 1.4. Since we produce Ψ effectively from α, by the Recursion Theorem we can assume that $\widetilde{\Psi} = \Psi$, so that α is also a reduction function for Ψ.
Given e, let $\widehat{e} = \alpha(e)$. At stage $s = 0$, Ψ is totally undefined. For $s > 0$, we distinguish two cases.

a) $g(\widehat{e}, s) = 0$. If $\Psi^A(e)[s-1] \uparrow$ and $J^A(e)[s] \downarrow = v$, define $\Psi^A(e)[s] = v$ with use $j(A_s, e, s)$.
b) $g(\widehat{e}, s) = 1$. If $\Psi^A(e)[s] \downarrow$ then enumerate $y = J^A(e)[s]$ into $T^{[e]}$.

Note that, since Ψ just copies computations of J at a later stage, when a new computation $J^A(e)[s]$ appears, then no computation $\Psi^A(e)[t]$ which was defined at $t < s$ still applies at stage s.
Suppose $J^A(e) = z$, and let s be the least stage where this (final) computation appears. We show $z \in T^{[e]}$. At a stage $t \geq s$, we may only define a new computation $\Psi^A(e)[t]$ in case $g(\widehat{e}, t) = 0$. Since $\Psi^A(e)[t]$ remains undefined till this happens, by the definition of α, in fact there must be such a stage $t \geq s$. Since the use for $\Psi^A(e)[t]$ is $j(A_s, e, s)$ and $A_s \restriction j(A_s, e, s)$ is stable, $\Psi^A(e) \downarrow$. Hence $g(\widehat{e}, r) = 1$ for some $r > t$, at which point we enumerate z into $T^{[e]}$.
Next we show T is a trace with bound $h(e) = \lfloor \frac{1}{2} b(\alpha(e)) \rfloor$. Suppose $q < r$ are stages where distinct elements y, z are enumerated into $T^{[e]}$. Then $y = J^A(e)[q], z = J^A(e)[r]$, and $g(\widehat{e}, q) = g(\widehat{e}, r) = 1$. Since $A_q \restriction j(A_q, e, q) \neq A_r \restriction j(A_q, e, q)$, no definition $\Psi^A(e)[q']$ issued at a stage $q' \leq q$ is valid at stage r. (Here is where we need that A is r.e.) So we must have made a new definition $\Psi^A(e)[t]$ at a stage t,

$q < t < r$, whence $g(\widehat{e}, t) = 0$. Since $g(\widehat{e}, s)$ can change from 1 to 0 and back at most $h(e)$ times, this proves that $|T^{[e]}| \leq h(e)$.
Using the Recursion Theorem with indices for g and b as parameters, we obtain T and h effectively in those indices. □

We obtain an interesting consequence which is not obvious from the definition.

COROLLARY 4.2. *$\{e : W_e$ super-low $\}$ is Σ^0_3-complete.*

PROOF. This follows from the corresponding fact for jump-traceability, Proposition 2.4. □

THEOREM 4.3. *There is a super-low real A which is not jump-traceable.*

PROOF. In [**7**] we show that no r.e. traceable set is diagonally non-computable. Since a ML-random set is diagonally non-computable, the Martin-Löf random real obtained in Corollary 3.2 is not jump-traceable. □

THEOREM 4.4. *There is an ω-r.e. jump-traceable real A which is not super-low.*

Notice however that A is low by Proposition 2.2.

PROOF. Fix an effective listing $(g_e, b_e)_{e \in \mathbb{N}}$ of all pairs consisting of a binary and a unary partial recursive function, such that for all w, $\{q : g_e(w,q) \downarrow\}$ is an initial segment of $\mathbb{N}$. Then we can assume the same property for the approximation at a stage s, $g_e(w,q)[s]$.
To ensure A is not super-low, we meet the requirements
$P_e :\ g_e, b_e$ *total* $\&\ \forall x\ g_e(x,q)$ *changes at most* $b_e(x)$ *times* $\Rightarrow$
$\quad \exists y\ \neg A'(y) = \lim_q g_e(y,q)$.
We define an auxiliary binary p.r. functional Ψ. As usual, by the Recursion Theorem, we are given a reduction function α such that $\Psi^X(e,y) = J^X(\alpha(\langle e,y\rangle))$. The strategy for P_e is as follows.

(1) Pick a fresh candidate y at stage t. Let $\widetilde{y} = \alpha(e,y)$. Wait till $b_e(\widetilde{y}) \downarrow$ at a stage t.
(2) Pick a fresh number z (thus, $z \geq b_e(\widetilde{y})$), called the *parameter* of P_e. From now on, ensure that
$$\Psi^A(e,y) \downarrow \Leftrightarrow\ z \in A.$$
To do so, for all strings σ of length z, define $\Psi^{\sigma 1}(e,y) = 1$. This is allowed, since there have been no definitions with arguments e, y so far.
Do the following at most $b_e(\widetilde{y})$ times at stages $s \geq t$: Whenever $g_e(\widetilde{y}, q)[s-1] \uparrow$, $g_e(\widetilde{y}, q)[s] \downarrow$, and $g_e(\widetilde{y}, q-1) \neq g_e(\widetilde{y}, q)$, then declare $A_s(z) = 1 - g_e(\widetilde{y}, q)$. Otherwise $A_s(z) = A_{s-1}(z)$.

Then, if the hypothesis of P_e is satisfied,
$$\widetilde{y} \in A' \ \Leftrightarrow\ \Psi^A(e,y) \downarrow \Leftrightarrow\ \lim_s g_e(\widetilde{y}, s) = 0$$
Moreover, $A_s(z)$ changes at most z times (since $z \geq b_e(\widetilde{y})$), so that A is ω-r.e. To ensure A is jump traceable, we enumerate a trace T. We meet the requirements
$$Q_e : |T^{[e]}| \leq h(e)\ \&\ (v = J^A(e) \downarrow \Rightarrow\ v \in T^{[e]}),$$

where $h(e)$ is a recursive bound to be determined below.
The priority ordering of requirements is $Q_0 < P_0 < Q_1 < \ldots$. The strategy for Q_e is simple: whenever a computation $J^A(e) = v$ appears at stage s which has not been seen before, then

(1) put v into $T^{[e]}$
(2) initialize the requirements P_i, $i \geq e$.

We say that Q_e *acts*. In that case, $A(z)$ retains its value, for any parameter z of a lower priority requirement P_j. Therefore, unless also a higher priority P_i is initialized, for $t \geq s$, $A_t \upharpoonright j(A_t, e, t)$ only depends on the values $A(z)$, where z is the parameter of a higher priority P_i, which gives at most 2^e possibilities for $A_t \upharpoonright j(A_t, e, t)$ (here we need that P_i only needs to change $A(z)$ for a single number z, which would fail if we had to make A r.e.).
Construction. Let $A_0 = \emptyset$. At stage $s > 0$, go through the requirements $Q_0, P_0, \ldots, Q_s, P_s$ and let them carry out one step of their strategies. At the end, if $y \leq s$ and no value has been assigned yet to $A_s(y)$, retain the value at stage $s - 1$.
Verification. Let $h(0) = 1$ and, for $e > 0$, let $h(e) = h(e-1)(2^e + 1)$

Lemma 4.5. *Let $e \geq 0$. Then*

(i) *Q_e is met*
(ii) *P_e is initialized at most $h(e)$ times and met.*

Proof. For $e = 0$, (i) and (ii) hold, since P_0 is initialized at most once, when $J^A(0)$ converges for the first time. Assume $e > 0$.
(i) While P_{e-1} is not initialized, the requirements P_i, $i < e$ pick at most one number z. If F_s is the set of such numbers at a stage s, then there are at most 2^e possibilities for $A_s \cap F_s$. Hence Q_e enumerates at most 2^e numbers into $T^{[e]}$ before P_{e-1} is initialized another time. Hence, by inductive hypothesis (ii) for $e-1$, $|T^{[e]}| \leq 2^e h(e-1) \leq h(e)$.
(ii) If P_e is initialized, then either P_{e-1} is also initialized, or Q_e acts. So P_e is initialized at most $h(e)$ times. Once it is no more initialized, P_e diagonalizes successfully. □

5. A construction of a K-trivial r.e. real

The following theorem was considered in discussions with Downey and Hirschfeldt.

Theorem 5.1. *For each low r.e. set B there is an r.e. K-trivial set A such that $A \not\leq_T B$.*

Proof. Let $\mathbb{N}^{\langle e \rangle}$ denote the set of numbers of the form $\langle y, e \rangle$. We meet the requirements

$$P_e : A \neq \{e\}^B,$$

by enumerating numbers $x \in \mathbb{N}^{\langle e \rangle}$ into A. To ensure A is K-trivial, we apply the criterion implicit in [**3**, Theorem 3.1] in the form presented in [**9**, Prop. 3.3.]. This is actually a characterization of $\mathcal{K}$, as proved in [**9**, Theorem 5.12]. We refer to those papers for motivation, and to [**9**] for a proof.
Note that $K(y) = \lim_s K_s(y)$, where $K_s(y) = \min\{|\sigma| : U_s(\sigma) = y\}$. One uses the "cost function"

$$c(x, s) = 1/2 \sum_{x < y \leq s} 2^{-K_s(y)},$$

which bounds the cost of changing $A(x)$ at stage s. Note that $c(x,s)$ is nondecreasing in s, $\lim_s c(x,s) \leq 1/2$ for each x, and $\lim_x \lim_s c(x,s) = 0$ by the definition of prefix Kolmogorov complexity.

FACT 5.2 ([**9**]). *Suppose that $A(x) = lim_s A_s(x)$ for a Δ^0_2–approximation (A_s) such that*

$$S = \sum \{c(x,s) : s > 0 \; \& \; x \textit{ is minimal s.t. } A_{s-1}(x) \neq A_s(x)\} \leq 1/2. \tag{3}$$

Then A is K-trivial.

To meet the requirements P_e, we use a Robinson type procedure, using the lowness of B to "certify" computations $\{e\}^B(x)[s] = 0$. We may ask a a $\Sigma^0_1(B)$-question about the enumeration of A, and we have a Δ^0_2-approximation to the answer. But which enumeration? We may assume that it is given, by the recursion theorem. Formally, an *enumeration* is an index for a partial recursive function $\mathcal{A}$ defined on an initial segment of $\mathbb{N}$ such that, where $\mathcal{A}(t)$, is interpreted as a strong index for the part of A enumerated by stage t, $\mathcal{A}(s) \subseteq \mathcal{A}(s+1)$ for each s. We write $\mathcal{A}_t$ for $\mathcal{A}(t)$. Given any (possibly partial) enumeration $\widetilde{\mathcal{A}}$, we effectively produce an enumeration $\mathcal{A}$, asking $\Sigma^0_1(B)$-questions about the given enumeration $\widetilde{\mathcal{A}}$. We must show that $\mathcal{A}$ is total in the interesting case that $\mathcal{A} = \widetilde{\mathcal{A}}$ (by the recursion theorem), where these questions are actually about $\mathcal{A}$.
Here is the $\Sigma^0_1(B)$-question for requirement P_e:
Is there a stage s and an $x \in \mathbb{N}^{\langle e \rangle}$ such that $\widetilde{\mathcal{A}}$ is defined up to $s-1$, and

- *$\{e\}^B(x) = 0[s]$, where $B_s \upharpoonright u(B_s,e,x,s) = B \upharpoonright u(B_s,e,x,s)$ (B is correct on the use of the computation), and*
- *$c(x,s) \leq 2^{-(e+n+3)}$,*

where $n = |\mathbb{N}^{\langle e \rangle} \cap \widetilde{A}(s-1)|$ is the number of enumerations for the sake of P_e prior to s.
Since B is low, there is a total computable function $g(e,s)$ such that $\lim g(e,s) = 1$ if the answers is YES, and $\lim g(e,s) = 0$ otherwise. (The function $g(e,s)$ actually depends on a further argument which we supress, an index for $\widetilde{\mathcal{A}}$.)
Construction. We define $\mathcal{A}_s$, assuming $\mathcal{A}_{s-1}$ has been defined or $s = 0$.
For each $e < s$, if there is an $x < s$, $x \in \mathbb{N}^{\langle e \rangle}$ satisfying

$$\{e\}^B(x) = 0[s] \; \& \; c(x,s) \leq 2^{-(e+n+3)},$$

where $n = |\mathbb{N}^{\langle e \rangle} \cap \mathcal{A}_{s-1}|$, then choose x least and search for the least $t \geq s$ such that $g(e,t) = 1$, or $B_t \upharpoonright u \neq B_s \upharpoonright u$, where $u = u(B_s,e,x,s)$ is the use at s. In the first case, enumerate x into A (at the current stage s). If the search does not end for some $e < s$, then we leave $\mathcal{A}_s$ undefined.
Verification. We may assume $\mathcal{A} = \widetilde{\mathcal{A}}$ by the recursion theorem.

LEMMA 5.3. *$\mathcal{A}$ is total.*

PROOF. Assume that $\mathcal{A}_{s-1}$ is defined or $s = 0$. Since $\mathcal{A} = \widetilde{\mathcal{A}}$ and by the correctness of $\lim_t g(e,t)$, the search at stage s ends for each e. So we define $\mathcal{A}_s$. $\square$

LEMMA 5.4. *A is K–trivial.*

Proof. We apply the Fact 5.2. At stage s, suppose x is minimal s.t. $\mathcal{A}_{s-1}(x) \neq \mathcal{A}_s(x)$. We enumerate x for the sake of some requirement P_e, which so far has enumerated n numbers. Then $c(x,s) \leq 2^{-(e+n+3)}$, hence $S \leq \sum_{0 \leq e,n} 2^{-(e+n+3)} = 1/2$. $\square$

Lemma 5.5. *Each requirement P_e is met.*

Proof. Suppose for a contradiction that $A = \{e\}^B$. First assume $\lim_s g(e,s) = 1$. Choose witnesses x, s for the affirmative answer to the $\Sigma^0_1(B)$ question for P_e. Since $B \upharpoonright u$ does not change after s where $u = u(B_s, e, x, s)$, we search for t till we see $g(e,t) = 1$. Then P_e enumerates x at stage s.
Now consider the case $g(e,s) = 0$ for all $s \geq s_0$. Then we do not enumerate numbers for the sake of P_e after stage s_0. Then there is n such that P_e puts just n numbers into A. Since $A = \{e\}^B$, there is $x \in \mathbb{N}^{\langle e \rangle}$ and $s \geq s_0$ such that $\{e\}^B(x) = 0[s]$ and $c(x,s) \leq 2^{-(e+n+3)}$, where $n = |\mathbb{N}^{\langle e \rangle} \cap A|$. So the answer to the $\Sigma^0_1(B)$ question for P_e is Yes, contradiction. $\square$

Note that the action of P_e may be infinitary, which is harmless here, but could be avoided by refining the $\Sigma^0_1(B)$ question.
Also note that the argument in the proof of Lemma 5.5, in the case $\lim_s g(e,s) = 1$ breaks down if B is merely Δ^0_2. The opponent can now present the correct computation $\{e\}^B(x) = 0$ at a stage s where the limit $\lim_s g(e,s)$ has not yet been reached. Then he temporarily changes B below the use at stage $t > s$ while keeping $g(e,t) = 0$, and we do not put x into A at s. At a later stage where the old computation $\{e\}^B(x) = 0$ comes back, he has increased the cost function above $2^{-(e+n+3)}$. Thus the following question remains:

Question 5.6. *Does Theorem 5.1 hold for Δ^0_2 low sets B?*

We may replace B by a u.r.e. sequence of uniformly low sets B_i and obtain a stronger result, which is proved by making the appropriate notational changes in the proof of Theorem 5.1.

Corollary 5.7. *For each u.r.e. sequence of uniformly low sets B_i, there is an r.e. K-trivial set A such that $A \not\leq_T B_i$ for each i.*

We apply this to a class first studied by Andrei Muchnik (1998).

Definition 5.8 ([**9**]). *A is* low for K *via a constant b if*

$$\forall y \ K(y) \leq K^A(y) + b.$$

Let $\mathcal{M}$ denote this class of reals.

In [**9**] it is proved as a main result that $\mathcal{K} = \mathcal{M}$.
Note that $\mathcal{M} \subseteq \mathcal{M}[p]$ for reach p as in Definition 1.3. Thus each $A \in \mathcal{M}$ is jump traceable. Here is a uniform version of this.

Proposition 5.9. *There is a fixed C such that, if A is low for K via b, then A is jump-traceable, where the trace T_b is obtained effectively in b and has bound $C 2^b i \log i$.*

Proof. Up to constants, for each i such that $J^A(i)$ is defined,

$$K(i) \geq K^A(i) \geq K^A(J^A(i)) \geq K(J^A(i)) - b,$$

and hence $J^A(i) \in \{y : K(y) \leq K(i) + b + d\}$, for some fixed d. Since $K(i) \leq \log i + 2 \log\log i + d'$ for some fixed constant d', the trace T_b given by $T_b^{[i]} = \{y : K(y) \leq \log i + 2 \log\log i + b + d + d'\}$ is as required. $\square$

The class $\mathcal{M}$ is Σ^0_3 on both the ω-r.e. and the r.e. sets. Since it includes all finite sets, there is a u.r.e. listing of the r.e. sets in $\mathcal{M}$. However, there is no way to determine a constant for being low for K:

THEOREM 5.10. *There is no effective sequence* (B_i, b_i) *of pairs of an r.e. set and a constant such that each* B_i *is low for* K *via* b_i*, and for each r.e. set* A*,*

$$A \in \mathcal{M} \;\Rightarrow\; \exists i\; A \leq_T B_i.$$

In particular, there is no such sequence listing all *the r.e. sets in* $\mathcal{M}$*.*

PROOF. By Proposition 5.9, each B_i is jump-traceable, where the trace and its bound are obtained effectively in b_i. By Theorem 4.1, we obtain a witness for the (super)-lowness of B_i, effectively in i. The result follows by Corollary 5.7 $\square$

We close with a further question.

QUESTION 5.11. *Find an elementary property which distinguishes the classes of low and super-low r.e. degrees. For instance, is there a noncappable (hence, low cuppable) degree which does not cup with a super-low r.e. degree to* $0'$*? Is there a degree which is not the supremum of two super-low degrees?*

It would also be interesting to see to what extent Theorem 4.1 holds for d.r.e. sets.
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Dept. of Computer Science, Univ. of Auckland, Web site http://www.cs.auckland.ac.nz/∼nies
E-mail address: andre@cs.auckland.ac.nz