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Lowness properties and randomness

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Abstract

The set A is low for (Martin-Löf) randomness if each random set is already random relative to A. A is K-trivial if the prefix complexity K of each initial segment of A is minimal, namely $\forall n\ K(A\upharpoonright n) \leqslant K(n) + \mathcal{O}(1)$. We show that these classes coincide. This answers a question of Ambos-Spies and Kučera in: P. Cholak, S. Lempp, M. Lerman, R. Shore, (Eds.), Computability Theory and Its Applications: Current Trends and Open Problems, American Mathematical Society, Providence, RI, 2000: each low for Martin-Löf random set is Δ^0_2. Our class induces a natural intermediate Σ^0_3 ideal in the r.e. Turing degrees, which generates the whole class under downward closure.

Answering a further question in P. Cholak, S. Lempp, M. Lerman, R. Shore, (Eds.), Computability Theory and Its Applications: Current Trends and Open Problems, American Mathematical Society, Providence, RI, 2000, we prove that each low for computably random set is computable.

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1. Introduction

Two classes of sets have been discovered independently by different researchers. We demonstrate that they coincide. This class also leads to the first example of a natural

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intermediate Σ^0_3 ideal in the r.e. Turing degrees. (All sets will be sets of natural numbers unless otherwise stated. They are identified with infinite strings over $\{0, 1\}$.)

- Chaitin [7] and Solovay [28] studied the class of K-trivial sets (which we denote $\mathcal{K}$). The set A is K-trivial if the prefix complexity of each initial segment of A is minimal. Solovay constructed a non-recursive K-trivial set.
- Zambella [30] introduced the class Low(MLR) of low for random sets, a property which says the set is computationally weak as an oracle: no regularity can be detected in a random set when using A as an oracle. Kučera and Terwijn [15] constructed a non-computable r.e. low for random set.

Muchnik (1998) defined the class $\mathcal{M}$ of low for K sets, which as an oracle do not reduce the prefix complexity of a string. In unpublished work, he constructed a non-recursive set in $\mathcal{M}$. By an easy argument, $\mathcal{M}$ is included in both $\mathcal{K}$ and Low(MLR), and all sets in $\mathcal{M}$ are low in the usual sense. We show that Low(MLR) $= \mathcal{M}$ and $\mathcal{K}$ is closed downward under Turing reducibility, which leads to a proof that $\mathcal{K}$ equals $\mathcal{M}$. Hence all three classes coincide. However, Low(MLR) and $\mathcal{K}$ represent very different aspects of the same notion. The class Low(MLR) expresses that the set is computationally weak, while $\mathcal{K}$ states that the set is far from random. The part $\mathcal{K} = \mathcal{M}$ is joint with D. Hirschfeldt, and can be proved by modifying the argument that $\mathcal{K}$ is closed downward.

Our results continue a line of research started by van Lambalgen, Kurtz and others, joining two areas of computability theory: the complexity of sets, and their randomness properties. To classify sets by their absolute complexity, one introduces a hierarchy of classes: computable, recursively enumerable, Δ^0_2, etc. $\mathcal{K}$ lies in between computable and Δ^0_2. The complexity of sets is compared via reducibilities, for example Turing reducibility $\leqslant_T$. To study the classes in this hierarchy and the degree structures arising from these reducibilities, increasingly difficult forcing arguments and priority constructions are needed.

The most commonly accepted notion of algorithmic randomness is the one introduced by Martin-Löf [16]. A MARTIN-LÖF TEST is a uniformly r.e. sequence (U_n) of open sets in Cantor Space 2^ω such that $\mu(U_n) \leqslant 2^{-n}$, where μ is the usual Lebesgue measure on 2^ω. A set X is MARTIN-LÖF RANDOM if it passes each test in the sense that $X \notin \bigcap_n U_n$. The class of such sets is denoted MLR. Schnorr [24] proved that a set X is random in this sense if and only if the algorithmic prefix complexity K of all its initial segments is large, namely $\forall n\ K(X \upharpoonright n) \geqslant n - \mathcal{O}(1)$. The methods used to study algorithmic randomness have been quite different from the ones mentioned above—they were effective measure theoretic or, when dealing with K-complexity, combinatorial.

The constructions below share elements from both approaches. The enumeration of a number into a set is replaced by the enumeration of certain objects (say, clopen sets of small measure) which can be subdivided arbitrarily much.

The class $\mathcal{K}$ induces a Σ^0_3 ideal in the r.e. Turing degrees, which generates the whole of $\mathcal{K}$ under Turing downward closure. As in computational complexity theory, such closure properties can be taken as further evidence that this common class $\mathcal{K}$ is a very natural one. $\mathcal{K}$ is the first known example of a natural intermediate Σ^0_3-ideal, and $\mathcal{K}$ also is the first Σ^0_3-ideal not obtained by a direct construction. The existence of such an ideal is surprising as Turing reducibility itself on the r.e. sets is only Σ^0_4. Moreover,

$\mathcal{K}$, as an operator, is *degree invariant*, namely, for Turing equivalent sets X, Y, the relativized classes $\mathcal{K}^X$ and $\mathcal{K}^Y$ coincide. This relates to Sacks' question whether there is a degree invariant solution to Post's Problem [23]. A degree invariant ideal which is also principal would give such a solution (at least as a Borel operator). However, we also prove that $\mathcal{K}^X$ is not a principal ideal in the r.e. degrees relativized to X.

The classes and concepts: In the following we discuss the relevant classes and concepts in an informal way, deferring the formal definitions to Section 2. More intuition on the concepts and techniques of this paper can be found in [10].

A LOWNESS PROPERTY of a set A expresses that, in some way, A has low computational power when used as an oracle. We require that such a property be downward closed under $\leqslant_T$. The usual lowness, $A' \equiv_T \emptyset'$, is an example. The lowness property Low(MLR) is itself based on relative randomness: A is LOW FOR MARTIN-LÖF RANDOM if each random set X is already random relative to A, i.e. X passes all A-r.e. tests. Terwijn and Kučera [15, submitted 1997] constructed a non-computable r.e. low for random set.

The class $\mathcal{K}$ of K-trivial sets embodies being far from random. While random sets have high initial segment complexity, for K-trivial sets this complexity is as low as possible, namely $\forall n\ K(X\restriction n) \leqslant K(n) + \mathcal{O}(1)$. Clearly each computable set is K-trivial. Chaitin [6] proved that $\mathcal{K} \subseteq \Delta^0_2$. Solovay (1975), in a widely circulated manuscript [28], gave the first, rather complicated construction of a non-computable set in $\mathcal{K}$, which was adapted by Calude and Coles [5] to the r.e. case. Kummer (unpublished) and Downey (see [9]) independently built an r.e. non-computable set in $\mathcal{K}$ via similar, very short and elegant constructions.

Let $K^A(y)$ be the prefix complexity of y relative to the oracle A. We call a set A LOW FOR K if $\forall y\ K(y) \leqslant K^A(y) + \mathcal{O}(1)$. In other words, the oracle A cannot be used to further compress the string y. The class of such sets is denoted $\mathcal{M}$. Andrei Muchnik (unpublished, 1998) constructed a non-computable r.e. set in this class.

Cost functions: The constructions of a non-recursive set in those apparently very different classes are quite similar, which was a first indicator that the classes are the same. We describe a common framework for those constructions, called the cost function method. A COST FUNCTION is a computable function $c : \mathbb{N} \times \mathbb{N} \mapsto \{q \in \mathbb{Q} : q \geqslant 0\}$ such that $\lim_x \lim_s c(x, s) = 0$. Suppose we are building a Δ^0_2-set A, via a Δ^0_2-approximation (A_r). At stage s, if x is least such that $A_s(x)$ changes (for instance to meet a requirement in a list of requirements ensuring that A is non-computable), the cost of this change is $c(x, s)$. The global restraining requirement is that the sum of the costs over all stages be finite.

One defines a cost function which ensures that the constructed set is in the relevant class. For $\mathcal{K}$, one uses $c(x, s) = d \sum_{x<y\leqslant s} 2^{-K_s(y)}$, where $K_s(y)$ denotes the prefix complexity of y by stage s (the particular choice of the constant $d > 0$ is irrelevant). This method is interesting because it has no injury to requirements, thereby giving a new injury free solution to Post's problem.

We will see that, conversely, if $A \in \mathcal{K}$, then this Δ^0_2 set can be viewed as being built via the cost function method for $\mathcal{K}$. From this characterization one obtains further information about $\mathcal{K}$. For instance, each set $A \in \mathcal{K}$ is truth-table below an r.e. set in $\mathcal{K}$.

Other randomness notions: A set Z is COMPUTABLY RANDOM if no computable betting strategy (martingale) which is monotone, i.e. bets on the bit positions in their natural order, succeeds on Z. If no strategy betting in *any* order succeeds, the set is called KOLMOGOROV–LOVELAND RANDOM. Denoting the classes of such sets by CR and KLR, respectively, the inclusions $\mathsf{MLR} \subseteq \mathsf{KLR} \subset \mathsf{CR}$ hold. A persistent open question is whether the first inclusion is strict as well [2,19].

Given randomness notions $\mathcal{C} \subseteq \mathcal{D}$, let $\mathrm{Low}(\mathcal{C}, \mathcal{D})$ denote the class of oracles A such that $\mathcal{C} \subseteq \mathcal{D}^A$. We write $\mathrm{Low}(\mathcal{C})$ for $\mathrm{Low}(\mathcal{C}, \mathcal{C})$. Note that one makes the class $\mathrm{Low}(\widetilde{\mathcal{C}}, \widetilde{\mathcal{D}})$ larger by decreasing $\mathcal{C}$ or increasing $\mathcal{D}$.

In Section 5 we prove that in fact $\mathrm{Low}(\mathsf{MLR}, \mathsf{CR}) = \mathcal{M}$, which implies both $\mathrm{Low}(\mathsf{MLR}) = \mathcal{M}$ and $\mathrm{Low}(\mathsf{KLR}) \subseteq \mathcal{M}$. However, it is unknown if non-recursive sets in $\mathrm{Low}(\mathsf{KLR})$ exist. If not, then at least for some oracle X, the relativized classes KLR^X and MLR^X are distinct.

Recent history of the results: Kučera and Terwijn [15] had asked if there is a low for random set outside Δ^0_2. (This is also Problem 4.4. in Ambos-Spies and Kučera [2]). The work of Terwijn and Zambella on Schnorr low sets [29] suggested the existence of such a set, since there are continuum many Schnorr low sets, and they are necessarily outside Δ^0_2. Stephan and Nies showed that $\{e : W_e \in \mathrm{Low}(\mathsf{MLR})\}$ is Σ^0_3. To do so they gave a characterization of $\mathrm{Low}(\mathsf{MLR})$. Using a modified form of this characterization, the author proved $\mathrm{Low}(\mathsf{MLR}) = \mathcal{M}$, which implies $\mathrm{Low}(\mathsf{MLR}) \subseteq \Delta^0_2$, and finally he strengthened this to $\mathrm{Low}(\mathsf{MLR}, \mathsf{CR}) = \mathcal{M}$, using martingales. Hirschfeldt made an important step towards understanding $\mathcal{K}$, proving that each $A \in \mathcal{K}$ is Turing incomplete (see [9, Theorem 4.1]). The author showed the stronger result that $\mathcal{K}$ is closed downwards under $\leqslant_T$, and gave the characterization of $\mathcal{K}$ via the cost function method. Hirschfeldt conjectured that $\mathcal{K} = \mathcal{M}$, and together they developed the modification of the proof that $\mathcal{K}$ is closed downwards which suffices.

Later the author answered Problem 4.8 in [2], showing that each $\mathrm{Low}(\mathsf{CR})$ set is computable. This was first proved directly, but is derived here from $\mathrm{Low}(\mathsf{MLR}, \mathsf{CR}) = \mathcal{M}$.

Related concepts: Only a few natural ideals are known in the r.e. degrees: the non-cuppable degrees, the non-promptly simple degrees (which by [1] coincide with the cappable degrees) and the almost deep degrees (**a** is almost deep if $\mathbf{a} \vee \mathbf{b}$ is low for each low r.e. degree **b** [8]). The latter two classes are interesting since, as is the case for $\mathcal{K}$, their defining property is not directly related to Turing reducibility. However, only for $\mathcal{K}$ is the ideal also Σ^0_3.

An example of a lowness property from the theory of inductive inference which is analogous to $\mathrm{Low}(\mathsf{MLR})$ is the class of sets of trivial EX-degree, i.e. the sets A such that $\mathrm{EX}[A] = \mathrm{EX}$, where $\mathrm{EX}[A]$ is the class of sets of computable functions which can be learned with an oracle A. Slaman and Solovay [26] proved that the non-recursive sets in this class coincide with the sets Turing equivalent to a 1-generic set in Δ^0_2. Thus, none of them is r.e.

Plan of the paper: In Section 2 we define the classes $\mathcal{K}$, $\mathrm{Low}(\mathsf{MLR})$ and $\mathcal{M}$, and study their basic properties. In Section 3 we discuss an important tool, the Kraft–Chaitin Theorem, based on [6]. The tool is first applied in Section 4, where we give an axiomatic formulation of the construction of a non-computable r.e. set in $\mathcal{K}$ from [9]. In

Section 5 we prove that Low(MLR, CR) $\subseteq \mathcal{M}$ (the converse inclusion is trivial), hence low for random equals low for K. In Section 6 we show that $\mathcal{K}$ is downward closed, that $\mathcal{K} = \mathcal{M}$ and that the construction from Section 3 provides a characterization of $\mathcal{K}$. In a final Section, we relativize $\mathcal{K}$, and we discuss reducibilities related to Low(MLR) and $\mathcal{M}$.

Subsequent work: After the submission of this paper, a further class that had been studied in the literature turned out to be equal to $\mathcal{K}$. Let us say that A is a *basis for ML-randomness* if $A \leqslant_T Z$ for some $Z \in \mathsf{MLR}^A$. This notion was first studied by Kučera [14], who constructed a non-recursive r.e. basis for ML-randomness via a variant of his injury-free solution to Post's problem. Each low for ML-random set A is a basis for ML-randomness. For, by the Kučera–Gács theorem, there is a ML-random Z such that $A \leqslant_T Z$. Then Z is ML-random relative to A.

In [11] it is proved that each basis A for ML-randomness if K-trivial. The proof is easily modified to directly reach the conclusion that A is low for K (see [20]). This gives an alternative, simpler proof of the inclusion Low(MLR) $\subseteq \mathcal{M}$. However, the proof in the present paper also applies to the cases of lowness for computable randomness and KL-randomness. The paper [11] also contains the following variant: if A is r.e., then $A \leqslant_T Z$ for some ML-random Turing incomplete Z implies that A is K-trivial. It is open whether the converse holds.

Notation: We identify a string σ in $2^{<\omega}$ with the natural number n such that the binary representation of $n+1$ is 1σ.

$K^A(y)$ is the length of a shortest prefix description of y using oracle A. More formally, an ORACLE MACHINE is a partial recursive functional $M : 2^\omega \times 2^{<\omega} \mapsto 2^{<\omega}$. We write $M^A(x)$ for $M(A, x)$. M is an ORACLE PREFIX MACHINE if the domain of M^A is an antichain under inclusion of strings, for each A. Let $(M_d)_{d \in \mathbb{N}^+}$ be an effective listing of all oracle prefix machines. The universal oracle prefix machine U is given by

$$U^A(0^{d-1}1\sigma) = M_d^A(\sigma).$$

If $U^A(\sigma) = y$, we say that σ is a U^A*-description* of y. Let $\Omega^A = \mu(\text{dom } U^A)$, and

$$K^A(y) = \min\{|\sigma| : U^A(\sigma) = y\}.$$

When the oracle is $\emptyset$, we obtain the usual notions of prefix machine and universal prefix machine. (We simply write Ω and $K(y)$.) Note that $K(y) = \lim_s K_s(y)$, where $K_s(y) = \min\{|\sigma| : U_s(\sigma) = y\}$ (if there is no such σ, we let $K_s(y) = \infty$). For a string y, $K(y)$ is not far greater than $|y|$, since a prefix code $\widehat{y}$ for y can serve as a description of y. Since there is such a code of length $|y| + 2\log|y|$ [4, Example 2.4], a computable upper bound is $K(y) \leqslant |y| + 2\log|y| + c_K$ for a certain constant c_K (which will be used below).

A Δ^0_2-APPROXIMATION $(A_r)_{r \in \mathbb{N}}$ of a set $A \in \Delta^0_2$ is an effective sequence of finite sets such that $A(x) = \lim_r A_r(x)$. Note that $A \leqslant_{tt} \emptyset'$ iff $A \leqslant_{wtt} \emptyset'$ iff there is such an approximation where the number of changes is recursively bounded. Reals with that property are called ω-r.e.

For a randomness notion $\mathcal{C}$, Non-$\mathcal{C}$ denotes the class of sets not in $\mathcal{C}$.

2. The classes and their basic properties

2.1. *Far from random: the class $\mathcal{K}$*

Note that $K(|y|) \leqslant K(y) + \mathcal{O}(1)$, since one can compute $|y|$ from y. Thus, the following definition expresses that, up to a constant, the K-complexity of initial segments of A is as small as possible.

Definition 2.1 (*Chaitin [6]*). A set A is K-TRIVIAL via a constant b if

$$\forall n \;\; K(A\restriction n) \leqslant K(n) + b.$$

Let $\mathcal{K}$ denote the class of K-trivial sets.

This notion is opposite to Martin-Löf-randomness since, by Schnorr [24], A is Martin-Löf-random iff, for some c, $\forall n \;\; K(A\restriction n) \geqslant n - c$. Thus, A is Martin-Löf-random if for each n, $K(A\restriction n)$ is close to its upper bound, and A is K-trivial if $K(A\restriction n)$ is within a constant of its lower bound $K(n)$. We list some properties of $\mathcal{K}$.

Theorem 2.2 (*Chaitin [6]*). $\mathcal{K} \subseteq \Delta^0_2$.

The proof uses trees of bounded width (also see [9]): the Δ^0_2 tree $T_b = \{\sigma : \forall \rho \subseteq \sigma \;\; K(\rho) \leqslant K(|\rho|) + b\}$ has width at most $\mathcal{O}(2^b)$. If A is K-trivial via the constant b, then A is a path on T_b. All paths on T_b are isolated, so $A \in \Delta^0_2$.

For sets A, B, let $A \oplus B = \{2x : x \in A\} \cup \{2x + 1 : x \in B\}$.

Theorem 2.3 (*Downey et al. [9], Theorem 6.2*). *If $A, B \in \mathcal{K}$, then $A \oplus B \in \mathcal{K}$.*

Let $(\Theta_e)_{e\in\mathbb{N}}$ be an effective listing of all tt-reduction procedures. The following is easily checked.

Fact 2.4. $\{e : \Theta_e \; total \;\; \& \;\; \Theta_e(\emptyset') \in \mathcal{K}\} \in \Sigma^0_3$.

As a consequence, there is a u.r.e. listing of all the r.e. K-trivial sets, and this class has a Σ^0_3 index set. Then, in fact, the index set is Σ^0_3-complete (it is easy to show that any non-trivial Σ^0_3 class of r.e. sets which is closed under finite differences and contains the computable sets has a Σ^0_3-complete index set).

2.2. *Low computational power: the class* Low(MLR)

Note that if $B \leqslant_T A$, then $K^A(y) \leqslant K^B(y) + \mathcal{O}(1)$. In particular, $K^A(y) \leqslant K(y) + \mathcal{O}(1)$. Relativizing the above-mentioned result of Schnorr [24], a set X is Martin-Löf-random relative to A iff, for some c, $\forall n \;\; K^A(X\restriction n) \geqslant n - c$. Let MLR^A denote this class of sets. Then $\mathsf{MLR}^A \subseteq \mathsf{MLR}$ for each A.

Definition 2.5 (*Kučera and Terwijn[15]*). A set A is LOW FOR RANDOM if $\mathsf{MLR}^A = \mathsf{MLR}$. In other words, MLR^A is as large as possible. Let Low(MLR) denote the class of low for random sets.

Note that this is a Π^1_1 definition, and Low(MLR) is closed downward under $\leqslant_T$. Recall that A is generalized low$_1$ (in brief, GL$_1$) if $A' \leqslant_T A \oplus \emptyset'$. A result of Kučera [14, Theorem 2] implies that each low for random A is GL$_1$.

2.3. Both: the class $\mathcal{M}$

We next consider Andrei Muchnik's class of sets which, when used as an oracle, do not decrease K.

Definition 2.6. A is LOW FOR K if $\forall y \;\; K(y) \leqslant K^A(y) + \mathcal{O}(1)$.
Let $\mathcal{M}$ denote this class of sets.

Note that $\mathcal{M} \subseteq$ Low(MLR), since MLR^X may be defined in terms of K^X. Moreover, $\mathcal{M} \subseteq \mathcal{K}$, since $\forall n \;\; K^A(A\restriction n) \leqslant K^A(n) + \mathcal{O}(1)$, and we may replace K^A by K if $A \in \mathcal{M}$. $\mathcal{M}$ is closed downward under $\leqslant_T$.

We show that the sets A in $\mathcal{M}$ satisfy a lowness property saying that $U^A(\sigma)$ has few possible values. (A related property, being recursively traceable, was used in [29] to characterize the oracles which are low for Schnorr tests.) Given $T \subseteq \mathbb{N}$, let $T^{[x]} = \{y : \langle y, x\rangle \in T\}$.

Definition 2.7. (i) A r.e. set $T \subseteq \mathbb{N}$ is a TRACE if for some computable h, $\forall x \;\; |T^{[x]}| \leqslant h(x)$. We say that h is a BOUND for the trace T.
(ii) The set A is U-TRACEABLE if there is an r.e. trace T such that

$$\forall \sigma \;\; (U^A(\sigma)\downarrow \Rightarrow U^A(\sigma) \in T^{[|\sigma|]}).$$

(Recall the identification of strings with numbers here.) Equivalently, one may require that there is a trace S such that $\{e\}^A(e)$ is in $S^{[e]}$ in case $\{e\}^A(e)$ defined. It is not hard to show that U-traceable sets are in GL$_1$ (see [22]).

Proposition 2.8. *If a set A is low for K, then A is U-traceable and low.*

Proof. For U-traceability, suppose $A \in \mathcal{M}$ via a constant b. Clearly, if $U^A(\sigma)$ is defined then $K^A(U^A(\sigma)) \leqslant K^A(\sigma) + \mathcal{O}(1)$. Since $A \in \mathcal{M}$, this implies $\forall \sigma \;\; K(U^A(\sigma)) \leqslant K(\sigma) + c_K$. Now $K(\sigma) \leqslant |\sigma| + 2\log_2(|\sigma|) + \mathcal{O}(1)$, so it is sufficient to let $T^{[n]} = \{y : K(y) \leqslant n + 2\log_2(n) + d\}$, for an appropriate constant d (which can in fact be determined effectively from b). T is a trace because $|T^{[n]}| = \mathcal{O}(2^n n^2)$.
Since A is in $\mathcal{M} \subseteq \mathcal{K} \subseteq \Delta^0_2$ and A is GL$_1$, A is low. $\square$

One may also prove that A is GL$_1$ in a direct way: for a stage s, using A we can check whether $\{e\}^A_s(e)\downarrow$. So all we need is a bound on the last stage where this can happen. If such a stage exists, then its K^A complexity, and hence its K-complexity, is at most $e + \mathcal{O}(1)$. Hence $\emptyset'$ can compute such a bound.

We summarize the properties of our classes we have seen so far.

	$\mathcal{K}$	Low(MLR)	$\mathcal{M}$
Closed under $\oplus$	yes	?	?
$\leqslant_T$- downward closure	?	yes	yes
Index set of r.e. members	Σ^0_3-complete	?	?
Superclasses	Δ^0_2	GL_1	Low, U-traceable

3. The Kraft–Chaitin theorem

In this section we review an important tool for our constructions.

Definition 3.1. An r.e. set $W \subseteq \mathbb{N} \times 2^{<\omega}$ is a Kraft–Chaitin set (KC set) if $\sum_{\langle r,y\rangle \in W} 2^{-r} \leqslant 1$.

If $X \subseteq \mathbb{N}$, the weight of X (in the context of W) is

$$wt(X) = \sum\nolimits_{n \in X} \sum \{2^{-r} : \ \langle r, n\rangle \in W\}.$$

The pairs enumerated into such a set W are called axioms.

Theorem 3.2 (*Chaitin [6], Theorem 3.2*). *From a Kraft–Chaitin set W one can effectively obtain a prefix machine M such that*

$$\forall \langle r, y\rangle \in W \exists w \ (|w| = r \ \& \ M(w) = y).$$

We say that M is a prefix machine for W.

The advantage of KC sets is that one only has to enumerate the length r of a desired M-description of y. The Kraft–Chaitin theorem takes care of actually providing the description. The following proof is based on [6].

Proof of Theorem 2.2. Let $\langle r_n, y_n\rangle_{n\in\mathbb{N}}$ be an effective enumeration of W. At stage n, we will find a string w_n of length r_n, and we set $M(w_n) = y_n$. We let $D_{-1} = \{\lambda\}$. At each stage $n \geqslant 0$ we have a finite set D_{n-1} of strings all of whose extensions are unused. We will think of a string x as the half-open interval $I(x) \subseteq [0, 1)$ of real numbers whose binary representation extends x. Let z_n be the longest string in D_{n-1} of length $\leqslant r_n$. Choose w_n so that $I(w_n)$ is the leftmost subinterval of $I(z_n)$ of length 2^{-r_n}, i.e., let $w_n = z_n 0^{r_n - |z_n|}$. To obtain D_n, first remove z_n from D_{n-1}. If $w_n \neq z_n$ then also add the strings $z_n 0^i 1$, $0 \leqslant i < r_n - |z_n|$.

One checks inductively that for each $n \geqslant 0$ the following hold:

(a) z_n exists,

(b) all the strings in D_n have different lengths,
(c) $\{I(z) : z \in D_n\} \cup \{I(w_i) : i \leqslant n\}$ is a partition of $[0, 1)$.

We prove (a) for $n \geqslant 0$, assuming (b) and (c) for $n-1$ (these are trivial statements for $n = 0$). If z_n fails to exist, then r_n is less than the length of each string in D_{n-1}, so that $2^{-r_n} > \sum\{2^{-|z|} : z \in D_{n-1}\}$ by (b) for $n-1$. Then $\sum_{i=0}^{n} 2^{-r_i} > 1$ since $\sum\{2^{-|z|} : z \in D_{n-1}\} + \sum_{i=0}^{n-1} 2^{-r_i} = 1$ by (c) for $n-1$. This contradicts the assumption that W is a KC-set.

Next, (b) for n holds if $w_n = z_n$. Otherwise $|z_n| < |w_n|$ but also $|w_n|$ is less than the next shortest string in D_{n-1}, so (b) holds by the definition of D_n. Finally, (c) is satisfied by the definition of D_n. $\square$

4. Constructing a K-trivial set

Suppose $A(x) = \lim_t A_t(x)$ for a Δ^0_2-approximation $(A_t)_{t\in\mathbb{N}}$. We will develop a sufficient condition on $(A_t)_{t\in\mathbb{N}}$ for the K-triviality of A (based on [9]). Then we meet this condition in order to construct an non-computable K-trivial r.e. set.

To show A is K-trivial, we enumerate a KC set W such that, for each $w \in \mathbb{N}$, $\langle K(w)+1, A\restriction w\rangle \in W$. Since neither $K(w)$ nor $A\restriction w$ are known, we have to work with approximations at stages r. If $K_t(w) < K_{t-1}(w)$, then we put an axiom $\langle K_t(w)+1, A_t\restriction w\rangle$ into W. Further, when $x < t$ is minimal such that $A_{t-1}(x) \neq A_t(x)$, then for each w, $x < w \leqslant t$ we put an axiom $\langle K_t(w)+1, A_t\restriction w\rangle$ into W. In this case, the axioms for descriptions of $A_{t-1}\restriction w$ we enumerated previously are “wasted”. Thus, each $A(x)$-change carries a *cost*, the weight wasted on descriptions of strings $A_{t-1}\restriction w$, $x < w \leqslant t$. Suppose we enumerated the axiom $\langle K_s(w)+1, A_s\restriction w\rangle$ into W at a stage $s < t$, adding a weight of $2^{-(K_s(w)+1)}$ to W. Since $2^{-K_s(y)} \leqslant 2^{-K_t(y)}$, the cost of changing $A(x)$ is at most

$$c(x,t) = 1/2 \sum_{x<y\leqslant t} 2^{-K_t(y)}.$$

Note that $c(x,t)$ is non-decreasing in t, $\lim_t c(x,t) \leqslant 1/2$ for each x, and $\lim_x \lim_t c(x,t) = 0$. Our sufficient condition for K-triviality implies that the sum of the costs of all changes is at most $1/2$.

Proposition 4.1. *Suppose that $A(x) = \lim_t A_t(x)$ for a Δ^0_2-approximation (A_t) such that*

$$S = \sum \{c(x,t) : t > 0 \ \&\ x \textit{ is minimal s.t. } A_{t-1}(x) \neq A_t(x)\} \leqslant 1/2. \qquad (1)$$

Then A is K-trivial.

Proof. We enumerate a KC set W in stages s:

Put the axiom $\langle K_s(w)+1, A_s\restriction w\rangle$ into W in case

(a) $K_s(w) < K_{s-1}(w)$, or
(b) $K_s(w) < \infty \ \&\ A_{s-1}\restriction w \neq A_s\restriction w$.

To show W is a KC set, suppose an axiom $\langle K_s(w)+1, A_s\upharpoonright w\rangle$ is put into W at stage s.

Stable case: $\forall t > s\ A_s\upharpoonright w = A_t\upharpoonright w$. The contribution of such axioms is at most $\Omega/2$, since at most one axiom is enumerated for each value $K_s(w)$.

Change case: $\exists t > s\ A_s\upharpoonright w \neq A_t\upharpoonright w$. Choose t minimal. Since $2^{-K_s(w)} \leqslant 2^{-K_t(w)}$, the contribution of such axioms for a single t is at most $c(x,t)$, where x is minimal such that $A_{t-1}(x) \neq A_t(x)$ (so that $x < w$). Our hypothesis in (1) is $S \leqslant 1/2$, so the total contribution is at most $1/2$.

Let M_e be the prefix machine for W obtained by the Kraft–Chaitin Theorem 3.2. We claim that, for each w, $K(A\upharpoonright w) \leqslant K(w)+e+1$. Given w, let s be greatest such that $s=0$ or $A_{s-1}\upharpoonright w \neq A_s\upharpoonright w$. If $s>0$ then the axioms in (b) at stage w cause $K_u(A\upharpoonright w) \leqslant K_s(w)+e+1$ for some $u > s$. If $K_s(w) = K(w)$, we are done. Otherwise (this includes the case $s=0$ as $K_0(w)=\infty$), the inequality is caused by an axiom in (a) at the greatest stage $t > s$ such that $K_t(w) < K_{t-1}(w)$. $\square$

Recall that an r.e. set A is PROMPTLY SIMPLE if A is co-infinite and there is an effective approximation $(A_s)_{s\in\mathbb{N}}$ of A such that, for each e, the requirement

$$S_e:\ |W_e| = \infty \Rightarrow \exists s \exists x\ [x \in W_{e,s} - W_{e,s-1}\ \&\ x \in A_s - A_{s-1}]$$

is met.

Theorem 4.2 (*Downey et al.* [9]). *There is a promptly simple K-trivial set A.*

Proof. Define an enumeration (A_r) as follows. Let $A_0=\emptyset$. At stage $s>0$, for each $e<s$, if S_e is not met yet and there is $x \geqslant 2e$ such that $x \in W_{e,s} - W_{e,s-1}$ and $c(x,s) \leqslant 2^{-(e+2)}$, then put x into A_s.

Condition (1) is satisfied since we need to make at most one change for each e. If W_e is infinite, there is an $x \geqslant 2e$ in W_e such that $c(x,s) \leqslant 2^{-(e+2)}$ for all $s > x$. Since $c(x,s)$ is non-decreasing in s, we enumerate $x \in W_e$ into A at the stage where x appears in W_e if S_e has not been met yet. Thus A is promptly simple. $\square$

One can combine this technique with the Robinson guessing method for low sets (see [27]) to obtain the following.

Theorem 4.3 (*Nies* [22]). *For each low r.e. set B, there is an r.e. $A \in \mathcal{K}$ such that $A \not\leqslant_T B$.*

Condition (1) is very restrictive. For instance:

Proposition 4.4. *A Δ^0_2-approximation $A_r(y)$ satisfying (1) changes at most $\mathcal{O}(y^2)$ times.*

Proof. Given $y < r$, when $A_{r-1}(y) \neq A_r(y)$, then S increases by at least $2^{-K_r(y)}$. Since $K_r(y) \leqslant 2\log_2(y) + \mathcal{O}(1)$, we have $2^{-K_r(y)} \geqslant \mathcal{O}(1)y^{-2}$. Since $S \leqslant 1/2$, the required bound on the number of changes follows. $\square$

A modification of the proof of Theorem 4.2 yields the existence of a promptly simple set in $\mathcal{M}$: we work with the cost function

$$c_M(x,r) = 1/2 \sum \{2^{-|\sigma|} : \ U^A(\sigma) \downarrow [r-1] \ \& \ x < \ \textit{the use of this computation}\}.$$

Running the construction in the proof of Theorem 4.2 with this new cost function, we obtain an r.e. set A in $\mathcal{M}$, via the KC set W defined as follows: when a new computation $U^A(\sigma) = y$ appears, then enumerate $\langle |\sigma| + 1, y \rangle$ into W. To see that W is a KC set, note that the computations where A is stable below the use contribute a weight of at most $\Omega^A/2$ (before, it was $\Omega/2$), while the others contribute at most $1/2$. Our enumeration into W causes $K(y) \leqslant K^A(y) + \mathcal{O}(1)$ for each y.

The cost function method in itself does not provide an injury-free construction. For instance, one can define a cost function encoding the restraint of the usual lowness requirements $\exists^\infty s \ \{e\}^A(e) \downarrow [s-1] \Rightarrow \ \{e\}^A(e) \downarrow$ in the canonical construction of a low simple set [27, Theorem VII.1.1]. If $\{e\}^A(e)$ converges at stage $s-1$, then one defines $c(x,s) = \max\{c(x,s-1), 2^{-(e+2)}\}$ for each x below the use of $\{e\}^A(e)$. Then this computation can only be destroyed by the finitely many simplicity requirements which are allowed to spend $2^{-(e+2)}$.

The construction in the proof of Theorem 4.2 can be considered injury free because $c(x,s)$ is defined in advance, rather than depending on A_{s-1}.

5. Low for random sets are low for K

Recall that, if $\mathcal{C} \subseteq \mathcal{D}$ are randomness notions, then $\mathrm{Low}(\mathcal{C},\mathcal{D})$ denotes the class of oracles A such that $\mathcal{C} \subseteq \mathcal{D}^A$. We review the definition of computable randomness, but see [2] for more details, and also for a definition of Kolmogorov–Loveland randomness.

Definition 5.1. A MARTINGALE is a function $M : \{0,1\}^* \mapsto \mathbb{R}_0^+$ such that, for all strings x, $M(x0) + M(x1) = 2M(x)$. M SUCCEEDS on a set Z if $\limsup_n M(Z\upharpoonright n) = \infty$. We write $S(M)$ for this success class. Z is COMPUTABLY RANDOM if no computable martingale M succeeds on Z. This class is denoted CR.

By a result of Schnorr [24], we can restrict ourselves to computable martingales with values in $\mathbb{Q}^+$; if Z is not computably random, then such a martingale succeeds on Z.

Theorem 5.2. *A is in* $\mathrm{Low}(\mathsf{MLR},\mathsf{CR})$ *if and only if A is low for K.*

If $\mathcal{C} \subseteq \widetilde{\mathcal{C}} \subseteq \widetilde{\mathcal{D}} \subseteq \mathcal{D}$ are randomness notions, then $\mathrm{Low}(\widetilde{\mathcal{C}},\widetilde{\mathcal{D}}) \subseteq \mathrm{Low}(\mathcal{C},\mathcal{D})$. So the following are immediate consequences of the theorem.

Corollary 5.3. *Each* Low(MLR) *set is low for* K.

Corollary 5.4. *Each* Low(KLR) *set is low for* K.

Proof of Theorem 5.2. As remarked after Definition 2.6, each low for K set is in Low(MLR). It remains to prove the other direction. We apply the usual topological notions for Cantor space 2^ω. For a set S of strings, $[S]$ denotes the open set $\{X : \exists y \in S\ y \prec X\}$, which is identified with the set of strings extending a string in S. So an open set R is called r.e. if the corresponding set of strings closed under extension is r.e. For a string y, we write $[y]$ instead of $[\{y\}]$ (so that $\mu[y] = 2^{-|y|}$). Given a string v, $\mu_v(X)$ denotes the measure of X within $[v]$, namely

$$\mu_v(X) = 2^{|v|}\mu(X \cap [v]).$$

A MARTINGALE FUNCTIONAL is a Turing functional L such that, for each oracle X, L^X is a (total) martingale. Let R be any r.e. open set such that $\mu R < 1$ and Non-MLRand $\subseteq R$ (for instance, let $R = \{z : \exists w \preccurlyeq z\ K(w) \leqslant |w| - 1\}$, then $\mu R \leqslant 1/2$). We will define a martingale functional L. If $A \in$ Low(MLR, CR) then $S(L^A) \subseteq$ Non-MLRand, and we may apply the following lemma to $N = L^A$.

Lemma 5.5. *Let N be any martingale such that $S(N) \subseteq$ Non-MLR. Then there are $v \in 2^{<\omega}$ and $d \in \mathbb{N}$ such that $v \notin R$ and*

$$\forall x \succcurlyeq v [N(x) \geqslant 2^d \Rightarrow x \in R]. \tag{2}$$

Proof. Suppose the lemma fails. Define a sequence of strings $(v_m)_{m\in\mathbb{N}}$ outside R, as follows: let v_0 be the empty string, and let v_{m+1} be some proper extension y of v_m such that $N(y) \geqslant 2^m$ but $y \notin R$. Then N succeeds on $Z = \bigcup_n v_n$ but $Z \notin R$, so $Z \in$ MLRand. $\square$

Note that $v \notin R$ implies that $\mu_v(R) < 1$ (otherwise let $X \notin R$ be a set extending v; then X is a random set in a Π^0_1 class of measure 0, which is impossible). In the following we fix an enumeration $(R_s)_{s\in\mathbb{N}}$ of R (viewed as a set of strings) such that R_s contains only strings up to length s and is closed under extension within those strings.

We will independently, but uniformly in m, build martingale functionals L_m for each $m \geqslant 1$ which have value 2^{-m} on any input of length $\leqslant m$. Then $L = \sum_{m\geqslant 1} L_m$ is a martingale functional (L is $\mathbb{Q}$-valued since the contributions of the L_m, $m > |w|$, add up to $2^{-|w|}$). We define L in order to ensure that for each A, if $N = L^A$ and $S(N) \subseteq$ Non-MLRand, then A is low for K. *Fix an effective listing $(\delta_m)_{m\geqslant 1}$ of all triples $\delta_m = \langle v, d, u\rangle$, where v is a string, and $d, u \in \mathbb{N}$. Given δ_m, we let $q = 2^{-u}$.* If δ_m represents witnesses v, d in Lemma 5.5 and $0 < q < 1 - \mu_v(R)$, then we will be able to define a KC set W showing A is low for K. So only L_m matters in the end. However, we need to consider all the possible witnesses δ_m, since we do not know the correct one in advance. Fix m. We will define an effective sequence $(T_s)_{s\in\mathbb{N}}$ of finite subtrees of $2^{<\omega}$ (viewed as characteristic functions). The limit tree T given by

$T(\gamma) = \lim_s T_s(\gamma)$ exists, and if δ_m is a witness, the set A is a path of T. Roughly speaking, γ is on T_s if condition (2) looks correct at stage s for $N = L_m^\gamma$ (the partial martingale where only γ is used as an oracle). Each path of T is low for K, since we enumerate a KC set W such that, for some constant c determined below, if $\gamma \in T$ and $K^\gamma(y) = r$, then $\langle r + c, y\rangle \in W$ (so that $K^\gamma(y) \leqslant r + \mathcal{O}(1)$ by the Kraft–Chaitin theorem).

Given $\delta_m = \langle v, d, u\rangle$, *let* $c = m + d + u + 3$. A PROCEDURE α is a triple $\langle \sigma, y, \gamma\rangle$, where $\sigma, y, \gamma \in 2^{<\omega}$, $|y| < |\gamma|$ and $|\sigma| \leqslant |y| + 2\log|y| + c_K$ (c_K was defined near the end of Section 1). We start α at a stage s which is least such that $\gamma \in T_s$ & $U_s^\gamma(\sigma) = y$, and γ is the shortest among such strings at s. Now α wants to put $\langle r + c, y\rangle$ into W, where $r = |\sigma|$. It first causes a clopen set $C \subseteq [v]$ of measure $\mu_v(C) = 2^{-(r+c)}$ to go into R. Simplifying, α chooses a clopen set $\widetilde{C} = \widetilde{C}(\alpha)$ of that measure, which is disjoint from R_s and the sets chosen by other procedures, and causes (in a way to be specified) $L_m^X(z) \geqslant 2^d$ for each $X \succcurlyeq \gamma$ and each string $z \in \widetilde{C}$ of minimal length. If at a stage $t > s$, once again $\gamma \in T_t$, then $\widetilde{C} \subseteq R_t$, and α now has permission to put $\langle r + c, y\rangle$ into W. In short, the weight of axioms put into W is charged against the measure of new enumeration into R. If the sets belonging to different procedures are disjoint, then W is a KC set.

We discuss how to guarantee disjointness. Suppose $\beta \neq \alpha$ is a procedure which chose its set $\widetilde{C}(\beta)$ at a stage before stage s. If $(\beta)_2$, the third component of β, has reappeared on the tree, then $\widetilde{C}(\beta) \subseteq R_s$, so there is no problem since α chooses its set disjoint from R_s. However, if $(\beta)_2$ has not reappeared (and it possibly never will), then β keeps away its set from assignment to other procedures. The solution to this problem is to build up the set $\widetilde{C}(\alpha)$ in small portions $\widetilde{D}$, whose measure is a fixed fraction of $2^{-(r+c)}$, and only assign a new set $\widetilde{D}$ once the old one is in R. If α always reappears on the tree after assigning such a set, then eventually $\widetilde{C}(\alpha)$ reaches the required measure $2^{-(r+c)}$, in which case α is allowed to enumerate the axiom $\langle r + c, y\rangle$ into W. Otherwise, α keeps away only one single set $\widetilde{D}$, whose measure is so small that the union (over all procedures) of sets kept away is at most $q/4$ (recall that $q = 2^{-u}$). In the formal construction, $\widetilde{E}_t$ denotes the union of sets of strings appointed by procedures by stage t. Then the measure of $\widetilde{E}_t - R_t$ is at most $q/4$ at any stage.

The procedure $\alpha = \langle \sigma, y, \gamma\rangle$ appoints certain strings z and ensures $L_m^X(z) \geqslant 2^d$, for each $X \succcurlyeq \gamma$. Once activated, namely when $U_s^\gamma(\sigma) = y$, the procedure α can claim the amount $\varepsilon = 2^{-(r+m)}$ of the initial capital 2^{-m} of L_m^X, for any oracle $X \succcurlyeq \gamma$ (recall that $r = |\sigma|$). So given X, the total capital claimed by all activated procedures is $2^{-m}\Omega^X < 2^{-m}$. The procedure appoints strings z of the form $x0^{1+r+m+d}$, and "withdraws" its capital at x, increasing $L_m^X(x0)$ by ε for oracles $X \succcurlyeq \gamma$. To maintain the martingale property, it also has to decrease $L_m^X(x1)$ by ε. Now it doubles the capital along z, always betting all the capital on 0, and reaches an increase of 2^d at z. Any string in $[y]$ is called USED by α.

The procedure α has to obey the following restrictions.

1. Choose the extension z outside $[\widetilde{E}_{s-1}]$, where $\widetilde{E}_{s-1}$ is the set of strings previously appointed by other procedures β, since the open sets generated by the strings appointed by different procedures need to be disjoint.

2. Let $C_t(\alpha)$ denote the set of strings x' used by α up to stage t. The procedure α must ensure $x \notin [C_{s-1}(\alpha)]$ so that α's capital is still available at x. Such a choice is possible for sufficiently many x, since for all t, $\mu_v[\widetilde{C}_t(\alpha)] \leqslant 2^{-(r+c)}$, so that $\mu_v[C_t(\alpha)] \leqslant 2^{-(r+c)} 2^{1+r+d+m} = q/4$.

There is no conflict between α and other procedures β as far as the capital is concerned: if $\gamma' = (\beta)_2$ is incomparable with γ then γ and γ' can only be extended by different oracles X. Otherwise α and β own different parts of the initial capital of L_m^X, for any X extending their third components.

We are now ready for the formal definition of the martingale functional L_m. The notation is summarized in a table below. For a procedure $\alpha = \langle \sigma, y, \gamma \rangle$, let $n_\alpha > \max(|\sigma| + m + d + 1, |\gamma|, |v|)$ be a natural number assigned to α in some effective one-one way. Each procedure α defines an auxiliary function $F_\alpha : 2^{<\omega} \mapsto \mathbb{Q}$. The set $\widetilde{C}(\alpha)$ of appointed strings coincides with the set of minimal strings in $\{w : F_\alpha(w) \geqslant 2^d\}$. For each oracle X, let

$$L_m^X(w) = 2^{-m} + \sum \{F_\alpha(w) : (\alpha)_2 \preccurlyeq X\}. \tag{3}$$

Given $\alpha = \langle \sigma, y, \gamma \rangle$, let $r = |\sigma|$. We ensure

(F1) $F_\alpha(w) = 0$ if $|w| \leqslant |\gamma|$,
(F2) $F_\alpha(w) \geqslant -2^{-(r+m)}$, and $F_\alpha(w) = 0$ unless $U_s^\gamma(\sigma) = y$,
(F3) $\forall w \; F_\alpha(w0) + F_\alpha(w1) = 2F_\alpha(w)$.

Based on those properties, we check that L_m^X is a martingale functional. Firstly, $L_m^X(w)$ is a rational for each X, since by (F1) only the finitely many procedures α such that $|(\alpha)_2| < |w|$ contribute to the sum in (3). Next, for $p = |w|$,

$$\begin{aligned} L_m^X(w0) + L_m^X(w1) &= 2^{-m+1} + \sum \{F_\alpha(w0) + F_\alpha(w1) : (\alpha)_2 \preccurlyeq X \upharpoonright p+1\} \\ &= 2(2^{-m} + \sum \{F_\alpha(w) : (\alpha)_2 \preccurlyeq X \upharpoonright p+1\}) \\ &= 2L_m^X(w) \end{aligned}$$

(for the last equality we used (F1)). Finally, $L_m^X(w) \geqslant 0$, since $F_\alpha(w) \geqslant -2^{-(r+m)}$, and α contributes to sum (3) only if the computation $U^\gamma(\sigma) = y$ converges, where $r = |\sigma|$ and $\gamma \preccurlyeq X$. So, for each w, $L_m^X(w) \geqslant 2^{-m}(1 - \Omega^X) \geqslant 0$.

Let $\delta_m = \langle v, d, u \rangle$, $q = 2^{-u}$. The construction for m works at stages which are powers of 2; letters s, t denote such stages. At stage s we define T_s and extend the functions $F_\alpha(w)$ to all w such that $s \leqslant |w| < 2s$. For each w such that $s \leqslant |w| < 2s$ and each string η (which may be shorter that w), by the end of stage s we may calculate

$$\overline{L}_m(\eta, w) = 2^{-m} + \sum \{F_\alpha(w) : (\alpha)_2 \preccurlyeq \eta\}.$$

We summarize the notation.

R	Given r.e. open set such that $\mu R < 1$, $\mathsf{Non\text{-}MLRand} \subseteq R$
δ_m	witness for Lemma 5.5, of the form $\langle v, d, u\rangle$
c	$m + d + u + 3$
q	2^{-u}
L_m	martingale functional for witness δ_m
α	procedure, of form $\langle \sigma, y, \gamma\rangle$ where $U^{\gamma}(\sigma) = y$, $r = \lvert\sigma\rvert$
n_α	code number of α
F_α	auxiliary function defined by α
$C_t(\alpha)$	set of strings x used by α up to (the end of) stage t
$\widetilde{C}_t(\alpha)$	set of strings appointed by α up to stage t, of form $x0^{r+m+d+1}$
T_t	tree for m at the end of stage t
W	KC set for m
$\widetilde{E}_t$	set of strings appointed by procedures up to stage t

Stage 1: Let T_1 contain only the empty string and let $F_\alpha(w) = 0$ for each α and each w, $|w| \leqslant 1$. Let $\widetilde{E}_1 = \emptyset$.

Stage $s > 1$: Suppose T_t has been determined for $t < s$, and the functions $F_\alpha(w)$ have been defined for all w, $|w| < s$. Let

$$T_s = \{\gamma : \forall w \succcurlyeq v[(|w| < s \ \&\ \overline{L}_m(\gamma, w) \geqslant 2^d) \Rightarrow w \in R_s]\}.$$

(1) If $\mu_v(R_s) > 1 - q$ goto (4) (If δ_m is a witness this case does not occur.)
(2) For each $\alpha = \langle \sigma, y, \gamma\rangle$, $n_\alpha < s$, if $U^{\gamma}_s(\sigma) = y$, $U^{\gamma}_{s/2}(\sigma)$ is undefined and, for σ, y, the string γ is the shortest such string, then START the procedure α.
(3) Carry out the following for each procedure $\alpha = \langle \sigma, y, \gamma\rangle$ in the order of $n_\alpha < s$. Let $r = |\sigma|$.
  (3a) If α has been started and $\gamma \in T_s$, first we check if the goal has been reached, namely $\mu_v \widetilde{C}_{s/2}(\alpha) = 2^{-(|\sigma|+c)}$. In that case we put $\langle |\sigma| + c, y\rangle$ into W, and we say that α ENDS. Otherwise we say that α ACTS, and we choose a set $D = D_\alpha \subseteq [v]$ of strings of length s such that $\mu_v D = 2^{-(n_\alpha+u+2)}$ and

$$[D] \cap [R_s \cup \widetilde{E}_{s/2} \cup G \cup C_{s/2}(\alpha)] = \emptyset,$$

  where $G = \bigcup\{D_\beta : \beta$ has acted at stage s so far$\}$. (We will verify that D exists.) Let $\widetilde{D} = \{x0^{m+d+r+1} : x \in D\}$, put D into $C_s(\alpha)$, and put $\widetilde{D}$ into $\widetilde{C}_s(\alpha)$ and $\widetilde{E}_s$. Note that $|w| < 2s$ for all strings $w \in \widetilde{D}$, since $m+d+r+1 < n_\alpha < s$.
  (3b) For each $x \in D$, let $F_\alpha(x) = 0$, $F_\alpha(x1) = -\varepsilon$ and $F_\alpha(x0) = \varepsilon$, where $\varepsilon = 2^{-(r+m)}$. Now we double the capital along $x0^{r+d+m+1}$: for each string p, $|p| \leqslant r + m$, let $F_\alpha(x0p) = \varepsilon 2^l$ if $p = 0^l$, and $F_\alpha(x0p) = 0$ otherwise. (This causes $\overline{L}_m(\gamma, w) \geqslant 2^d$ for each $w \in \widetilde{D}$.)

Go on to the next α.

(4) For each string w, $s \leqslant |w| < 2s$ such that $F_\alpha(w)$ is still undefined, let $F_\alpha(w) = F_\alpha(w')$, where $w' \preccurlyeq w$ is longest such that $F_\alpha(w')$ is defined.

End of Stage s.

Verification: We go through a series of Claims. Let $\alpha = \langle \sigma, y, \gamma \rangle$.

Claim 1. *Properties* (F1)–(F3) *are satisfied.*

(F1) holds because when we assign a non-zero value to $F_\alpha(w)$ at stage s, then $|w| \geqslant s > n_\alpha > |\gamma|$. (F2) and (F3) are satisfied since each x chosen in (3b) goes into $C(\alpha)$. So by choice of D in (3a), no future definition of F_α on extensions of x is made except for by (4)

Claim 2. α *is able to choose* D_α in (3a).

- By definition of T_s, for each $\beta = \langle \sigma', y', \gamma' \rangle$ and each $t \geqslant 2$, if $\gamma' \in T_t$, then $\widetilde{C}_{t/2}(\beta) \subseteq R_t$. Thus for each procedure β, $\mu_v(\widetilde{C}_t(\beta) - R_t) \leqslant 2^{-(n_\beta+u+2)}$ as $\widetilde{C}_t(\beta) - R_t$ consists of a single set $\widetilde{D}_\beta$. Then, letting $t = s/2$, $\mu_v(\widetilde{E}_{s/2} - R_s) \leqslant 2^{-(u+2)} = q/4$.
- Each set D_β chosen during stage s satisfies $\mu_v(D_\beta) \leqslant 2^{-(n_\beta+u+2)}$, hence $\mu_v G$ never exceeds $q/4$.
- For each s, $\mu_v \widetilde{C}_s(\alpha) \leqslant 2^{-(r+c)}$, and hence $\mu_v C_s(\alpha) \leqslant 2^{r+d+m+1} 2^{-(r+c)} = q/4$.

Since the test in (1) failed, $\mu_v(R_s) \leqslant 1-q$, so relative to $[v]$ a measure of $q/4$ is available outside $[R_s \cup \widetilde{E}_{s/2} \cup G \cup C_{s/2}]$ for choosing D_α. All strings in $R_s \cup \widetilde{E}_{s/2} \cup G \cup C_{s/2}$ have length $< s$ (for strings in $\widetilde{E}_{s/2}$, this holds by the comment at the end of (3a)), so the strings in D_α can be chosen of length s.

Claim 3. *Each procedure* α *acts only finitely often.*

Each time α acts at s and $s' > s$ is least such that $\gamma \in T_{s'}$, we have increased $\mu_v(\widetilde{C}(\alpha))$ by the fixed amount $2^{-(n_\alpha+c+r)}$. So eventually γ is not on the tree or α ends.

Claim 4. *For each string* η, *there is a stage* s_η *such that no procedure* α, $(\alpha)_2 \preccurlyeq \eta$, *acts at any stage* $\geqslant s_\eta$. *Moreover, for each* $w \succcurlyeq v$, $|w| \geqslant s_\eta$, $\overline{L}_m(\eta, w) = \overline{L}_m(\eta, w')$ *for some* w' *such that* $v \preccurlyeq w' \preccurlyeq w$ *and* $|w'| < s_\eta$.

This follows because there are only finitely many procedures α such that $(\alpha)_2 \preccurlyeq \eta$. By Claim 3, there is a stage s_η by which those procedures have stopped acting, and further definitions $F_\alpha(w)$ are only made in (4)

Claim 5. $T(\eta) = \lim_s T_s(\eta)$ *exists.*

Suppose $s \geqslant s_\eta$ is least such that $\eta \in T_s$. We show $\eta \in T_t$ for each $t \geqslant s$. Suppose $v \preccurlyeq w$, $|w| \leqslant t$ and $\overline{L}_m(\eta, t) \geqslant 2^d$. By Claim 4, $\overline{L}_m(\eta, w) = \overline{L}_m(\eta, w')$ for some $w' \preccurlyeq w$ of length $< s_\eta$. Then $w' \in R_s$ since $\eta \in T_s$, and hence $w \in R_t$.

In the following we assume δ_m is a witness for Lemma 5.5, where $N = L^A$.

Claim 6. *A is on T.*

Given l, let $\eta = A\upharpoonright l$. Suppose $|w'| < s_\eta$ and $\overline{L}_m(w', \eta) \geqslant 2^d$. Then $L^A(w') \geqslant 2^d$, since $L^A(w') \geqslant L^A_m(w') \geqslant \overline{L}_m(w, \eta)$. By (2), $w' \in R$. Let s be a stage so that all such w' are in R_s. Then by Claim 4, $\eta \in T_t$ for all $t \geqslant s$.

Claim 7. *Each path of T is low for K.*

We first verify that W is a KC set. Note that

$$\sum_s \sum \{2^{-(|\sigma|+c)} : \langle |\sigma| + c, y\rangle \text{ is put into } W \text{ by } \langle \sigma, y, \gamma\rangle \text{ at stage } s\} \leqslant \mu_v R.$$

For, when α ends at s then $\mu_v \widetilde{C}_{s/2}(\alpha) = 2^{-(|\sigma|+c)}$ and $\widetilde{C}_{s/2}(\alpha) \subseteq R$. The sets $[\widetilde{C}(\alpha)]$ are pairwise disjoint by the choice of D in (3a). Hence the required inequality holds.

Let M_e be a prefix machine for W according to Theorem 3.2. We claim that, for each path X of T and each string y, $K(y) \leqslant K^X(y) + c + e$. For choose a shortest U^X-description σ of y, and choose $\gamma \subseteq X$ shortest such that $|\gamma| > y$ and $U^\gamma(\sigma) = y$. Since $\gamma \in T$, at some stage t, we start the procedure $\langle \sigma, y, \gamma\rangle$, and the procedure ends. At this stage we put $\langle |\sigma| + c, y\rangle$ into W, causing $K(y) \leqslant K^X(y) + c + e$. $\square$

By Theorem 5.2 and Proposition 2.8, each $A \in \text{Low}(\mathsf{MLR}, \mathsf{CR})$ is low in the usual sense. This answers Problem 4.4 in [2] in the negative. It was first asked in [15, p.1400].

Corollary 5.6. *Any low for Martin-Löf random set is low, and hence Δ^0_2.*

The following answers Problem 4.8 in [2] in the negative.

Theorem 5.7. *Each* Low(CR) *set is computable.*

Proof. On the one hand, $\text{Low}(\mathsf{CR}) \subseteq \text{Low}(\mathsf{MLR}, \mathsf{CR}) = \mathcal{M} \subseteq \Delta^0_2$. On the other hand, Bedregal and Nies [3] have shown that if A is Low(CR) then A has hyperimmune-free degree (also see [13]). The only Δ^0_2 sets of hyperimmune-free degree are the computable ones, by Miller and Martin [18]. $\square$

The author first gave a direct proof of Theorem 5.7, which will appear in [20]. Its advantage is that it can be extended to the resource bounded setting, and also to show that in fact each set in Low(PrecR, CR) is computable. Here PrecR is the class of sets on which not even a partial recursive martingale succeeds (i.e., no martingale that may choose to be undefined on strings off the set succeeds).

6. K-trivial sets

We prove that the class $\mathcal{K}$ is closed downward under Turing reducibility, and give the modifications needed to prove that $\mathcal{K} = \mathcal{M}$. The first version of the proof also shows that Proposition 4.1 in fact provides a characterization of the K-trivial sets. This yields some corollaries which further restrict the sets in $\mathcal{K}$.

Theorem 6.1. *If A is K-trivial and $B \leqslant_T A$, then B is K-trivial.*

As noted in [9], the corresponding fact is easily verified for weak truth table reducibility: Suppose $B \leqslant_T A$ via a Turing reduction Γ such that the use of Γ is bounded by a recursive function g. Then, up to constants,

$$K(B\restriction n) \leqslant K(A\restriction g(n)) \leqslant K(g(n)) = K(n).$$

Hirschfeldt and Nies modified the proof of Theorem 6.1 to obtain a stronger result. However, the original version of the proof is also needed for the characterization of $\mathcal{K}$.

Theorem 6.2 (*with Hirschfeldt*). *Each K-trivial set A is low for K.*

We note the modifications needed to obtain a proof of Theorem 6.2 in brackets [...].

Proof of Theorem 6.2. Suppose A is K-trivial via a constant b. For Theorem 6.1, let $B = \Gamma^A$, where Γ is a Turing functional whose use is non-decreasing in the input. Let $(A_r)_{r\in\mathbb{N}}$ be a Δ^0_2-approximation of A. For each s, one can effectively determine an $f(s) > s$ such that $\forall n < s\ K(A\restriction n) \leqslant K(n) + b\ [f(s)]$, i.e., the inequality holds at stage $f(s)$. Let $s_0 = 0$ and $s_{i+1} = f(s_i)$. The construction is restricted to *stages* in $\{s_i : i \in \mathbb{N}\}$. We use italics to emphazise this. In the following, s, t, u always will denote such *stages*. We may modify the approximation (A_r) so that $A_r(x) = A_{s_i}(x)$ for all r, $s_i \leqslant r \leqslant s_{i+1} - 1$. We say that $A(x)$ CHANGES AT s if $A_{s-1}(x) \neq A_s(x)$.

We will determine a KC set W in order to show that B is K-trivial [A is low for K]. We also enumerate an auxiliary KC set L to exploit the hypothesis that A is K-trivial. For certain n, an axiom $\langle r, n\rangle$ will be enumerated into L (at most one for each n). Putting $\langle r, n\rangle$ into L causes $K(n) \leqslant r + \mathcal{O}(1)$ and hence $K(A\restriction n) \leqslant r + \mathcal{O}(1)$.

We may assume that an index d for a machine M_d is given, and we can think of M_d as being a prefix machine for L: From an index for an r.e. set $Q \subseteq \mathbb{N} \times 2^{<\omega}$, we can effectively obtain an index for a KC set $\widetilde{Q}$ such that $\widetilde{Q} = Q$ in case Q already is a KC set. Let M_d be the machine effectively obtained from $\widetilde{Q}$ via the Kraft–Chaitin theorem. Our construction effectively produces a KC set L from d. Thus, if $Q = L$, which will happen for some Q by the recursion theorem, then Q is a KC set and M_d is a machine for L. Of course, first we have to show that L is a KC-set, no matter what d is.

For the remainder of this proof, let $c = b + d$ and $k = 2^{c+1}$. When we put $\langle r, n\rangle$ into L, then $K(n) \leqslant r + d$ and hence $K(A\restriction n) \leqslant r + c$, assuming M_d is a machine for L.

To gain some intuition, we first give a direct proof that no K-trivial set A satisfies $\emptyset' \leqslant_{wtt} A$ (which also follows from the downward closure of $\mathcal{K}$ under $\leqslant_{wtt}$ and the fact that the wtt-complete set Ω is not K-trivial). Suppose $\emptyset' \leqslant_{wtt} A$. Now we *build* an r.e. set B, and by the Recursion Theorem we can assume we are given a total wtt-reduction Γ such that $B = \Gamma^A$, whose use is bounded by a computable function g. We wait till $\Gamma^A(k)$ converges, let $n = g(k)$ and put the single axiom $\langle r, n\rangle$ into L, where $r = 1$. Our total cost is $1/2$. Each time the opponent has a U-description of $A \upharpoonright n$ of length $\leqslant r + c$ we force $A \upharpoonright n$ to change, by putting into B the largest number $\leqslant k$ which is not yet in B. If we reach $k + 1$ such changes, then his total cost is $(k+1)2^{-(r+c)} > 1$, which is a contradiction.

In the proof of Hirschfeldt's more general result that K-trivial sets are T-incomplete (see [9, Theorem 4.1]), there is no recursive bound on the use of $\Gamma^A(k)$. The problem now is that the opponent might, before giving a description of $A_s \upharpoonright n$, move this use beyond n, thereby depriving us of the possibility to cause further changes of $A \upharpoonright n$. The solution is to carry out many attempts, based on different computations $\Gamma^A(m)$. Each time the use of such a computation changes, the attempt is cancelled. What we placed in L for this attempt now becomes "garbage" but as the reduction Γ is total, this does not happen always. We have to ensure that the total weight of the garbage produced by all attempts is limited, otherwise L is not a KC set. □

Ingredients: Our proof combines three main ideas. The essence of the first one, and some elements of the third, first appeared in the proof of Hirschfeldt's result. Roughly speaking, for an axiom $\langle r, n\rangle \in L$, either n reaches a k-set (as defined below) or n is garbage. The weight of numbers of either type is at most $1/2$. The first idea is already present in the proof of wtt-incompleteness above. The third is a way to deal with the garbage. Both together ensure that L is a KC set.

The second idea is needed to identify W. We use a tree of runs of procedures, where the branchings are determined by U-descriptions (U^A-descriptions). At branching nodes the construction of a K-trivial (low for K) set is emulated. That is, $B \in \mathcal{K}$ [$A \in \mathcal{M}$] for the same reason as in the proof of Theorem 4.2 (in the construction outlined after Proposition 4.4). We discuss these ideas in detail.

6.1. The concept of a j-set

Recall that $k = 2^{c+1}$. For $1 \leqslant j \leqslant k$, we say that a finite set $E \subseteq \mathbb{N}$ is a j-SET at *stage* t if, for all $n \in E$, at some *stage* $u < t$ we put an axiom $\langle r_n, n\rangle$ into L, and now there are j distinct strings z of the form $A_v \upharpoonright n$ for some *stage* v, $u \leqslant v \leqslant t$, such that $K_v(z) \leqslant r_n + c$. An r.e. set with an enumeration $E = \bigcup E_t$ is a j-set if E_t is a j-set at each *stage* t. Since the opponent has to match a description of n we provide via L by descriptions that are at most c longer of strings of length n, we have the following important fact.

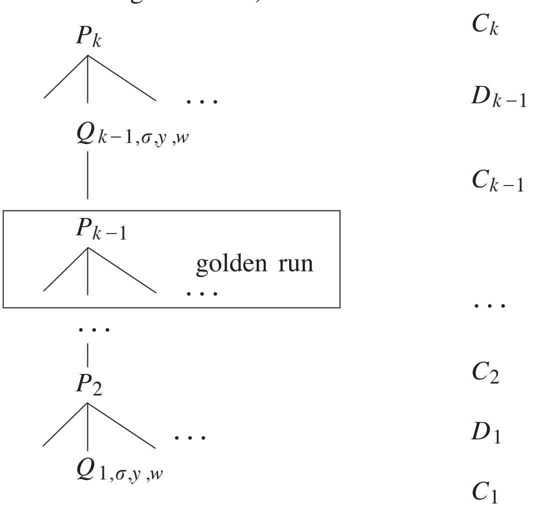
Fact 6.3. *If the r.e. set E is a k-set, then $wt(E) \leqslant 1/2$.*

Proof. For all $n \in E$, there is an axiom $\langle r_n, n\rangle$ in L and there are k distinct strings z of length n such that $K(z) \leqslant r_n + c$. Hence $1 \geqslant \Omega = \mu(\mathrm{dom}(U)) \geqslant k \sum_{n \in E} 2^{-(r_n+c)} =$

$k2^{-c}wt(E)$. Because $k = 2^{c+1}$, this implies $wt(E) \leqslant 1/2$. Note that we did not assume here that M_d is a machine for L. $\square$

6.2. The golden run, and indexing procedures by descriptions

As in the proof of wtt-incompleteness, we attempt to enumerate a k-set C_k of weight 1. Now we use a tree of runs of procedures. The successor relation is given by recursive calls. Each run of a procedure enumerates a set and has a goal, the weight this set has to reach so that the run can end. Runs may also be cancelled by runs of procedures which are above this run on the tree. The root procedure is P_k, which has goal 1. It calls several procedures of type Q_{k-1}. These call a single procedure P_{k-1} and so on till we reach the bottom level, consisting of procedures of type Q_1. All procedures have further indices or parameters, discussed below. The failure of P_k to reach $wt(C_k) = 1$ implies that there is a level i and a run of a procedure of type P_i which does not return, though all its subprocedures (of type Q_{i-1}) return unless they are cancelled. Using this "golden run" we are able to define a KC set W as desired. (However, one cannot effectively determine a golden run.)



To reach C_k, a number has to pass through j-sets C_j $(1 \leqslant j < k)$ and j-sets D_j $(1 \leqslant j < k)$, in the order $C_1, D_1, \ldots, D_{k-1}, C_k$. The procedures of type P_i $(1 < i \leqslant k)$ move numbers n from D_{i-1} into C_i upon $A \upharpoonright n$ change. This adds an ith string z of length n as in the definition of j-sets, hence C_i is an i-set. The procedures of type Q_j $(1 \leqslant j < k)$ enumerate C_1 for $j = 1$, and move numbers from C_j to D_j. C_1 simply is the right domain of L, namely $\{n : \exists r \langle r, n\rangle \in L\}$.

We index the procedures of type Q_j by descriptions σ, and also by the object y being described and a certain A-use w. Each procedure P_i may call procedures $Q_{i-1,\sigma,y,w}$ for all σ such that $U(\sigma) = y$ $[U^A(\sigma) = y]$. Ultimately we want to show $K(B \upharpoonright y) \leqslant |\sigma| + \mathcal{O}(1)$ $[K(y) \leqslant |\sigma| + \mathcal{O}(1)]$, provided the run of P_i is a golden one, since this would make B K-trivial (it would make A low for K). We prove the K-triviality of B by emulating the construction of a K-trivial set. The failure of P_i to reach its goal means that there are few A-changes, hence the weight of axioms placed in W for which the change case in Proposition 4.1 applies is small.

To give an outline of the procedures, let us pretend that $k = 2$. Now the single run of the root procedure P_2 attempts to enumerate a 2-set C_2 of weight 1, but never completes this task. It proceeds as follows. Each string σ is AVAILABLE in the beginning. At a *stage* s, for each available σ, if $U(\sigma) = y$ and $\Gamma^A(y')$ converges for each $y' < y$ [if $U^A(\sigma) = y$], then declare σ UNAVAILABLE. Let $w = \gamma^A(y-1)$. Start a procedure $Q_{1,\sigma,y,w}$ attempting to obtain a 1–set D, $w \leqslant \min(D)$, of weight 2^{-r}, where $r = |\sigma|$. In this simplified outline, D is a singleton. The procedure $Q_{1,\sigma,y,w}$ picks a large number $n > w$ and puts the axiom $\langle r, n\rangle$ into L. Then at some later *stage* s, $D = \{n\}$ is a 1-set (since we see a description of $A_s \upharpoonright n$ of length $\leqslant r + c$). If $A \upharpoonright w$ has not changed by stage s, then $Q_{1,\sigma,y,w}$ returns the set D. Now P_2 waits for an $A \upharpoonright w$ change, since this would make D a 2-set. If it obtains the change, then it puts D into C_2 and declares σ available again. If $A \upharpoonright w$ changes before we see such a description, we cancel the run of $Q_{1,\sigma,y,w}$ and declare σ available.

The KC set W is defined as follows. When a run $Q_{1,\sigma,y,w}$ returns at stage s, then put the axiom $\langle |\sigma| + 1, B_s \upharpoonright y\rangle$ into W [put $\langle |\sigma| + 1, y\rangle$ into W]. Note that $\Gamma^A \upharpoonright y$ did not change, hence still $w = \gamma^A(y-1)$. We have the same two cases as in the construction of a K-trivial set in Proposition 4.1.

Stable case: $A \upharpoonright w$ is stable from s on. Then $B \upharpoonright y$ is stable [the computation $U^A(\sigma) = y$ is stable]. So the axiom is as desired, assuming that σ is a shortest description. For each σ, this case can occur at most once, so the total contribution to W in this case is $\leqslant \Omega/2$ [$\leqslant \Omega^A/2$].

Change case. $A \upharpoonright w$ changes after s. Then $B \upharpoonright y$ may change [$U^A(\sigma) = y$ may be destroyed], in which case the axiom we placed into W is wasted. However, its weight is added to C_2, so that in the construction, P_2 makes progress towards reaching its goal. Assuming that $wt(C_2)$ never exceeds $1/2$, the contribution of those axioms is bounded by $1/2$.

We now discuss the general case where $k = 2^{c+1}$. At each stage we have a finite tree with $2k-2$ levels of runs of procedures. The leaves are the runs of procedures of type Q_1, which act in the way indicated above. Each n enumerated by such a procedure into C_1 at stage t corresponds to a unique run of a procedure at each level at stage t (we say n BELONGS to that run). Since n is chosen large, it is bigger than the parameter w of any run of a Q-type procedure n belongs to. Thus $A \upharpoonright w$-changes contribute to the aim that n reach the k-set C_k.

A procedure P_i has a parameter p, its GOAL, which is the weight it wants to transfer from D_{i-1} to C_i. Similarly, a procedure Q_j has goal q, the weight it wants to transfer from C_j to D_j. P_i calls several procedures $Q_{i-1,\sigma,y,w}$ which enumerate $i-1$-sets $D \subseteq D_{i-1}$ where $\min(D) \geqslant w$. Eventually such a Q_{i-1} type procedure may reach its goal and return its set D. In this case P_i waits for an $A \upharpoonright w$ change, and then puts D into C_i. Note that D is now an i-set. If $A \upharpoonright w$ changes before $Q_{i-1,\sigma,y,w}$ returns, then this very change turns the current set D into an i-set, so P_i is entitled to put D into C_i. However, P_i also has to cancel the run of $Q_{i-1,\sigma,y,w}$.

Identifying strings with numbers, we may view the tree at stage s as a subtree of $\{\gamma \in \omega^{<\omega} : |\gamma| \leqslant 2k-3 \ \&\ \forall i < k \ \gamma(2i+1) = 0\}$.

6.3. Waste management

A number $n \in C_j$ may not be promoted to D_j if the run $Q_{j,\sigma,y,w}$ during which it was placed into C_j is canceled. Similarly, a number from D_{i-1} may fail to go into C_i if the required $A \upharpoonright w$-change does not occur. These 'garbage numbers' jeopardize the requirement that L be a KC set. To avert this, each run of a procedure is equipped with a GARBAGE QUOTA, assigned in an effective (if somewhat arbitrary) way during the construction. A procedure P_i has as a further parameter a garbage quota α, the amount it is allowed to waste by leaving it in $D_{i-1} - C_i$. Similarly, $Q_{j,\sigma,y,w}$ has a garbage quota β, the amount it may leave in $C_j - D_j$. All goals and all garbage quotas are of the form 2^{-l}, $l \in \mathbb{N}$. We denote runs of P_i-procedures with such parameters by $P_i(p, \alpha)$, and runs of Q_j-procedures by $Q_{j,\sigma,y,w}(q, \beta)$. The goal parameter of a run must be chosen small enough to meet the garbage quota of the run immediately above on the tree which called it.

The procedures proceed as follows, making sure not to exceed their garbage quotas.

$Q_{j,\sigma,y,w}(q, \beta)$: If $j = 1$, the procedure chooses n large, puts an axiom $\langle r, n\rangle$ into L, where $2^{-r} = \beta$, and waits for $K_t(n) \leqslant r + d$ at a later stage t, at which point n is put into D_1. This is repeated till the goal has been reached. If $j > 1$, while the goal q has not been reached, the run $Q_{j,\sigma,y,w}(q, \beta)$ continues to call a single procedure $P_j(\beta, \alpha)$ for decreasing values of α, and waits till it returns a set C', at which time C' is put into D_j. Thus the amount of garbage left in $C_j - D_j$ is produced during a single run of a procedure P_j, which does not reach its goal β. So it is bounded by β.

$P_i(p, \alpha)$: this procedure calls procedures $Q_{j,\sigma,y,w}(2^{-|\sigma|}\alpha, \beta)$ for an appropriate value of β. Then the weight left in $D_{i-1} - C_i$ by all the returned runs of Q_{i-1}-procedures which never receive an A-change adds up to at most $\Omega\alpha$ $[\Omega^A\alpha]$, since this is a one-time event for each σ. The runs of procedures Q_{i-1} which are cancelled and have enumerated D so far do not contribute to the garbage of P_i, since D goes into C_i upon cancellation.

To assign the garbage quotas, at any substage of *stage s*, let

$$\alpha_i^* = 2^{-(2i+3+n_{P,i})}, \tag{4}$$

where $n_{P,i}$ is the number of runs of P_i-procedures started prior to this substage of *stage s*. Let

$$\beta_j^* = 2^{-(2j+2+n_{Q,j})}, \tag{5}$$

where $n_{Q,j}$ is the number of runs of Q_j-procedures started so far. Then the sum of all the α_i^* values and β_j^* values is $\leqslant 1/2$. When P_i is called at a substage of *stage s*, its garbage quota α is at most α_i^*. Similarly, Q_j's garbage quota β is at most β_j^*. This ensures $wt(C_1 - C_k) \leqslant 1/2$. Since $wt(C_k) \leqslant 1/2$ by Fact 6.3, L is a KC set.

The construction in the proof of Theorem 5.2 is similar to the portion of the present construction consisting of the Q_j-type procedures called by a run of P_{j+1}. The procedure $\alpha = \langle \sigma, y, \gamma\rangle$ closely corresponds to a procedure $Q_{j,\sigma,y,w}$. Both are based on a description of y, $U^{\gamma}(\sigma) = y$ in the first case, and $U^A_s(\sigma) = y$ in the second. Both are stopped when their guess about A turns out wrong. Both carry out their actions in small bits to avert too much damage in case this happens. Reserving only a small set D_α of measure $2^{-(n_\alpha+r+c)}$ at a time corresponds to calling a procedure Q_j with a small goal. A procedure waiting to reappear on a tree T_s corresponds to P_{j+1}'s waiting for an $A \upharpoonright w$ change after $Q_{j,\sigma,y,w}$ returned.

We give the formal description of the procedures and the construction.

The procedure $P_i(p, \alpha)$ $(1 < i \leqslant k,\ p = 2^{-l}, \alpha = 2^{-r}$ for some $r \geqslant l)$.

It enumerates a set C. Begin with $C = \emptyset$.

At *stage* s, declare each σ, $|\sigma| = s$, available (availability is a local notion for each run of a procedure). For each σ, $|\sigma| \leqslant s$, do the following.

($\mathrm{P1}_\sigma$) If σ is available, and $U(\sigma) = y$ for some y, $y < s$, $\Gamma^A(y')[s] \downarrow$ for each $y' < y$ [and $U^A(\sigma)[s] = y$ for some $y < s$] let $w = \gamma^A(y-1)$ [let w be use of this computation] and call the procedure $Q_{i-1,\sigma,y,w}(2^{-|\sigma|}\alpha, \beta)$, where $\beta = \min(2^{-|\sigma|}\alpha, \beta^*_{i-1})$. Declare σ *unavailable*.

($\mathrm{P2}_\sigma$) If σ is unavailable due to a run $Q_{i-1,\sigma,y,w}(q, \beta)$, and $A_s \upharpoonright w \neq A_{s-1} \upharpoonright w$, declare σ available.

- (a) Say the run is RELEASED. If $wt(C \cup D_{i-1,\sigma}) < p$, then put $D_{i-1,\sigma}$ into C and go on to (b). Otherwise, choose a subset $\widetilde{D}$ of $D_{i-1,\sigma}$ such that $wt(C \cup \widetilde{D}) = p$, and put $\widetilde{D}$ into C. *Return the set* C, cancel all runs of subprocedures and end this run of P_i. ($\widetilde{D}$ exists since $p = 2^{-l}$ for some l, and $r_n > l$ for each $n \in D$-now order the numbers r_n in a non-decreasing way.) If we inductively assume that $D_{i-1,\sigma}$ was an $i-1$-set already at the last *stage*, then C is an i-set, as will be verified below.
- (b) If the run $Q_{i-1,\sigma,y,w}$ has not returned yet, cancel this run and all the runs of subprocedures it has called.

The procedure $Q_{j,\sigma,y,w}(q, \beta)$ $(0 < j < k,\ \beta = 2^{-r}, q = 2^{-l}$ for some $r \geqslant l)$.

It enumerates a set $D = D_{j,\sigma}$. Begin with $D = \emptyset$.

(Q1) *Case* $j = 1$. Pick a number n larger than any number used so far. Put n into C_1, and put $\langle r_n, n\rangle$ into L, where $2^{-r_n} = \beta$. Wait for a *stage* $t > n$ such that $n < t' < t$ for some *stage* t' and $K_t(n) \leqslant r_n + d$, and go to (Q2). (If M_d is a machine for L, then t exists.)
Case $j > 1$. Call $P_j(\beta, \alpha)$, where $\alpha = \min(\beta, \alpha^*_j)$, and goto (Q2).

(Q2) *Case* $j = 1$. Put n into D (D remains a 1-set).
Case $j > 1$. Wait till $P_j(\beta, \alpha)$ returns a set C'. Put C' into D.
In any case, if $wt(D) < q$ then goto (Q1). Else *return the set* D. (Note that in this case, necessarily $wt(D) = q$. Also, D is a j-set, assuming inductively that the sets C' are j-sets if $j > 1$.)

At *stage* 0, we begin the construction by calling $P_{k,0}(1, \alpha^*_k)$. At each *stage*, we descend through the levels of procedures of type P_k, $Q_{k-1} \dots P_2, Q_1$. At each level we start or continue finitely many runs of procedures. This is done in some effective order, say from left to right on that level of the tree of runs of procedures, so that the values α^*_i and β^*_j are defined at each substage. Since we descend through the levels, a possible termination of a procedure in ($P2_\sigma$) (b) occurs before the procedure can act.

Verification: Before Lemma 6.6, we do not assume that M_d is a machine for L. C_1, the right domain of L, is enumerated in (Q1). For $1 \leqslant j < k$, let $D_{j,t}$ be the union of sets D enumerated by runs of a procedure $Q_{j,\sigma,y,w}$ up to the end of stage t. Let $C_{i,t}$ be the union of sets C enumerated by runs of a procedure P_i ($1 < i \leqslant k$) by the end of stage t.

Lemma 6.4. *The r.e. sets C_i are i-sets.*

Proof. By the wait in (Q1) and the definition of *stages*, D_1 is a 1-set: if n enters D at stage t, then $K(A_t \upharpoonright n) \leqslant K_t(n) + b \leqslant r_n + d + b$. For $2 \leqslant i \leqslant k$, assume inductively that D_{i-1} is an $i-1$-set. To see that C_i (and hence D_i in case $i < k$) is an i-set, assume that during *stage* s a number n is moved from D_{i-1} to C_i. This happens at $(P2_\sigma)$ for some σ. Let s' be the last *stage* before s. Then $n \in D_{i-1,\sigma}[s']$ since no Q_{i-1}-type procedure has been active yet at s. Also $\min D_{i-1,\sigma} > w$, and inductively $D_{i-1,\sigma}$ was an $i-1$-set already at s'. Thus at a *stage* $t \leqslant s'$, $\langle r, n\rangle$ was enumerated into L by a sub-procedure of type Q_1 of this run $Q_{i-1,\sigma,y,w}$, and there are $i-1$ distinct strings z of the form $A_v \upharpoonright n$ for some *stage* v, $t \leqslant v < s$ such that $K_v(z) \leqslant r + c$. Moreover, $n < s'$ and hence $K_s(A_s \upharpoonright n) \leqslant r + c$ by the definition of *stages*, the wait in (Q1) and because $n \in D_1$. Also, $A \upharpoonright w$ did not change from t to $s-1$, else the run of $Q_{i-1,\sigma,y,w}$ would have been canceled before s. Since $A_{s-1} \upharpoonright w \neq A_s \upharpoonright w$, we have a new string $z = A_s \upharpoonright n$ as required in order to show that C_i is an i-set. (Informally, we have verified that the change at s is not a change back to a previous configuration). $\square$

We next verify that L is a KC set. First we show that no procedure exceeds its garbage quota.

Lemma 6.5. (a) *Let $1 \leqslant j < k$. The weight of the numbers in $C_j - D_j$ which belong to a run $Q_{j,\sigma,y,w}(q, \beta)$ is at most β.*

(b) *Let $1 < i \leqslant k$. The weight of the numbers in $D_{i-1} - C_i$ which belong to a run $P_i(p, \alpha)$ is at most α.*

Proof. We actually obtain the bounds at any stage of the run. This suffices for the lemma, even if the run gets cancelled.

(a) For $j = 1$ the bound holds since the run has at most one number n in $C_1 - D_1$ at any given stage. So if the run gets stuck waiting at (Q1), it has left weight β in $C_1 - D_1$. If $j > 1$, all numbers as in (a) of this Lemma belong to a single run of a procedure $P_j(\beta, \alpha)$ called by $Q_{j,\sigma,y,w}(q, \beta)$, because, once such a run returns a set C', this set is put into D_j. The run of P_j does not return, so it does not reach its goal β. Thus the weight of such numbers is $\leqslant \beta$ at any stage of the run of $Q_{j,\sigma,y,w}$.

(b) Suppose n belongs to a run $P_i(p, \alpha)$ and $n \in D_{i-1,t}$ at *stage* t. Then n was put there during a run of a procedure $Q_{i-1,\sigma,y,w}(2^{-|\sigma|}\alpha, \beta)$ called by P_i. We claim that, *if n does not reach C_i, then no further procedure $Q_{i-1,\sigma,y',w'}$ is called after stage t*

during the run of P_i. Firstly assume that $A_s\upharpoonright w \neq A_{s-1}\upharpoonright w$ for some *stage* $s > t$. The only possible reason that n does not reach C_i is that the run of P_i did not need n to reach its goal in (P2$_\sigma$) (i.e., $n \notin \widetilde{D}$), in which case the run of P_i ends at s. Secondly, assume there is no such s. Then the run of P_i, as far as it is concerned with σ, keeps waiting at (P2$_\sigma$), and σ does not become available again. This proves the claim.

The claim implies that, for each σ there is at most one run $Q_{i-1,\sigma,y,w}(2^{-|\sigma|}\alpha, \beta)$ called by $P_i(p,\alpha)$ which leaves numbers in $D_{i-1} - C_i$. The sum of the weights of such numbers over all such σ is at most $\Omega\alpha$. (For Theorem 6.2, we distinguish two cases. If the run of P_i returns at *stage* s, then the sum of the weights is bounded by the value of $\Omega^A\alpha$ at the last *stage* before s. Otherwise the sum is bounded by $\Omega^A\alpha$.) $\square$

By the previous lemma and the definitions of the values α_i^*, β_j^* at substages,

$$wt(C_1 - C_k) \leqslant \sum\nolimits_{j=1}^{k-1} wt(C_j - D_j) + \sum\nolimits_{i=2}^{k} wt(D_{i-1} - C_i) \leqslant 1/2.$$

By Fact 6.3, $wt(C_k) \leqslant 1/2$. We conclude that $wt(C_1) \leqslant 1$, and hence that L is a KC-set.

From now on we may assume that M_d is a machine for L, using the recursion Theorem as explained above.

Lemma 6.6. *There is a run of a procedure* P_i, *called a* GOLDEN RUN, *such that*

(i) *the run is not cancelled*,
(ii) *each run of a procedure* $Q_{i-1,\sigma,y,w}$ *started by* P_i *returns unless cancelled*,
(iii) *the run of* P_i *does not return.*

Proof. Assume no such run exists. We claim that *each run of a procedure returns unless cancelled*. This yields a contradiction, since we call P_k with goal 1, this run is never cancelled, but if it returns, it has enumerated weight 1 into C_k, contrary to Fact 6.3.

To prove the claim we use induction on levels of procedures of type $Q_1, P_2, Q_2, \ldots, Q_{k-1}, P_k$. Suppose the run of a procedure is not cancelled.

$Q_{j,\sigma,y,w}(q,\beta)$: In case $j = 1$, by the hypothesis we always reach (Q2) after putting n into C_1, because the run is not cancelled and M_d is a machine for L. In case $j > 1$, inductively each run of a procedure P_j called by $Q_{j,\sigma,y,w}$ returns, as it is not cancelled. In any case, each time the run is at (Q2), the weight of D increases by β. Therefore $Q_{j,\sigma,y,w}$ reaches its goal and returns.

$P_i(p,\alpha)$: The run satisfies (i) by hypothesis, and (ii) by inductive hypothesis. Thus, (iii) fails, i.e., the run returns. $\square$

Lemma 6.7. *B is K-trivial*. (A is low for K).

Proof. Choose a golden run of a procedure $P_i(p,\alpha)$ as in Lemma 6.6. We enumerate a KC set W. Note that $p/\alpha = 2^g$ for some $g \in \mathbb{N}$. At *stage* s, when a run $Q_{j,\sigma,y,w}(2^{-|\sigma|}\alpha, \beta)$ returns, then put $\langle |\sigma| + g + 1, B_s\upharpoonright y\rangle$ into W (put $\langle |\sigma| + g + 1, y\rangle$

into W) . We prove that W is a KC-set, namely,

$$S_W = \sum_s \sum \{2^{-r} : \langle r, z\rangle \in W_s - W_{s-1}\} \leqslant 1.$$

Suppose $\langle r, z\rangle$ enters W at *stage* s due to a run $Q_{i-1,\sigma,y,w}(2^{-|\sigma|}\alpha, \beta)$ which returns.

Stable case: The contribution to S_W of those axioms $\langle r, z\rangle$ where $A\restriction w$ is stable from s on is bounded by $2^{-(g+1)}\Omega$ $[2^{-(g+1)}\Omega^A]$, since for each σ such that $U(\sigma)$ is defined ($U^A(\sigma)$ is defined) this can only happen once.

Change case: Now suppose that $A\restriction w$ changes after *stage* s. Then the set D returned by $Q_{i-1,\sigma,y,w}$, whose weight is $2^{-|\sigma|}\alpha$, went into C_i. Since the run of P_i does not return,

$$\sum_s \sum \{2^{-|\sigma|} : Q_{i-1,\sigma,y,w} \textit{ returns at } s \;\&\; A\restriction w \textit{ changes at some } t > s\} < 2^g,$$

otherwise the run of P_i reaches its goal $p = 2^g\alpha$. Thus the contribution of the corresponding axioms to S_W is less than $1/2$.

Let M_e be the machine for W according to the Kraft–Chaitin theorem. We claim that, for all y,

$$K(B\restriction y) \leqslant K(y) + g + e + 1$$

$[K(y) \leqslant K^A(y) + g + e + 1]$. Suppose that s is the minimal *stage* such that $U_s(\sigma) = y$, $\Gamma^A\restriction y \downarrow [s]$ and $A_s\restriction\gamma(y-1)$ is stable (a stable computation $U_s^A(\sigma) = y$ appears), where σ is a shortest description of y. Let w be as in (P1$_\sigma$), namely, $w = \gamma^A(y-1)$ (let w be the use of this computation). Then σ is available at s: otherwise some run $Q_{i-1,\sigma,y',w'}$ is waiting to be released at (P2$_\sigma$). In that case, $A\restriction w'$ has not changed since that run was started. Then $w = w'$ and $y = y'$, contrary to the minimality of s. So we call $Q_{i-1,\sigma,y,w}$. Since $A\restriction w$ is stable and the run of P_i is not cancelled, this run is not cancelled, so it returns by (ii) of Lemma 6.6. At this *stage* we put $\langle |\sigma| + g + 1, B_s\restriction y\rangle$ into W [we put $\langle |\sigma| + g + 1, y\rangle$ into W], causing the required inequality. □

7. Further results on $\mathcal{K}$

In this section we study further properties of $\mathcal{K}$ and its role within the Turing degrees. We also show that any proof of Theorem 6.1 is necessarily non-uniform. First we show that Proposition 4.1 actually provides a characterization of K-trivial sets.

Theorem 7.1. *The following are equivalent.*

(i) *A is K-trivial,*
(ii) *There is a Δ^0_2-approximation $(\widetilde{A}_r)_{r\in\mathbb{N}}$ of A such that*

$$S = \sum \{c(x, r) : x \textit{ is minimal s.t. } \widetilde{A}_{r-1}(x) \neq \widetilde{A}_r(x)\} < 1/2, \tag{6}$$

where $c(x, r) = 1/2 \sum_{x<y\leqslant r} 2^{-K_r(y)}$.

By (ii) and Fact 4.4, any $A \in \mathcal{K}$ is ω-r.e.

Proof of Theorem 7.1. (ii) $\Rightarrow$ (i) is Proposition 4.1, with $(\widetilde{A}_r)_{r\in\mathbb{N}}$ instead of $(A_r)_{r\in\mathbb{N}}$.

(i)$\Rightarrow$ (ii). We extract some additional information from the proof of Theorem 6.1, for the special case that $B = A$ and Γ is the identity functional, where $\gamma(y)$ is defined to be $y + 1$. Let (A_s) be the modified Δ^0_2-approximation from the beginning of the proof of Theorem 6.1. We first prove that there a constant $g \in \mathbb{N}$ and a recursive sequence of stages $q(0) < q(1) < \cdots$ such that

$$\widehat{S} = \sum \{\widehat{c}(x, r) : x \text{ is minimal s.t. } A_{q(r+1)}(x) \neq A_{q(r+2)}(x)\} < 2^g, \tag{7}$$

where $\widehat{c}(z, r) = \sum_{z<y\leqslant q(r)} 2^{-K_{q(r+1)}(y)}$.

By Lemma 6.6, choose a golden run $P_i(p, \alpha)$.

Claim 7.2. *For each stage s, there is a stage $t > s$ such that, for all $y < s$, if σ is a shortest description of y at t, then a run $Q_{i-1,\sigma,y,y+1}$ has returned by t and is not released yet, that is, P_i waits at* (P2$_\sigma$).

Such a t exists because, for each y, there are only finitely many possible σ's. Once $A\restriction y + 1$ has settled, a run of a procedure $Q_{i-1,\sigma,y,y+1}$ is not canceled, therefore it returns by property (ii) of golden runs. This proves the claim. Note that the least such t can be determined effectively. Let $q(0) = 0$. If $s = q(r)$ has been defined, let $q(r + 1)$ be the least t satisfying this condition for s.

As before, let $g \in \mathbb{N}$ be the number such that $p/\alpha = 2^g$. We show that $\widehat{S} < 2^g$. Suppose x is minimal such that $A_{q(r+1)}(x) \neq A_{q(r+2)}(x)$. Then $A_{s-1}(x) \neq A_s(x)$ for some *stage* s with $q(r + 1) < s \leqslant q(r + 2)$. No later than s, the runs of procedures $Q_{i-1,\sigma,y,y+1}$ with $x \leqslant y < q(r)$ which are still waiting at (P2$_\sigma$) are released. This adds a weight of at least $\widehat{c}(x, r)$ to C_i. Thus $\widehat{S} < 2^g$, otherwise the run of P_i reaches its goal.

We obtain the required Δ^0_2-approximation $\widetilde{A}_r(x)$ after some manipulations. First let $A^*_r(x) = A_{q(r+2)}(x)$. Note that, for $z < r$, $c(z, r) = 1/2 \sum_{z<y\leqslant r} 2^{-K_r(y)} \leqslant \widehat{c}(z, r)$, so that $\sum\{c(x, r) : x$ *is minimal s.t.* $A_{r-1}(x) \neq A_r(x)\} \leqslant \widehat{S} < 2^g$. Now choose r_0 so large that the sum over all $r \geqslant r_0$ is at most 1/2. Let $\widetilde{A}_r(x) = A^*_{r_0}(x)$ for $r \leqslant r_0$, and $\widetilde{A}_r(x) = A^*_r(x)$ else. This shows (i)$\Rightarrow$ (ii). $\square$

In [9] it is shown that there is a uniform listing of $\mathcal{K}$ that includes the constants via which K-triviality holds. The proof is based on recursive sequences of stages satisfying (7).

Theorem 7.3 (*Downey et al.* [9]). *There is an effective list $((B_{e,s}(x))_{s\in\mathbb{N}}, d_e)$ of Δ^0_2-approximations and constants such that each K-trivial set occurs as a set $B_e = \lim_s B_{e,s}$, and each B_e is K-trivial via the constant d_e.*

Each K-trivial set is truth-table below an r.e. one:

Theorem 7.4. *For each K-trivial set A, there is an r.e. K-trivial set D such that $A \leqslant_{tt} D$, via a polynomial time tt-reduction.*

Proof. Let $(\widetilde{A}_r)_{r\in\mathbb{N}}$ be the Δ^0_2-approximation from (ii) of Theorem 7.1. Recall that, by Proposition 4.4, one may choose a constant c such that $\widetilde{A}_r(y)$ changes at most cy^2 times. Let $f(x) = c\sum_{0\leqslant z<x} z^2$. Define the r.e. set D as follows: when $\widetilde{A}_r(x) \neq \widetilde{A}_{r+1}(x)$ for the $i+1$st time, then enumerate $f(x)+i$ into D. Then $A \leqslant_{tt} D$, via a polynomial time tt-reduction (where numbers are identified with strings): if the greatest $i < cy^2$ such that $f(y)+i \in D$ is even, then $A(y) = 1-\widetilde{A}_0(y)$. If the greatest $i < cy^2$ such that $f(y)+i \in D$ is odd or there is no such i, then $A(y) = \widetilde{A}_0(y)$.

To see that D is K-trivial, note that for each r and each $x < r$,

$$D_{r-1}\upharpoonright x \neq D_r\upharpoonright x \Rightarrow \widetilde{A}_{r-1}\upharpoonright x \neq \widetilde{A}_r\upharpoonright x.$$

Thus the sum in (6) for (D_r) is no greater than the sum for $(\widetilde{A}_r)$. $\square$

Definition 7.5. The set A is SUPER-LOW if $A' \leqslant_{tt} \emptyset'$.

Of course, super-low sets A are ω-r.e., that is, $A \leqslant_{tt} \emptyset'$. In Nies [22] it is proved that super-lowness and U-traceability coincide on the r.e. sets, in a uniform way (but no inclusion holds between the classes on the ω-r.e. sets).

The following could also be proved directly via a modification of the proof of Theorem 6.1 (see [20]). However, we prefer to use Theorem 6.2 and Proposition 2.8.

Corollary 7.6. *Each K-trivial set A is super-low.*

Proof. It suffices to show that the r.e. set D obtained via Theorem 7.4 is super-low. D is low for K by Theorem 6.2, hence U-traceable by Proposition 2.8 (which is uniform). Thus D is super-low by [22]. $\square$

It is not hard to show that there are super-low r.e. sets A,B such that $A \oplus B$ is Turing complete [22]. Thus not all super-low r.e. sets are K-trivial.

The following corollary shows that some non-uniformity as the one in the proof of Theorem 6.2 is necessary.

Corollary 7.7. *There is no effective way to obtain from a pair (A, b), where A is an r.e. set that is K-trivial via b, a constant c such that A is low for K via c.*

Proof. Otherwise, by Theorem 7.3 above we would obtain a listing (B_e, c_e) of all the low for K sets with appropriate constants. But such a listing does not exist: If A is an r.e. set in $\mathcal{M}$, then an index of a reduction showing the super-lowness can be obtained uniformly from an index for A and the constant via which $A \in \mathcal{M}$ (by the uniformity of Theorem 2.8 and of the equivalence of U-traceability and super-lowness for r.e.

sets). So one could extend the listing to include (super-) lowness indices. But an easy extension of Theorem 4.3 gives a set $C \in \mathcal{K} = \mathcal{M}$ not Turing below any B_e. The details are in [22, Theorem 5.9]. $\square$

Note that, by a similar argument, Theorem 7.6 is non-uniform, even for lowness instead of super-lowness. The non-uniformity in the proof of Theorem 6.2 is easily detected: the constant via which A is low for K given by that proof is $g + e + 1$, and g depends on what the golden run is. Thus we cannot determine the golden run effectively.

The restriction on the number of changes in Proposition 4.4 can be improved. Each K-trivial set has a Δ^0_2-approximation which changes as little as desired (we thank Frank Stephan for pointing this out).

Corollary 7.8. *Let $A \in \mathcal{K}$. Given a non-decreasing recursive h such that $\lim_n h(n) = \infty$, there is a Δ^0_2-approximation $(A_r)_{r\in\mathbb{N}}$ of A such that $A_r(y)$ changes at most $h(y)$ times.*

Proof. By Theorem 7.4, there is an r.e. $D \in \mathcal{K}$ such that $A = \Phi^D$ for a tt-reduction Φ with recursive use φ. D is U-traceable by Theorem 7.6 and [22]. By the method of [29, Fact1], there is an r.e. *trace* T with bound h for the total D-recursive function $p(y) = \mu s\ \ D_s\restriction\varphi(y) = D\restriction\varphi(y)$, that is, $\forall y\ \ p(y) \in T^{[y]}$. Now let $A_r(y) = 1$ if $\Phi^{D_v}(y) = 1$ where $v = \max\ T_r^{[y]}$, and let $A_r(y) = 0$ otherwise. $\square$

Theorem 7.9. *The K-trivial sets form a Σ^0_3 ideal in the ω-r.e. T-degrees, which is generated by its r.e. members. Moreover, this ideal is non-principal.*

Proof. By Theorems 6.1, 2.3 and 7.4 the K-trivial sets induce an ideal generated by the r.e. members. This ideal is Σ^0_3 by Fact 2.4. Suppose the ideal equals $[\mathbf{0}, \mathbf{b}]$ for some degree $\mathbf{b}$. Then $\mathbf{b}$ is r.e. and low by Theorem 7.6. This contradicts Theorem 4.3. $\square$

Corollary 7.10. *There is an r.e. low_2 set E such that $A \leqslant_T E$ for each K-trivial set A.*

Proof. By Theorem 7.4, it suffices to give such a bound E for the r.e. K-trivial sets. By [21], any proper Σ^0_3 ideal in the r.e. degrees has a low_2 upper bound. $\square$

By Theorem 4.3, no such E is low_1.

8. Relativizations, operators, and reducibilities

We review some extensions and related notions.

Operators: Let $\mathcal{K}(X)$ be the class of sets which are K-trivial relative to X, that is, $\mathcal{K}(X) = \{A :\ \forall n\ \ K^X(A\restriction n) \leqslant K^X(n) + \mathcal{O}(1)\}$. The relativization of the class of low for K sets is $\mathcal{M}(X) = \{A :\ \forall y\ \ K^X(y) \leqslant K^{A\oplus X}(y) + \mathcal{O}(1)\}$.

We show that $\mathcal{K}$ is an operator with good closure properties and a very simple representation. Firstly, $\mathcal{K}$ is degree invariant as an operator, since

$$X \equiv_T Y \;\Rightarrow\; \forall z\; |K^X(z) - K^Y(z)| \leqslant \mathcal{O}(1) \;\Rightarrow\; \mathcal{K}(X) = \mathcal{K}(Y).$$

All the results on $\mathcal{K}$ we have discussed relativize (the coincidence with Low(MLR) will be addressed later on).

Theorem 8.1. (i) *$\mathcal{K}(X)$ is closed under $\oplus$ and closed downward under $\leqslant_T$.*
(ii) *There is an r.e. index e such that, for each X, $W_e^X \in \mathcal{K}(X)$ and $X <_T W_e^X$.*
(iii) *$\mathcal{M}(X) = \mathcal{K}(X)$.*
(iv) *$A \in \mathcal{K}(X) \;\Rightarrow\; A$ is tt-below some $D \in \mathcal{K}(X)$ which is r.e. in X, via a polynomial time tt-reduction as in Theorem 7.4.*
(v) *$A \in \mathcal{K}(X) \;\Rightarrow\; A' \leqslant_{tt} X'$.*

Proof. One obtains (i)–(iv) by examining the proofs of Theorems 2.3, 6.1, 6.2, 4.2 and 7.4. For (v), suppose that $A \in \mathcal{K}(X)$. By (iv) we can suppose A is r.e. in X. Since $A \oplus X \in \mathcal{K}(X)$, $A \oplus X \in \mathcal{M}(X)$ by (iii). Relativizing Proposition 2.8, $A \oplus X$ is U-traceable relative to X. Then, relativizing the fact from [22] that each U-traceable set is super-low, $A' \leqslant_{tt} X'$. $\square$

Theorem 8.2. *There is an effective listing $(\Gamma_e)_{e \in \mathbb{N}}$ of tt-reduction procedures such that, for each X, $\mathcal{K}(X) = \{\Gamma_e(X') : \; e \in \mathbb{N}\}$.*

Proof. Since $\{e : W_e^X \in \mathcal{K}(X)\}$ is $\Sigma^0_3(X)$ via a fixed index, there is an effective listing (V_j) of oracle enumeration procedures such that for each X, $\{V_j^X : j \in \mathbb{N}\}$ equals the class of sets in $\mathcal{K}(X)$ which are r.e. in X. Let (Φ_i) be an effective listing of the tt-reduction procedures needed in (iv) of Theorem 8.1. For each pair i, j we can effectively determine a tt-reduction Γ_e, $e = \langle i, j \rangle$ such that $\Gamma_e(X') = \Phi_i(V_j^X)$. $\square$

Slaman [25] studied Borel operators $\mathcal{S} : \mathcal{P}(\mathbb{N}) \mapsto \mathcal{P}(\mathcal{P}(\mathbb{N}))$ such that, for each X, Y, $\mathcal{S}(X)$ is an ideal in the Turing degrees containing X, but not all sets, and $\mathcal{S}$ is MONOTONE, that is, for each X, Y, $X \leqslant_T Y \;\Rightarrow\; \mathcal{S}(X) \subseteq \mathcal{S}(Y)$, a property stronger than degree invariance. Slaman proves that, on an upper cone in the Turing degrees, any such operator is given by (possibly transfinite) iterates of the jump. Possibilities for $\mathcal{S}(X)$ include $\{Y : Y \leqslant_T X\}$, $\{Y : Y \leqslant_T X'\}$, or $\{Y : \exists n \in \mathbb{N}\; Y \leqslant_T X^{(n)}\}$.

Since the operator $\mathcal{K}$ is not given by such iterates, it fails to be monotone. An explicit example of non-monotonicity was pointed out by R. Shore: By Theorem 4.2, let A be a promptly simple set in $\mathcal{K}(\emptyset) = \mathcal{K}$. Then A is low cuppable, i.e. there is a low r.e. G such that $\emptyset' \leqslant_T A \oplus G$ (see [27, Theorem XIII.4.2]). Hence $A \in \mathcal{K}(\emptyset) - \mathcal{K}(G)$, otherwise $A \oplus G \in \mathcal{K}^G$ and hence $(A \oplus G)' \leqslant_T G'$ by Theorem 8.1 (v), which is a contradiction.

Reducibilities: For sets A, B, let $A \leqslant_{LK} B \;\Leftrightarrow\; \forall y\; K^B(y) \leqslant K^A(y) + \mathcal{O}(1)$, and $A \leqslant_{LR} B \;\Leftrightarrow\; \mathsf{MLR}^B \subseteq \mathsf{MLR}^A$. Clearly, $\leqslant_T$ implies $\leqslant_{LK}$, which in turn implies $\leqslant_{LR}$. Note that the result Low(MLR) $= \mathcal{K}$ relativizes, as follows: $A \oplus X \leqslant_{LR} X \;\Leftrightarrow\; A \in \mathcal{K}(X)$. In [20] it is shown that, for r.e. A, B, $A \leqslant_{LR} B$ implies $A' \leqslant_{tt} B'$. Moreover, applying the technique of pseudo-jumps in [12] to the r.e. operator given by the construction of a low for K set, there is an r.e. A which is T-incomplete but $\leqslant_{LK}$-complete.

Let $\mathit{Left}_{LK}(X) = \{A : A \leqslant_{LK} X\}$ and $\mathit{Left}_{LR}(X) = \{A : A \leqslant_{LR} X\}$.

Proposition 8.3. (i) $\mathcal{K}(X) \subseteq \mathit{Left}_{LK}(X)$.
(ii) *If G is as above, then $\mathcal{K}(G)$ is a proper subclass of $\mathit{Left}_{LK}(G)$.*
(iii) $A \equiv_{LK} B \;\Leftrightarrow\; A \in \mathcal{K}(B)$ & $B \in \mathcal{K}(A)$.

Proof. For (i), since $\mathcal{K}(X) = \mathcal{M}(X)$, for each $A \in \mathcal{K}(X)$, $\forall y\; K^X(y) = K^{A\oplus X}(y) \leqslant K^A(y)$ up to additive constants. (ii) holds since $A \in \mathit{Left}_{LK}(\emptyset) = \mathcal{M}(\emptyset)$, so that $A \in \mathit{Left}_{LK}(G) - \mathcal{K}(G)$. For (iii), by (i) it only remains to show the direction from left to right: for each n, $K^A(A\restriction n) \leqslant K^A(n) + \mathcal{O}(1)$. If $A \equiv_{LK} B$ we may replace the oracle A by B, which shows that $A \in \mathcal{K}(B)$. $\square$

By relativizing (ii), we see that for each X there is a $G \geqslant_T X$ such that $\mathcal{K}(G)$ is a proper subclass of $\mathit{Left}_{LK}(G)$. Then, since the operators $\mathcal{K}$ and Left_{LK} are degree invariant, by arithmetic determinacy, this holds on an upper cone of Turing degrees.

Just as $\mathcal{K}$, Left_{LK} and Left_{LR} are Σ^0_3 operators (see [20] for Left_{LR}), but unlike $\mathcal{K}$, they are monotone in the sense of Slaman. Since each image is downward closed under $\leqslant_T$, by Slaman's result, the image of X cannot be an ideal for all X. The explicit counterexample used above for $\mathcal{K}$ works once again (say for $\leqslant_{LR}$): note that $A \in \mathit{Left}_{LR}(\emptyset)$. Thus $A \leqslant_{LR} G$, and trivially $G \leqslant_{LR} G$, but $\emptyset' \equiv_T A \oplus G \not\leqslant_{LR} G$ by the result in [20], since G is low. In particular, $\oplus$ does not determine a supremum in the r.e. $\leqslant_{LR}$-degrees.

Using Theorem 8.1 (v) and Proposition 8.3 (ii), $A \equiv_{LK} B$ implies $A' \equiv_{tt} B'$ for all sets A, B. We do not know if this holds for $\equiv_{LR}$ in place of $\equiv_{LK}$. The recent paper [17] contains further work on reducibilities, for instance that $A \leqslant_K B$ implies $A \geqslant_{LR} B$ for $A, B \in \mathsf{MLR}$ (here $X \leqslant_K Y \Leftrightarrow \forall n\, K(X\restriction n) \leqslant K(Y\restriction n) + \mathcal{O}(1)$).

Many other questions remain. For instance, is $\mathcal{K}$ definable in the (r.e.) Turing degrees?

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