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Undecidable fragments of elementary theories

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Abstract. We introduce a general framework to prove undecidability of fragments. This is applied to fragments of theories arising in algebra and recursion theory. For instance, the $\forall\exists\forall$-theories of the class of finite distributive lattices and of the p.o. of recursively enumerable many-one degrees are shown to be undecidable.

1. Introduction

A fragment of a first-order theory T is a set $T \cap S$ for some set S of sentences with a simple prescribed syntactical structure. Since such a set S is decidable, undecidability of a fragment of a theory implies undecidability of the whole theory. Here we address the converse question: if T is undecidable, which fragments of T are undecidable?

A formula is Σ_k if it has the form $(\exists \cdots \exists)(\forall \cdots \forall)(\exists \cdots \exists)(\cdots)\psi$, with $k-1$ quantifier alternations and ψ quantifier free, and Π_k if it has the form $(\forall \cdots \forall)(\exists \cdots \exists)(\forall \cdots \forall)(\cdots)\psi$. We only consider the case that S is the set of Σ_k- or Π_k-sentences. There are several reasons for investigating the decidability question for such fragments: firstly, theories consist mostly of sentences which have too many quantifier alternations to be "comprehensible"; so, even if undecidability of a theory is known, the question remains which feasible fragments are undecidable. Secondly, it is desirable to obtain a sharp classification of the quantifier level where a theory T becomes undecidable, i.e., to determine a number k such that e.g. $T \cap \Sigma_k$ is undecidable but $T \cap \Sigma_{k-1}$ is decidable. This is the best result one can hope for if T is a complete theory, since $T \cap \Sigma_k$ is as complex as $T \cap \Pi_k$; but for incomplete T one can even try to determine k such that e.g. $T \cap \Sigma_k$ undecidable but $T \cap \Pi_k$ decidable. Thirdly, sometimes one can give an algebraic interpretation of the sentences in a set S as above, and then a decision procedure usually gives algebraic information about the class. For instance, the Π_1-theory of a variety is closely connected to the word problem of its finitely presented members.

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In this paper we develop a general framework to prove undecidability of fragments. This incorporates many ideas used before for plain proofs of undecidability of theories, e.g. [Tra53], [Er ea 65] and [Bu, Sa75], as well as in undecidability proofs for fragments of special theories in [Ler83] and [A, S93]. The method is applied to the classes of finite undirected graphs, finite distributive lattices and finite partition lattices. Furthermore, building on these results we obtain undecidability of fragments of theories arising in recursion theory. For instance we give a sharp classification in the above sense for the theory of the recursively enumerable many-one degrees. We begin with some examples.

Undecidable fragments

(1) Let $T = Th(\mathbf{N}, +, x)$. Then $T \cap \Sigma_1$ is undecidable. To prove this, one uses that, by the solution to Hilbert's 10th problem, the set of polynomials

$$\{p \in \mathbf{Z}|\bar{x}| : (\exists \bar{n} \geq \bar{0})[p(\bar{n}) = 0]\}$$

is undecidable ("$\bar{x}$" denotes a tuple $x_1, \ldots, x_k$ of variables). Moreover, for $p \in \mathbf{Z}|\bar{x}|$, $p(\bar{n}) = 0 \Leftrightarrow p_+(\bar{n}) = p_-(\bar{n})$, where p_+ is the part of p with positive and $(-1)p_-$ the part of p with negative coefficients. Since the sentence $(\exists \bar{n})[p_+(\bar{n}) = p_-(\bar{n})]$ is Σ_1, this shows that $\Sigma_1 \cap Th(\mathbf{N}, +, x, 1)$ is undecidable. To eliminate the constant 1 from the language, note that 1 can be defined existentially as an element x satisfying $(\exists z)[z + z \neq z \wedge xz = z]$.

(2) If T is the theory of groups, then $T \cap \Pi_1$ undecidable. This follows from the fact that there is a finitely presented group with unsolvable word problem.

(3) Let T be the theory of modular lattices and S be the set of universally quantified equations $(\forall \bar{x})[t(\bar{x}) = s(\bar{x})]$, for lattice terms s, t. By [Fre80], the fragment $T \cap S$ is undecidable (even if we restrict the number of universal quantifiers to 5).

Decidable fragments

(4) Let L be (a) a finite relational language, (b) the language of lattices, and let T be (a) the theory of all L-structures (b) the theory of all finite distributive lattices. Then $T \cap \Pi_2$ is decidable: to determine whether a sentence $(\forall x_1, \ldots, x_n)(\exists y_1, \ldots, y_m)\psi$ is in T, for (a), it suffices to test L-structures of cardinality $\leq n$. For (b), it suffices to test the distributive lattices of cardinality $\leq 2^{2^n}$.

(5) Let V be a variety such that the finitely presented algebras in V have solvable word problem. Suppose further that such a decision procedure is uniform in a presentation. Then $Th(V) \cap \Pi_1$ is decidable. An example of such a variety is the variety of lattices (Kinsey; see [Fre, Na79]). To see that this fragment is decidable, note that, using conjunctive normal form of ψ, any Π_1-sentence $(\forall \cdots \forall)\psi$ in the language of V is equivalent to a conjunction of sentences of the form

$$(\forall \bar{x})[\bigwedge_{1 \leq i \leq n} e_i \rightarrow \bigvee_{1 \leq j \leq m} f_j], \tag{1}$$

where e_i and f_j are equations of the form $t(\bar{x}) = s(\bar{x})$. Thus it suffices to check $\sigma \in Th(V)$ for each such conjunct σ. But $\sigma \in Th(V)$ iff at least one equation f_j holds in the V-algebra presented by $\langle \bar{x} : e_1, \ldots, e_n \rangle$.

Other fragments of theories which have been studied to some extent are given by taking as S the set of implications $\varphi_0 \rightarrow \varphi$, for some fixed sentence φ_0 (e.g. the conjunction of some finite system of axioms), or the set of relativizations of the form $\varphi^{[\{x : \psi(x)\}]}$, for some fixed formula ψ.

We now give an outline of the method to prove undecidability of fragments, which is developed in detail in the Sections 2 and 3. For a language L, let L-Valid be the set of sentences which hold in all L-structures. A set of sentences U is *hereditarily undecidable* (h.u.) if $L\text{-Valid} \cap U \subseteq X \subseteq U$ implies that X is undecidable. First it is shown that, for some finite relational language L, the set of Σ_2-sentences which hold in all *finite* L-structures is undecidable. Then a notion of definability with parameters of a class $\mathbf{C}$ of structures in a second $\mathbf{D}$ class is given, which is derived from [Bu, McK81], but takes into account the number of quantifier alternations in the defining formulas. This definability relation makes it possible to transfer undecidability of fragments of $Th(\mathbf{C})$ to undecidability of fragments of $Th(\mathbf{D})$.

We use a notion from recursion theory: disjoint sets of natural numbers X, Y are called recursively inseparable (r.i.) if there is no recursive set R such that $X \subseteq R$ and $Y \cap R = \emptyset$. We will make use of the following fact several times.

Recursion theoretic fact

If U_0, U_1 are recursively inseparable, f is a (total) recursive function and V_0, V_1 disjoint, then (2)

$$f(U_i) \subseteq V_i \ (i = 0, 1) \quad \Rightarrow \quad V_0, V_1 \text{ recursively inseparable.}$$

To verify this, note that if a recursive set R separates V_0 from V_1, then $f^{-1}(R)$ is recursive and separates U_0 from U_1. Observe that a set of sentences U is h.u. iff L-Valid$\cap U$ and $\bar{U}$ are r.i.

Conventions

We use self-explanatory names for the classes of structures under discussion. *LStructures* is the class of L-structures, and *SymGraphs* is the class of structures (V, E) such that E is a symmetric irreflexive relation on V. *P*Orders is the class of partial orders. A class $\mathbf{C}$ of structures is always understood in the context of a language L, which may or may not contain equality. If $\mathbf{C}$ is a class of p.o., we can ignore this difference since equality can be defined without quantifiers. F-$\mathbf{C}$ denotes the class of finite members of $\mathbf{C}$. $Th(\mathbf{C})$ denotes the set of sentences in the language L which hold in every structure in $\mathbf{C}$, and $\Sigma_k - Th(\mathbf{C})$ stands for $\Sigma_k \cap Th(\mathbf{C})$. Lattices are normally viewed as structures in the language of partial orders. The theory of a class $\mathbf{C}$ of lattices in the full language is denoted by $Th_{\vee\wedge}(\mathbf{C})$.

2. A relational language L such that $\Sigma_2 - Th(F\text{-}L\text{Structures})$ is hereditarily undecidable

In this section we give a direct coding of computations in models of a relational language L. The proof of Theorem 2.1 will not be needed in later sections, but is crucial for a deeper understanding of undecidability proofs by the method of coding classes of structures.

THEOREM 2.1. *There exists a finite relational language L with equality such that the Σ_2-theory of the class of finite L-structures is h.u.*

Proof. We will represent computations of a 3-register machine M with additional input and output registers in L-structures. Such a machine is given by m instructions which have one of the forms

i : Reg := Reg + 1; *goto* j,

i : Reg := max(Reg − 1, 0); *goto* j,

i : *If* Reg = 0 *then goto* j *else goto* k,

where $0 \leq i, j, k < m$ and Reg is one of the registers. We refer to $\{0, \ldots, m-1\}$ as the set of states of M. A configuration of M is given by 6 natural numbers *state*, *input*, *output*, *reg0*, *reg1*, *reg2* which denote the current state and contents of the registers. The initial configuration has state 0, and only the input register may have nonzero content. The state 1 is called the halting state; if an M-computation reaches this state, the computation is finished and the content of the output register is regarded as the output.

A *partial successor model* is a structure $(X, f, 0)$ such that f is a 1-1 partial map and $f(x) \neq 0$ for $x \in X$. The *standard part* of such a model is the set of elements obtained from 0 by iterated applications of f. The L-structures representing computations will be certain finite disjoint unions of partial successor models, corresponding to time, state and contents of the registers. If an M-computation on input k converges, the corresponding L-structure will be finite. The relation symbols in L are the following.

(a) The unary symbols Time, State, Input, Output, Reg0, Reg1 and Reg2,

(b) A binary relation symbol f_X and unary relation symbols Zero_X, where X is one of the unary symbols listed in (a). These give the model of partial successor on X. (We use a unary predicate Zero_X to avoid introducing constants.)

(c) A 7-ary relation symbol R, intended to hold for $(t, x_1, \ldots, x_6)$ if M reaches the configuration $(x_1, \ldots, x_6)$ at step t.

We avoid the explicit use of the symbols in (a) by variable conventions: t ranges over those x such that Time(x) holds, and the auxiliary variable c ranges over coded configurations, i.e., 6-tuples of variables $(x_1, \ldots, x_6)$ such that State(x_1), ..., Reg2(x_6) holds. We write Input(c) etc. for the respective components of c, and Zero(Input(c)) instead of $\mathrm{Zero}_{\mathrm{Input}}(\mathrm{Input}(c))$.

We will obtain a Π_2-sentence γ_k which expresses that an L-structure codes the computation on input k stepwise correctly in the following sense:

$R(t, c) \wedge \mathrm{Zero}_{\mathrm{Time}}(t)$ implies that c is the initial configuration corresponding to the input k and, if $R(t, x)$ holds, then either State(c) is the halting state, or $f_{\mathrm{Time}}(t)$ is defined and $R(f_{\mathrm{Time}}(t), d)$ holds for a unique configuration d. Suppose first we only want to prove that L-Valid$\cap\Sigma_2$ is undecidable. Let M be a 3-register machine such that $k \in C \Leftrightarrow M(k) = 0$, for some undecidable recursively enumerable set C. Let φ_k be the sentence

$$\gamma_k \rightarrow \text{“the computation stops with output 0”}.$$

Note that φ_k has the form “$\Pi_2 \rightarrow \Sigma_1$” and therefore is a Σ_2-sentence. If $M(k) = 0$ then φ_k is valid, since each L-structure satisfying γ_k must code the actual M-computation on its standard part (i.e., the union of the standard parts of the partial successor models). On the other hand, if not $M(k) = 0$, then φ_k fails in the structure coding the (possibly diverging) M-computation with input k. Thus $k \in C \Leftrightarrow \varphi_k$ is valid, and $\Sigma_2 - Th(L\text{Structures})$ is undecidable, since the decidability of this fragment would imply the decidability of C.

To obtain the full result, fix a machine M such that $\{x : M(x) = 0\}$ and $\{x : M(x) = 1\}$ are recursively inseparable. As before,

$$M(k) = 0 \quad \Rightarrow \quad \varphi_k \text{ is valid.}$$

Now

$$M(k) = 1$$

$$\Rightarrow \quad \varphi_k \text{ fails in the } \textit{finite} \text{ structure coding this } M\text{-computation}$$

$$\Rightarrow \quad \varphi_k \notin Th(F\text{-}L\text{Structures}).$$

By the recursion theoretic fact (2), $\Sigma_2 - Th(F\text{-}L\text{Structures})$ in h.u.

It remains to define the sentences γ_k and φ_k formally. To obtain γ_k, one has to express the following.

Uniqueness of the configuration at each step $t(\Pi_1)$:

$$(\forall t)(\forall c)(\forall d)[R(t, c) \wedge R(t, d) \rightarrow c = d].$$

Axioms of partial successor $(\Pi_1 \wedge \Sigma_1)$:

"*f_X defines a partial 1-1 map from X into X*" $(\Pi_1) \wedge$

$$(\forall x)(\forall y)[\mathrm{Zero}_X(x) \wedge \mathrm{Zero}_X(y) \rightarrow x = y](\Pi_1) \wedge$$

$$(\exists x)[\mathrm{Zero}_X(x)](\Sigma_1) \wedge (\forall x)(\forall y)[f_X(x, y) \rightarrow \neg \mathrm{Zero}_X(y)](\Pi_1).$$

Existence of a correct initial configuration (Σ_1):

$$(\exists c)(\exists t[\mathrm{Zero}_{\mathrm{Time}}(t) \wedge R(t, c) \wedge \mathrm{Zero}(\mathrm{State}(c)) \wedge \mathrm{Zero}(\mathrm{Output}(c)) \wedge \cdots \wedge$$

$$\mathrm{Zero}(\mathrm{Reg2}(c)) \wedge (\exists i_0) \cdots (\exists i_k)[\mathrm{Zero}_{\mathrm{input}}(i_0) \wedge \mathrm{Input}(c) = i_k \wedge$$

$$\textstyle\bigwedge_{0 \le r < k} f_{\mathrm{Input}}(i_r, i_{r+1})]].$$

Domain of the successor map for Time and stepwise correctness of the computation (to express this by a Π_2 formula, we need to quantify universally over all the possible states of M):

$$(\forall s_0)\cdots(\forall s_{m-1})[\mathrm{Zero}_{\mathrm{state}}(s_0) \wedge \bigwedge_{0 \le i < m-1} f_{\mathrm{state}}(s_i, s_{i+1}) \rightarrow$$
$$(\forall t)(\forall c)[R(t, c) \rightarrow (\mathrm{State}(c) = s_1 \vee$$
$$(\exists s)(\exists d)[f_{\mathrm{Time}}(t) = s \wedge R(s, d) \wedge \mathrm{Succ}(c, d)]]].$$

Here the quantifier free formula Succ(c, d) which describes the successor relation induced by M on configurations is given by

$$\mathrm{Succ}(c, d) \quad \Leftrightarrow \quad \bigwedge_{0 \le i < m} [\mathrm{State}(c) = s_i \rightarrow \psi_i(c, d)],$$

where ψ_i depends on the instruction i as follows:

Instruction	ψ_i
i : *If* Reg0 = 0 *then* j *else* k	$\mathrm{Input}(c) = \mathrm{Input}(d) \wedge \cdots \wedge \mathrm{Reg2}((c) = \mathrm{Reg2}(d)$ $\wedge\ (\mathrm{Zero}(\mathrm{Reg0}(c)) \wedge \mathrm{State}(d) = s_j) \vee$ $(\neg \mathrm{Zero}(\mathrm{Reg0}(c)) \wedge \mathrm{State}(d) = s_k)$.
Tests on other registers	*Similar*
i : Reg0 := Reg0 + 1; *goto* j	$\mathrm{Input}(c) = \mathrm{Input}(d) \wedge \mathrm{Output}(c) = \mathrm{Ouptut}(d) \wedge$ $f_{\mathrm{Reg0}}(\mathrm{Reg0}(c), \mathrm{Reg0}(d)) \wedge$ $\mathrm{Reg1}(d) = \mathrm{Reg1}(c) \wedge \mathrm{Reg2}(d) = \mathrm{Reg2}(c) \wedge$ $\mathrm{State}(d) = s_j)$
i : Reg0 := max(Reg0 − 1, 0); *goto* j	$\mathrm{Input}(c) = \mathrm{Input}(d) \wedge \mathrm{Output}(c) = \mathrm{Output}(d) \wedge$ $(f_{\mathrm{Reg0}}(\mathrm{Reg0}(d), \mathrm{Reg0}(c)) \vee (\mathrm{Zero}(\mathrm{Reg0}(c)) \wedge$ $\mathrm{Zero}(\mathrm{Reg0}(d)))) \wedge$ $\mathrm{Reg1}(d) = \mathrm{Reg1}(c) \wedge \mathrm{Reg2}(d) = \mathrm{Reg2}(c) \wedge$ $\mathrm{State}(d) = s_j)$.

Note that, in any L-structure $\mathbf{A}$, Succ(c, d) and $f_{\mathrm{Time}}(t, s)$ implies that t, c are in the standard part of $\mathbf{A}$ iff s, d are in the standard part. Using this, by induction on the standard elements in the partial successor model for time, one can show that an L-structure satisfying γ_k codes the actual M-computation on its standard part.

Let φ_k be the Σ_2-sentence

$$\gamma_k \to (\exists c)(\exists t)[R(t, c) \wedge (\exists s)[\mathrm{Zero}_{\mathrm{state}}(s) \wedge f_{\mathrm{state}}(s, \mathrm{State}(c))] \wedge \mathrm{Zero}(\mathrm{Output}(c))]. \qquad \square$$

REMARKS.

(1) A similar proof shows that for some language L with finitely many function symbols and equality, $\Sigma_1 - Th(L)$ is h.u.

(2) Let 0-Stop be the sentence used above expressing "the computation stops with output 0". It cannot be avoided that for some k and some L-structure $\mathbf{A}$, $\gamma_k \wedge$0-Stop holds, but not $M(k) = 0$. This means that $\mathbf{A}$ describes a computation which stops at a nonstandard stage. Otherwise, for each k,

$$\neg M(k) = 0 \quad \Leftrightarrow \quad \gamma_k \to \neg \text{0-Stop} \in L\text{-Valid}.$$

Then the complement of $\{k : M(k) = 0\}$ would be r.e. and hence $\{k : M(k) = 0\}$ would be decidable.

3. How to transfer undecidability of fragments

Consider relational languages L_0 and L_1 of finite type. Intuitively, to define an L_0-structure A in an L_1-structure D, we represent the set of elements of A by a subset S of D. Then the relations in the L_0-structure A (including equality) give rise to corresponding relations on this set. It is required that the set S as well as these relations and their complements are definable in D with a collection of Σ_k-formulas and some fixed parameter list. Such a collection is called a Σ_k*-scheme* (for defining L_0-structures in L_1-structures). A class $\mathbf{C}$ of L_0-structures is Σ_k*-elementarily definable with parameters* (Σ_k*-e.d.p.*) in a class $\mathbf{D}$ of mdoels for L_1 if there is a scheme s of Σ_k-formulas such that every model $A \in \mathbf{C}$ is definable in some model $D \in \mathbf{D}$ via s.

We now give a more formal definition. We first assume that one relation symbol of L_0 is the equality symbol. A Σ_k-scheme s consists of Σ_k-formulas in L_1

$\varphi_U(x; \bar{\mathbf{p}})$, and

$\varphi_R(x_1, \ldots, x_n; \bar{\mathbf{p}})$, $\varphi_{\neg R}(x_1, \ldots, x_n; \bar{\mathbf{p}})$ for each n-ary R relation symbol in L_0

(including equality). The following correctness conditions on an L_1 structure and parameters $\bar{\mathbf{p}}$ ensure that an L_0-structure is defined:

"$\{x : \varphi_U(x; \bar{\mathbf{p}}\} \neq \emptyset$",

"*On* $\{x : \varphi_U(x; \bar{\mathbf{p}})\}$, *the relations defined by* φ_R *and* $\varphi_{\neg R}$ *are complements*" and

"$\varphi_=$ *defines an equivalence relation which is compatible with the relations defined by the formulas* φ_R *and* $\varphi_{\neg R}$"

(i.e. φ_R and $\varphi_{\neg R}$ depend only on equivalence classes). These conditions can be expressed as a universally quantified Boolean combination of Σ_k-formulas, and therefore by a Π_{k+1}-formula $\alpha(\bar{\mathbf{p}})$. We say that $\mathbf{C}$ is Σ_k-e.d.p. in a class $\mathbf{D}$ if for some Σ_k-scheme s, the following holds: for each $A \in \mathbf{C}$ there is $D \in \mathbf{D}$ and a list of parameters $\bar{\mathbf{d}}$ in D such that $D \models \alpha(\bar{\mathbf{d}})$, and, if $S = \{x : D \models \varphi_U(x; \bar{\mathbf{d}})\}$ and, for an n-ary relation symbol R,

$$\tilde{R} = S^n \cap \{x_1, \ldots, x_n : D \models \varphi_R(x_1, \ldots, x_n; \bar{\mathbf{d}}\}, \quad \text{then}$$

$$A \cong (S, (\tilde{R})_{R \text{ relation symbol of } L_0}) / \{x, y : D \models \varphi_=(x, y; \bar{\mathbf{d}})\}. \tag{3}$$

If s is a scheme without parameters, $\mathbf{C}$ is called Σ_k-*elementarily definable* in $\mathbf{D}$. For a simple example, see Proposition 4.1. The idea to define the complements of relations separately goes back to [Ler83].

TRANSFER LEMMA 3.1. *Let* $r \geq 2$ *and* $k \geq 1$, *and suppose that the language of* $\mathbf{C}$ *contains equality.*

(i) *If* $\mathbf{C}$ *is* Σ_k-*elementarily definable in* $\mathbf{D}$, *then*

$$\Sigma_r - Th(\mathbf{C}) \ h.u. \quad \Rightarrow \quad \Sigma_{r+k-1} - Th(\mathbf{D}) h.u.$$

(ii) *If* $\mathbf{C}$ *is* Σ_k-*elementarily definable with parameters in* $\mathbf{D}$, *then*

$$\Pi_{r+1} - Th(\mathbf{C}) \ h.u. \quad \Rightarrow \quad \Pi_{r+k} - Th(\mathbf{D}) \ h.u.$$

Proof. The idea is to define an effective map F from L_0-sentences to L_1-sentences and apply the recursion theoretic fact (2). Given an L_0-sentence φ in normal form, the translation $\tilde{\varphi}(\bar{\mathbf{p}})$ ($\tilde{\varphi}$ if no parameters are used in the scheme) is obtained by relativizing the quantifiers to $\{x : \varphi_U(x; \bar{\mathbf{p}})\}$ and replacing atomic formulas $R(x_1, \ldots, x_n)$ and $\neg R(x_1, \ldots, x_n)$ by $\varphi_R(x_1, \ldots, x_n : \bar{\mathbf{p}})$ and $\varphi_{\neg R}(x_1, \ldots, x_n; \bar{\mathbf{p}})$ in a way to get a minimum number of alternations of quantifiers: if the innermost quantifier in φ is existential, replace R by φ_R and $\neg R$ by $\varphi_{\neg R}$. Otherwise, replace

R by the Π_k-formula $\neg\varphi_{\neg R}$ and replace $\neg R$ by the Π_k-formula $\neg\varphi_R$. For instance, if φ is

$$(\exists x)(\forall y)[Rxy \vee \neg Ryx], \quad \text{then } \tilde{\varphi}(\bar{\mathbf{p}}) \text{ is}$$

$$(\exists x)[\varphi_U(x;\bar{\mathbf{p}}) \wedge (\forall y)[\varphi_U(y;\bar{\mathbf{p}}) \rightarrow (\neg\varphi_{\neg R}(x,y,\bar{\mathbf{p}}) \vee \neg\varphi_R(y,x,\bar{\mathbf{p}})]].$$

Note that the translation of a Σ_r-sentence is a Σ_{r+k-1}-formula and the translation of a Π_{r+1}-sentence is a Π_{r+k}-formula.

Let $F(\varphi) = (\forall\bar{\mathbf{p}})[\alpha(\bar{\mathbf{p}}) \rightarrow \tilde{\varphi}(\bar{\mathbf{p}})]$ (and $F(\varphi) = \alpha \rightarrow \tilde{\varphi}$ if there are no parameters). Clearly,

$$\varphi \in L_0\text{-Valid} \quad \Rightarrow \quad F(\varphi) \in L_1\text{-Valid}. \quad \text{Moreover}$$

$$\varphi \notin Th(\mathbf{C}) \quad \Rightarrow \quad F(\varphi) \notin Th(\mathbf{D}), \tag{4}$$

since, if φ fails in some structure $A \in \mathbf{C}$, then $F(\varphi)$ fails in some $D \in \mathbf{D}$ coding A. The counterexample for $(\forall\bar{\mathbf{p}})[\cdots]$ is provided by the list of parameters used for the coding.

For the proof of (i), note that, if φ is Σ_r, then $F(\varphi)$ is in an obvious way equivalent to a Σ_{r+k-1}-sentence, since $r+k-1 \geq k+1$ and α is a Π_{k+1}-formula. Then, an application of (2) to the recursively inseparable set $\Sigma_r \cap L_0$-Valid and $\{\varphi : \varphi \notin \Sigma_r\text{-}Th(\mathbf{C})\}$ yields the desired result. For (ii) we argue similarly, using the fact that $\varphi \in \Pi_{r+1}$ implies that $F(\varphi)$ is equivalent to a Π_{r+k}-sentence. □

The equivalent of the Transfer Lemma also holds if the language of **C** does not contain equality. In this case we need a somewhat different notion of Σ_k-e.d.p. Let $\varphi_{eq}(x, y)$ be the formula expressing that x and y behave in the same way w.r.t. all elements in the structure, i.e. $\varphi_{eq}(x, y)$ is the conjunction of formulas of the kind

$$(\forall z)[(Rxz \Leftrightarrow Ryz) \wedge (Rzx \Leftrightarrow Rzy)]$$

for each relation symbol R of L_0. Given an L_0-structure A, let $eq(A)$ be $\{\langle x, y\rangle : A \models \varphi_{eq}(x, y)\}$ and let $A/eq(A)$ be the structure defined on equivalence classes in the obvious way. By induction on the number of quantifiers, it is easy to verify that

$$A \models \phi \quad \Leftrightarrow \quad A/eq(A) \models \phi \qquad \text{for each } L_0\text{-formula } \phi. \tag{5}$$

In defining the notion of Σ_k-e.d.p., we omit the part of α concerned with equality and replace (3) by

$$A/eq(A) \cong D/eq(D), \qquad \text{where}$$

$$D = (S, \tilde{R})_{R \text{ relation symbol of } L_0}.$$

Then (5) implies that (4) still holds. This proves the following.

TRANSFER LEMMA WITHOUT EQUALITY 3.2. *The equivalent of the Transfer Lemma holds if the language of* **C** *does not contain equality.* □

4. Applications to classes of algebraic structures

In this section we apply the method developed in the previous section to give undecidability results for fragments of elementary theories given by classes of algebraic structures. The main applications are concerned with classes of finite lattices. Recall that we view lattices as partial orders.

PROPOSITION 4.1. *F-POrders is Σ_2-e.d. in F-DistrLattices.*

Proof. By [Gr78], for any $\mathbf{P} \in F\text{-}P\text{Orders}$, there exists a finite distributive lattice L with least element 0 such that $\mathbf{P} \cong (\{x \in L - \{0\} : x \text{ is join irreducible}\}, \leq\}$. Since we can take $\varphi_{\leq}(x, y) \equiv x \leq y$ and $\varphi_{\not\leq}(x, y) \equiv x \not\leq y$, it suffices to define the nonzero join irreducible elements in any finite (distributive) lattice by a Σ_2-formula in the language of partial orders: if $x \neq 0$, then

$$x \text{ is join irreducible} \iff (\forall u)(\forall v)[x = u \vee v \rightarrow x = u \ \vee \ x = v]$$

$$\iff \bigvee \{y : y < x\} \neq x \iff (\exists z)(\forall y)[y \leq z \leftrightarrow y < x]. \qquad \square$$

We now consider undirected graphs (in the language without equality). The following Theorem apparently follows from [Tra53]. In [Ler83] a similar Theorem for directed graphs with equality was proved. Definability in directed graphs without equality was recently considered in [Pa93].

THEOREM 4.2. *$\Sigma_2 - Th$(F-SymGraphs) is h.u.*

Proof. Using some ideas from [Ler83], we show that $F\text{-}L$Structures is Σ_1-e.d. in *F-SymGraphs*, for any finite relational language L. Then, Theorem 2.1 and (i) of the Transfer Lemma imply the result.

Suppose the non-equality relation symbols of L are $R_1, \ldots, R_m$. We write $R_0(x, y)$ for $x = y$. Given an L-structure $\mathbf{A}$, if $0 \le n \le m$, we denote the complement of the relation $R_n^{\mathbf{A}}$ by $R_{n+m+1}^{\mathbf{A}}$.

For each $k \ge 3$, let $\mathrm{Cyc}_k(x)$ express that x is an element of a $2k$-cycle:

$$\mathrm{Cyc}_k(x) \equiv (\exists u_1) \cdots (\exists u_{2k-1})$$
$$[xu_1 \wedge u_1u_2 \wedge \cdots \wedge u_{2k-1}x \wedge$$
$$\text{“}\textit{no other edge relations hold on } \{x, u_1, \ldots, u_{2k-1}\}\text{”}].$$

(We write xy instead of Exy.) Note that all the vertices $x, u_1, \ldots, u_{2k-1}$ must be unequal, otherwise some further edge relation would hold. The restriction to cycles of even cardinality is necessary for later extensions. The formulas of this kind make it possible to define sets in an (undirected) graph existentially as

$$\{x : \mathrm{Cyc}_k(x) \wedge \mathrm{Cyc}_h(x)\},$$

for some fixed numbers $k \ne h$: to distinguish a set of elements $\{v_1, \ldots, v_m\}$ in a graph without $2k$-cycles or $2h$-cycles, add new vertices and edges so that $v_1, \ldots, v_m$ are the unique elements in the intersection of a $2k$-cycle and a $2h$-cycle. We write $\mathrm{Cyc}_{k,h}(x)$ for $\mathrm{Cyc}_k(x) \wedge \mathrm{Cyc}_h(x)$.

A graph $G_{\mathbf{A}}$ coding an L-structure $\mathbf{A}$ is constructed as follows (see Fig. 1):

– define a set $U_{\mathbf{A}}$ corresponding to the universe of $\mathbf{A}$ by

$$\varphi_U(x) \equiv \mathrm{Cyc}_{3,4}(x)$$

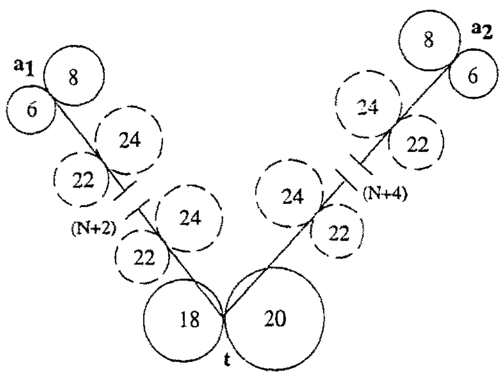

Figure 1. Coding the fact that $\langle a_1, a_2 \rangle \in R_1^{\mathbf{A}}$.

for each n, $0 \le n \le 2m+1$, add a set of elements T_n and appropriate cycles so that T_n is defined by the formula $\mathrm{Cyc}_{4n+5,4n+6}(x)$. The elements of T_n represent the tuples in $R_n^{\mathbf{A}}$.

- for each $t \in T_n$ representing a tuple $(a_1, \ldots, a_k) \in R_n^{\mathbf{A}}$ and for each $i \le k$, add a chain of length $N + 2i$ from t to x_i, where N is some sufficiently large fixed number depending only on the language L. To be able to distinguish between elements of chains for different relations $R_n^{\mathbf{A}}$, we attach cycles of cardinality $4n+7$ and $4n+8$ to each element of a chain for $R_n^{\mathbf{A}}$ which is not an endpoint. We choose $N = 4 \cdot (2m+1) + 8$ as the largest cardinality of cycles used to define a set $U_{\mathbf{A}}$ or T_n, or to mark elements of a chain. In this way, we achieve that by coding several relations, no new cycles are introduced which have so small cardinality that they might interfere with the definitions of the sets T_n or of $U_{\mathbf{A}}$ or the marking.

For each n, $0 \le n \le 2m+1$, if $R_n^{\mathbf{A}}$ is k-ary, the following Σ_1-formula defines $R_n^{\mathbf{A}}$ in G_A:

$$\begin{aligned}\varphi_n(x_1, \ldots, x_k) \equiv {} & (\textstyle\bigwedge_{1 \le i < k} \mathrm{Cyc}_{3,4}(x_i)) \wedge (\exists t)[\mathrm{Cyc}_{4n+5,4n+6}(t) \wedge \\ & \textstyle\bigwedge_{1 \le i \le k} (\exists y_1) \cdots (\exists y_{N+2i-1}) \\ & [\textstyle\bigwedge_{1 \le j \le N+2i-1} \mathrm{Cyc}_{4n+7,4n+8}(y_j) \wedge \\ & t y_1 \wedge \cdots \wedge y_{N+2i-1} x_i \wedge \text{"}\textit{no other relations} \\ & \textit{hold on } \{t, y_1, \ldots, y_{N+2i-1}, x_i\}\text{"}].\end{aligned}$$

We complete the Σ_1-scheme by setting, for $0 \le n \le m$,

$$\varphi_{R_n}(x_1, \ldots, x_k) \equiv \varphi_n(x_1, \ldots, x_k) \quad \text{and}$$

$$\varphi_{\neg R_n}(x_1, \ldots, x_k) \equiv \varphi_{n+m+1}(x_1, \ldots, x_k).$$

Clearly, in $G_{\mathbf{A}}$ the formula φ_{R_0} gives the identity on $U_{\mathbf{A}}$. Hence the structure defined in $G_{\mathbf{A}}$ by this scheme is isomorphic to $\mathbf{A}$. □

THEOREM 4.3 (J. Schmerl). *$\Sigma_2 - Th(F\text{-}Lattices)$ and $\Sigma_2 - Th(F\text{-}POrders)$ is h.u.*

Proof. Let F-SymGraphs$^{\ge 3}$ be the class of finite undirected graphs (V, E) such that $|V| \ge 3$. The Proof of Theorem 2.1 shows that F-LStructures is Σ_1-e.d. in F-SymGraphs$^{\ge 3}$. By Appendix A in [Ler83], the class F-SymGraphs$^{\ge 3}$ is Σ_1-e.d. in F-Lattices and therefore also in F-POrders. □

COROLLARY 4.4. *$\Sigma_3 - Th$(F-DistrLattices) is h.u.*

Proof. By Proposition 4.1 and Theorem 4.3, applying (i) of the Transfer Lemma. □

We next show that $\Pi_3 - Th(F\text{-DistrLattices})$ is also h.u., a result which will have an application in computability theory. We introduce the auxiliary class of bipartite graphs. A bipartite graph is a structure for the language $L(Le, Ri, E)$ where Le, Ri are unary and E is a binary predicate symbol, which satifies the axioms

$$(\forall x)[(Le\,x \leftrightarrow \neg Ri\,x)] \quad \text{and}$$

$$(\forall x)(\forall y)[E\,xy \rightarrow (Le\,x \wedge Ri\,y)].$$

The predicates Le and Ri denote the left and the right domain of the graph. Given a bipartite graph G, we write e instead of Le^G etc. Let BiGraphs denote the class of bipartite graphs.

COROLLARY 4.5. *$\Sigma_2 - Th$(F-BiGraphs) is h.u.*

Proof. It suffices to build the graph $G_{\mathbf{A}}$ in the proof of Theorem 4.2 as a bipartite graph. The only change is that we construct U_A and the sets T_n as subsets of the left domain of $G_{\mathbf{A}}$. This causes no problems since all the cycles we use have even cardinality and all the chains have even length. □

The graph $G(U, V)$ associated with equivalence relations U, V on a set F was introduced in [Ore42] (also see [Gr78], p. 200). This graph has the disjoint union of F/U and F/V as vertex set, F/U and F/V as left and right domain, respectively, and the set of edges is

$$\{\langle C, D\rangle : C \text{ is } U\text{-class} \wedge D \text{ is } V\text{-class} \wedge C \cap D \neq \emptyset\}.$$

LEMMA 4.6. *Let $G = (Le, Ri, E)$ be a finite bipartite graph without isolated points. Then there exist a finite set F and equivalence relations U, V on F such that $G \cong G(U, V)$ via an isomorphism that maps Le to F/U and Ri to F/V. Moreover, given a number $N \geq 1$, it can be achieved that all the U- and the V-classes have cardinality $\geq N$.*

Proof. First let $F = E$ and let U, V be the equivalence relations on F induced by the projections $F \rightarrow Le$ and $F \rightarrow Ri$, i.e,

$$\langle x, y\rangle U \langle x', y'\rangle \quad \Leftrightarrow \quad x = x' \quad \text{and}$$

$$\langle x, y\rangle V \langle x', y'\rangle \quad \Leftrightarrow \quad y = y'.$$

Since G has no isolated points, the projections are onto. Hence

$$\alpha(x) = \{\langle x, y\rangle : \langle x, y\rangle \in E\} \quad \text{and} \quad \beta(y) = \{\langle x, y\rangle : \langle x, y\rangle \in E\}$$

give bijections $Le \to F/U$ and $Ri \to F/V$. Clearly

$$\langle x, y\rangle \in E \quad \Leftrightarrow \quad \alpha(x) \cap \beta(y) \neq \emptyset.$$

To obtain equivalence classes of cardinality $\geq N$, note that, by expanding F, it is possible to add arbitrarily many new elements to a nonempty intersection of an U-class with a V-class. Doing this successively for all the nonempty intersections, we obtain the desired result. □

For the proofs of the following two Theorems, we work with a subclass of BiGraphs. Let BiGraphs* denote the class of bipartite graphs $G = (Le, Ri, E)$ such that $|Le| \geq 3$, $|Ri| \geq 3$, each vertex of G is connected to at least two other vertices and the complement graph $(Le, Ri, Le \times Ri - E)$ has no isolated vertices. Note that the graph $G_{\mathbf{A}}$ in the proof of Corollary 4.5 is actually in F-BiGraphs*, since each vertex of $G_{\mathbf{A}}$ is in some cycle of cardinality ≥ 6. This implies the following

COROLLARY 4.7. *$\Sigma_2 - Th(F\text{-}BiGraphs^*)$ is h.u.* □

THEOREM 4.8. *$\Pi_3 - Th(F\text{-}DistrLattices)$ is h.u.*

Proof. We show that F-BiGraphs* is Σ_1-e.d.p. in F-DistrLattices. Since $\Pi_3 - Th(F\text{-BiGraphs*})$ is h.u. by Corollary 4.7, the Theorem follows by an application of (ii) of the Transfer Lemma.

(1) As a first approximation, we give a *quantifier free* formula $\varphi(Y, P, Q)$ with the property that, for each $n \geq 2$ there exists a finite distributive lattice L and some P, $Q \in L$ such that φ is satisfied by exactly n incomparable elements of L. A modification of φ will lead to the formula φ_U. Let

$$\varphi(Y, P, Q) \equiv P \nleq Y \wedge Y \nleq Q.$$

In this first approximation actually $P = Q$, but for the full proof it will be the case that $P < Q$. Given $n \geq 2$, we determine finite sets A_i $(1 \leq i \leq n)$ and $P = Q$, and let L be the distributive lattice generated by these sets under union and intersection. It will be the case that, in L, $A_1, \ldots, A_n$ are precisely those Y satisfying the formula $\varphi(Y; P, Q)$. The finite sets generating L will be subsets of a disjoint union D of two copies of $\{1, \ldots, n\}$, denoted by $\{1_I, \ldots, n_I\}$ and $\{1_{II}, \ldots, n_{II}\}$. If $S \subseteq \{1, \ldots, n\}$, S_I denotes the corresponding subset of $\{1_I, \ldots, n_I\}$ etc. Let $N = \{1, \ldots, n\}$, and let

$$A_i = (N_I - \{i_I\} \cup \{i_{II}\}$$
$$P = Q = N_I.$$

The distributive lattice generated by these sets consists of two copies of the 2^n-element Boolean algebra, one on top of the other, and for each i, $1 \leq i \leq n$, an additional element A_i inserted between the i-th coatom of the lower and the i-th atom of the upper Boolean algebra (see Fig. 2).

Since $i_I \in P - A_i$ and $i_{II} \in A_i - Q$, $L \models \varphi(A_i; P, Q)$. Now suppose that $L \models \varphi(Y; P, Q)$. We show that, for some k, h, $A_k \leq Y$ and $Y \leq A_h$. Since A_k and A_h are incomparable for $k \neq h$, this implies that $Y \in \{A_1, \ldots, A_n\}$.

Note that for each sequence of elements $(X_i)_{1 \leq i \leq k}$ such that $X_i \in \{A_1, \ldots, A_n, P\}$ and $X_i \neq X_j$ for $i \neq j$, the following holds:

$$\bigcap\nolimits_{1 \leq i \leq k} X_i \nleq Q \quad \Rightarrow \quad k = 1 \quad \text{and} \quad P \nleq \bigcup\nolimits_{1 \leq i \leq k} X_i \quad \Rightarrow \quad k = 1. \tag{6}$$

Now suppose $P \nleq Y \wedge Y \nleq Q$. To show that $(\exists k)$ $[A_k \leq Y]$, let $Y = \bigcup_i \bigcap_{1 \leq i \leq n_i} R_{i,j}$, where the sets $R_{i,j}$ are among $\{A_1, \ldots, A_n, P\}$ and $R_{i,j} \neq R_{j,j'}$ for $j \neq j'$. Since $Y \nleq Q$, there is an $\tilde{i}$ such that

$$\bigcap\nolimits_{1 \leq j \leq n_{\tilde{i}}} R_{\tilde{i},j} \nleq Q. \tag{7}$$

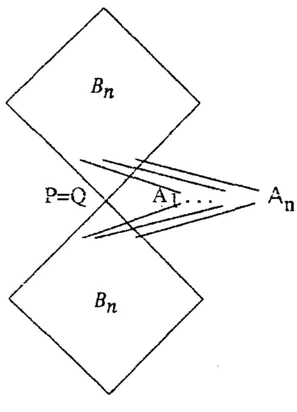

Figure 2. The lattice L.

Therefore $n_{\tilde{i}}=1$ and $R_{\tilde{i},1}\neq Q$. Hence $R_{\tilde{i},1}=A_k$ for some k and $A_k\leq Y$. The argument for $(\exists h)\ [Y\leq A_h]$ is similar: let $Y=\bigcap_i \bigcup_{1\leq j\leq n_i} R_{i,j}$, where again the sets $R_{i,j}$ are among the generating sets and $R_{i,j}\neq R_{i,j'}$ for $j\neq j'$. Since $P\not\leq Y$, for some $\tilde{i}$, h, $n_{\tilde{i}}=1$ and $R_{\tilde{i},1}=A_h$, so $Y\leq A_h$.

(2) Now suppose a graph $G=(Le, Ri, E)$ in FBiGraphs* is given. W.l.o.g. assume that $Le=\{1,\dots,n\}$, $Ri=\{1',\dots,n'\}$. Let the variables i, j, k range over $\{1,\dots,n\}$ and let i', j', k' range over $\{1',\dots,n'\}$. Lemma 4.6, applied to (Le, Ri, E) and $(Le, Ri, \bar{E})$ (where $\bar{E}=Le\times Ri-E$) gives equivalence relations U_E, V_E on a set F_1 and $U_{\bar{E}}$, $V_{\bar{E}}$ on a set F_2 such that $(Le, Ri, E)\cong G(U_E, V_E)$ and $(Le, Ri, \bar{E})\cong G(U_{\bar{E}}, V_{\bar{E}})$. We can assume that F_1, F_2 are disjoint. Under the canonical isomorphism, $i\in Le$ corresponds to a U_E-class C_i and a $U_{\bar{E}}$-class D_i. Similarly, $j'\in Ri$ corresponds to a V_E-class $C_{j'}$ and a $V_{\bar{E}}$-class $D_{j'}$. The distributive lattice L coding G will consist of certain subsets of a disjoint union of finite sets $D\cup D'\cup F_1\cup F_2$. Here D is as in (1), $N'=\{1',\dots,n'\}$ and D' is a disjoint union of two copies of N'. We refer to the sets D, D', F_1 and F_2 as *regions*. (The regions are not elements of L.) The sets generating L are given in Table 1 by their intersections with the regions. We let $\bar{i}=N-\{i\}$ $(1\leq i\leq n)$ and $\bar{j'}=N'-\{j'\}$ $(1'\leq j'\leq n')$.

The left domain of G is represented by the sets $A_1,\dots,A_n$, which are defined from the parameters P, Q as in (1), using the region D. Similarly, the right domain of G is represented by $A_{1'},\dots,A_{n'}$, using the parameters P', Q' and the region D'. To code the edge relation, we use the region F_1: we let

$$A_i\cap F_1=F_1-C_i,$$

$$A_{j'}\cap F_1=F_1-C_{j'}.$$

Table 1

	D		D'			
	I	II	I	II	F_1	F_2
A_i	$\bar{i}$	i	N'	$\varnothing$	F_1-C_i	F_2-D_i
$A_{j'}$	N	$\varnothing$	$\bar{j'}$	j'	$F_1-C_{j'}$	$F_2-D_{j'}$
P	N	$\varnothing$	$\varnothing$		$\varnothing$	$\varnothing$
Q	N	$\varnothing$	D'		F_1	F_2
P'	$\varnothing$		N'	$\varnothing$	$\varnothing$	$\varnothing$
Q'	D		N'	$\varnothing$	F_1	F_2
C_E	N	$\varnothing$	N'	$\varnothing$	F_1	$\varnothing$
$C_{\bar{E}}$	N	$\varnothing$	N'	$\varnothing$	$\varnothing$	F_2

Then

$$\begin{aligned}\langle i,j'\rangle \in E &\Leftrightarrow C_i \cap C_{j'} \neq \emptyset \\ &\Leftrightarrow (F_1 - C_i) \cup (F_1 - C_{j'}) \neq F_1 \\ &\Leftrightarrow F_1 \nsubseteq A_i \cup A_{j'}. \end{aligned} \tag{8}$$

To code $\bar{E}$, we use the region F_2 in a similar way. The actual parameters used to recover E and $\bar{E}$ are sets $C_E \supseteq F_1$ and $C_{\bar{E}} \supseteq F_2$ which are elements of L. These sets are defined in a way that (8) remains true, i.e.,

$$\begin{aligned}\langle i,j\rangle &\Leftrightarrow C_E \nsubseteq A_i \cup A_{j'}. \\ &\Leftrightarrow (\exists Z)[A_i, A_{j'} \subseteq Z \wedge C_E \nsubseteq Z]\end{aligned}$$

and similarly for $\bar{E}$. Thus, the relations E and $\bar{E}$ can be defined in L by a Σ_1-formula. However, C_E and $C_{\bar{E}}$ must both contain N_I and be disjoint from N_{II} in order to ensure an analog of (6). In a similar way we define $A_{1'}, \ldots, A_{n'}$ on the region D. Also, to maintain the first-order definition of $\{A_1, \ldots, A_n\}$ while extending these sets to the regions F_1 and F_2, we must P small and Q big on these regions. Thus P has empty intersection with the regions and Q contains them. Moreover we ensure that $P \leq A_{j'} \leq Q$ for each j'.

We verify that the formulas

$$\begin{aligned}&\varphi_{Le}(Y, P, Q) \equiv \varphi(Y, P, Q), \\ &\varphi_{Ri}(Y, P', Q') \equiv \varphi(Y, P', Q'), \\ &\varphi_U(Y, P, Q, P', Q') \equiv \varphi_{Le}(Y, P, Q) \vee \varphi_{Ri}(Y, P', Q') \\ &\varphi_E(Y, Y', C_E) = (\exists Z)[Y \leq Z \wedge Y' \leq Z \wedge C_E \nleq Z] \quad \text{and} \\ &\varphi_{\bar{E}}(Y, Y', C_{\bar{E}}) = \varphi_E(Y, Y', C_{\bar{E}})\end{aligned}$$

give a Σ_1-scheme for defining F-BiGraphs* in F-DistrLattices. It $G \in F$-BiGraphs*, construct L as above. First we have to check that

$$L \models \varphi_{Le}(Y, P, Q) \quad \Leftrightarrow \quad Y \in \{A_1, \ldots, A_n\}.$$

We proceed as in (1), keeping track of the effect of the additional generating elements. Note that Q is above each additional generating element except for Q'. To obtain an analog of (6), if $(X_i)_{1 \leq i \leq k}$ is a sequence of generating elements such that $X_i \neq X_j$ for $i \neq j$, then $\bigcap_{1 \leq i \leq k} X_i \nleq Q$ implies that the expression $\bigcap_{1 \leq i \leq k} X_i$ is A_k or $A_k \cap Q'$ for some k, $1 \leq k \leq n$. But $A_k \cap Q' = A_k$. Thus, in (7), if $\bigcap_{1 \leq j \leq n_{\bar{i}}} R_{\bar{i},j} \nleq Q$, as before this implies $A_k \subseteq Y$ for some k.

To show $(\exists h)[Y \subseteq A_h]$, we use a similar modification. P is below each additional generating element except P'. Then $P \nleq \bigcup_{1 \leq i \leq k} X_i$ implies that the expression $\bigcup_{1 \leq i \leq k} X_i$ is A_h or $A_h \cup P'$ for some k. Since $A_h \cup P' = A_h$, we can now argue as before.

A similar argument shows that $L \models \varphi_{Ri}(Y, P', Q') \Leftrightarrow Y \in \{A_{1'}, \ldots, A_{n'}\}$. Finally, we ensured that, for each i, j', $C_E \cap (D \cup D') \subseteq A_i \cup A_{j'}$. Then, by (8),

$$\varphi_E(A_i, A_{j'}, C_E) \quad \Leftrightarrow \quad C_E \nsubseteq A_i \cup A_{j'} \quad \Leftrightarrow \quad C_E \cap F_1 \nsubseteq (A_i \cup A_{j'}) \cap F_1 \quad \Leftrightarrow \quad \langle i, j' \rangle \in E.$$

Hence, φ_E defines the edge relation correctly, and, by the same argument, $\varphi_{\bar{E}}$ defines the complement of the edge relation correctly. □

If $k \geq 1$, let $\mathbf{P}_k$ be the lattice of equivalence relations on $\{1, \ldots, k\}$ (we also use the term "partition" and the notion for partitions). In [Bu, Sa75] it is shown that $Th(\{\mathbf{P}_k : k \geq 1\})$ is h.u. In the following theorem, using an efficient coding we obtain undecidability of $\Pi_4 - Th(\{\mathbf{P}_k : k \geq 1\})$.

THEOREM 4.9. *$\Pi_4 - Th(\{\mathbf{P}_k : k \geq 1\})$ is h.u.*

Proof. We show that F-BiGraphs* is Σ_2-e.d.p. in $\{\mathbf{P}_k : k \geq 1\}$. Given a bipartite graph $G = (Le, Ri, E)$ in F-BiGraphs*, let U, V be the equivalence relations on a set F (w.l.o.g. $F = \{1, \ldots, k\}$) given by Lemma 4.6 such that $G \cong G(U, V)$. We define G in $\mathbf{P}_k$, using the parameters U, V and additional parameters $\tilde{U}$, $\tilde{V}$. As before, suppose that $Le = \{1, \ldots, n\}$ and $Ri = \{1', \ldots, n'\}$. Let $C_i (D_{i'})$ be the U-class (V-class) corresponding to i (i'). We verify that, by the definition of the class F-BiGraphs*, if C is a U-class and D is a V-class, then

$$C \cap \bar{D}, \qquad \bar{C} \cap D, \qquad \bar{C} \cap \bar{D} \neq \varnothing. \tag{9}$$

Suppose $C = C_i$ and $D = D_{j'}$ ($1 \leq i \leq n$, $1' \leq j' \leq n'$). Since the vertex i is connected to a vertex other than j', $C \cap \bar{D} \neq \varnothing$. Similarly $\bar{C} \cap D \neq \varnothing$. Since there is an edge between vertices other than i and j', $\bar{C} \cap \bar{D} \neq \varnothing$.

Note that coatoms in $\mathbf{P}_k$ are partitions of the form $\{X, \bar{X}\}$, $\varnothing \subset X \subset \{1, \ldots, k\}$. We represent the set Le by the set of coatoms of the form $\{C_i, \bar{C}_i\}$, and Ri by the coatoms $\{D_{i'}, \bar{D}_{i'}\}$. Since $|Le| \geq 3$ and $|Ri| \geq 3$, the sets Le and Ri are in 1-1 correspondence with the appropriate sets of coatoms. Let the variables H, K range over coatoms and let

$$H_i = \{C_i, \bar{C}_i\} \quad \text{and} \quad H_{i'} = \{D_{i'}, \bar{D}_{i'}\}.$$

We say that H, K are *compatible* if $H \cap K$ has at most 3 equivalence classes, and *incompatible* else, i.e. if $H \cap K$ has 4 equivalence classes. By (9),

$$\langle i, i' \rangle \in E \quad \Leftrightarrow \quad H_i, H_{i'} \quad \text{are incompatible.}$$

We obtain the desired definability result by giving Σ_2-definitions with parameters of the sets of coatoms

$$\{H_i : 1 \leq i \leq n\} \quad \text{and} \quad \{H_{i'} : 1' \leq i' \leq n'\}, \tag{10}$$

as well as of

$$\{\langle H, K \rangle : H, K \text{ incompatible}\} \quad \text{and} \tag{11a}$$

$$\{\langle H, K \rangle : H, K \text{ compatible}\}. \tag{11b}$$

First note that, in the absence of a constant symbol for the greatest element of $\mathbf{P}_k$, to formulate "X is a coatom" one needs $\Sigma_1 \wedge \Pi_1$. Then the formula

$$\text{Closed}(P, H) \equiv P \leq H \wedge \text{“}H \text{ is a coatom”}$$

is $\Sigma_1 \wedge \Pi_1$ as well. Clearly, $\mathbf{P}_k \models \text{Closed}(P, H)$ iff $H = \{X, \bar{X}\}$ for some nontrivial set X which is the union of P-classes. To give Σ_2-definitions for the sets (11a) and (11b), note that H, K are incompatible iff there are $7 = 4 + 1/2\binom{4}{2}$ distinct coatoms above $H \cap K$. In the language of p.o., this statement is equivalent to

$$(\exists H_1) \cdots (\exists H_7)[|\{H_1, \ldots, H_7\}| = 7 \wedge (\forall R)[R \leq H, K \rightarrow \textstyle\bigwedge_{i=1\cdots 7} R \leq H_i]].$$

Moreover, H, K are compatible iff the interval $[H \cap K, H]$ contains at most two elements, i.e.

$$(\exists I)(\forall R)[(R \leq I \leftrightarrow R \leq H, K) \wedge (I \leq R \leq H \rightarrow I = R \vee H = R)].$$

For defining the sets in (10) with parameters, we need the following combinatorical Lemma.

LEMMA. *Suppose that* $\{X_1, \ldots, X_r\}$ *is a partition of a set* E *such that* $|X_i| \geq r$ *for each* i. *Then there is a partition* $\{Y_1, \ldots, Y_{r+1}\}$ *of* E *such that* $X_i \cap Y_j = \emptyset \Leftrightarrow i = j$.

Proof. Inductively, for j, $1 \leq j \leq r$, put into Y_j exactly one new element of each set X_i, $i \neq j$. Put the remaining elements of E into Y_{r+1}. Since $|X_i| \geq r$, $Y_{r+1} \cap X_i \neq \emptyset$ for each i. □

By Lemma 4.6, we can assume that the U- and the V-classes are sufficiently large. Let $\tilde{U}$ and $\tilde{V}$ be the equivalence relations on F obtained by an application of the preceding Lemma to U and V. We claim that

$$H = \{X, \bar{X}\} \text{ for some } U\text{-class } X \quad \Leftrightarrow$$

$$\text{Closed}(U, H) \wedge (\exists K)[\text{Closed}(\tilde{U}, K) \wedge H, K \text{ compatible}]. \tag{12}$$

The direction from left to right is obvious. For the other direction, suppose that Closed(U, H), $H = [Z, \bar{Z}\}$, but neither Z nor $\bar{Z}$ is a U-class. Then X_1, $X_2 \subseteq Z$, X_3, $X_4 \subseteq \bar{Z}$ for distinct U-classes $X_1, \ldots, X_4$ and, by the definition of $\tilde{U}$, each $\tilde{U}$-class meets Z and $\bar{Z}$. Hence for each K that Closed$(\tilde{U}, K)$, $H \wedge K$ has 4 equivalence classes, and (12) is violated.

By the Σ_2-definability of (11b), (12) can be expressed by a Σ_2-formula. □

REMARK. The same proof shows the Π_4-theory of each infinite subclass of $\{\mathbf{P}_k : k \geq 1\}$ is h.u.

5. Applications to structures arising in recursion theory

We use the results of the previous section to obtain undecidability of fragments for the theories of degree structures which arise in recursion theory. Studies of this kind were initiated by M. Lerman and J. Schmerl. They showed that the Π_3-theory of the p.o. $\mathbf{D}$ of all Turing degrees is undecidable. Lerman also gives a decision procedure for the Π_2-theory of $\mathbf{D}$, thereby obtaining a sharp classification in the sense of the introduction. By [Jo, Sl93], even $\Pi_2 - Th_\vee(\mathbf{D})$, the fragment of the theory in the language of upper semilattice, is decidable. We will consider similar problems for r.e. degree structures. First we review some basic concepts.

A reducibility gives a method to compare sets of natural numbers w.r.t. their computational complexity. Turing-reducibility is the most general one considered here: for sets of natural numbers X, Y, $X \leq_T Y$ if some oracle Turing machine computes X with oracle Y. The finest reducibility we consider here is m-reducibility: $X \leq_m Y$ if $z \in X \Leftrightarrow f(z) \in Y$ for some recursive function f. In between them are bounded truth table (btt), truth-table (tt) and weak truth-table (wtt) reducibility.

Given a reducibility $\leq_r$, write $Y \equiv_r X$ if $X \leq_r Y \wedge Y \leq_r X$. The *r-degree* $\deg_r(X)$ of a set X is the equivalence class $\{Y : Y \equiv_r X\}$, and $\leq_r$ induces a partial order on the r-degrees. This p.o., denoted by $\mathbf{D}_r$, forms an upper semilattice, since

$$\sup(\deg_r(X), \deg_r(Y)) = \deg_r(X \oplus Y),$$

and possesses a least element $\mathbf{0} = \deg_r(\{0\})$.

$\mathbf{R}_r$ is the p.o. of r-degrees of r.e. sets. This p.o. is a subsemilattice of $\mathbf{D}_r$ and possesses also a greatest element. (Here we ignore the degrees of $\emptyset$ and ω in the case of m-degrees.) For more information on reducibilities, see Ch. 3 of [Od89].

In [De79], A. Degtev proves that $\Pi_2 - Th_\vee(\mathbf{D}_m)$ and $\Pi_2 - Th_\vee(\mathbf{R}_m)$ is decidable. We show that the theory of these degree structures in the language of p.o. becomes undecidable at the next level, thereby answering a question in [Od84] about $\mathbf{D}_m$.

THEOREM 5.1. *The fragments $\Pi_3 - Th(\mathbf{D}_m)$ and $\Pi_3 - Th(\mathbf{R}_m)$ are h.u.*

Proof. F-DistrLattices is Σ_1-e.d.p. in $\mathbf{D}_m$ and $\mathbf{R}_m$ by the results of Lachlan in [La70] and [La72] that each finite distributive lattice is isomorphic to an interval $[\mathbf{a}, \mathbf{b}]$ of the degree structure (actually, Lachlan shows this for $\mathbf{a} = \mathbf{0}$). The result follows by Theorem 4.8 and an application of (ii) of the Transfer Lemma. □

THEOREM 5.2. *$\Pi_4 - Th(\mathbf{R}_{tt})$, and $\Pi_4 - Th(\mathbf{R}_{btt})$ are h.u.*

Proof. In [Ht, S90] it is shown that for each $k \geq 1$, $\mathbf{P}_k$ with the reverse partial order is isomorphic to an interval $[\mathbf{a}, \mathbf{b}]$ in the r.e. tt-degrees. In [N92], the same result is obtained for r.e. btt-degrees. Hence $\{\mathbf{P}_k : k \geq 1\}$ is Σ_1-e.d.p. in $\mathbf{R}_{tt}$ and $\mathbf{R}_{btt}$. The result follows now by Theorem 4.9 and (ii) of the Transfer Lemma. □

[Lem, N ta] contains a proof that F-BiGraphs is Σ_2-e.d.p. in $\mathbf{R}_{\mathrm{wtt}}$. This proof extends to $\mathbf{R}_T$. In [Lem, N ta] we conclude that these two degree structures have hereditarily undecidable Π_4-theory. For $\mathbf{R}_T$ this also follows from an unpublished proof of Slaman and Woodin, as noted in [A, S93]. It is known that $\Pi_2 - Th(\mathbf{R}_{\mathrm{wtt}})$ is decidable ([A e.a. ta]).

6. Open problems and a final note on Boolean pairs

We summarize how close one can presently come to the goal formulated in the introduction to determine the quantifier level where the theories considered here become undecidable. For most classes, the exact level remains unknown. We also briefly consider the lattice E of r.e. sets under inclusion (see [So89]), where a gap of 5 quantifier alternations remains. As in [Bu, Sa81], BP denotes the class of Boolean pairs.

Table 2

Class	Decidable	Undecidable	Best possible?
F-Graphs	Π_2	Σ_2	$\surd$
F-POrders	Π_2	Σ_2	$\surd$
Groups	Σ_1	Π_1	$\surd$
Lattices, F-Lattices	$\Pi_1 - Th_{\vee\wedge}$, $\Sigma_1 - Th_{\vee\wedge}$	Σ_2	
F-DistrLattices	$\Pi_2 - Th_{\vee\wedge}$	Σ_3, Π_3	
$\{\mathbf{P}_k : k \geq 1\}$	$\Pi_1 - Th_{\vee\wedge}$, $\Sigma_1 - Th_{\vee\wedge}$	Π_4	
$\mathbf{R}_m$	$\Pi_2 - Th_{\vee}$	Π_3	$\surd$
$\mathbf{R}_{tt}, \mathbf{R}_{btt}, \mathbf{R}_T$	$\Pi_1 - Th_{\vee}$	Π_4	
$\mathbf{R}_{wtt}$	Π_2	Π_4	
E	$\Pi_2 - Th_{\vee\wedge}$	Π_8	

We indicate how to obtain the results in Table 2 which are not covered by the previous sections. The decidability of $\Sigma_1 - Th(\text{Groups})$ is immediate since a Σ_1-sentence holds in all groups iff it holds in the one-element group. A similar argument applies to classes of lattices containing the one-element lattice. The decidability results for the Π_1-theories of classes of finite lattices and finite partition lattices follow from Example 5 in Section 1 and the fact that

$$\Pi_1 - Th_{\vee\wedge}(\{\mathbf{P}_k : k \geq 1\}) = \Pi_1 - Th_{\vee\wedge}(F\text{-Lattices}) = \Pi_1 - Th_{\vee\wedge}(\text{Lattices}).$$

The first equation uses the result of Pudlak and Tuma that every finite lattice can be embedded in a finite partition lattice. For the second, by the argument used for Example 5 in Section 1, it suffices to show that, if (1) holds in all finite lattices, then $\bigvee_{1 \leq j \leq m} f_j$ holds in the lattice L presented by $\langle \bar{x} : e_1, \ldots, e_n \rangle$ (the converse is obvious). Suppose f_j is the equation $d_j(\bar{x}) = \tilde{d}_j(\bar{x})$. By a slight extension of Lemma 1 in [Fre, Na 79], there exists a finite lattice B and a homomorphism f of L onto B

such that $f^{-1}(f(d_j)) = \{d_j\}$ for each j. Since some equation f_j is satified in B, this implies that f_j holds in L for some j. We note that, by Example 3 in Section 1, the fragment of $Th_{\vee\wedge}$(Lattices) given by sentences of the form $\Pi_1 \to \Pi_1$ is undecidable.

The decidability of $\Pi_2 - Th_{\wedge\vee}(E)$ was shown in [La 68]. The hereditary undecidability of $\Pi_8 - Th(E)$ is obtained as follows. Let RBP be the class of effective Boolean pairs. The proof of Theorem 6.1 in [Bu, McK81] can be used to show that $\Sigma_5 - Th(RBP)$ is h.u. By a result of L. Harrington (see [So89], p 382) RBP is Σ_3-e.d.p. in E. Then (ii) of the Transfer Lemma gives the result.

We summarize the main questions left open in Table 2.

6.1. *Open questions*

(i) *Is* $\Pi_2 - Th$(Lattices) *undecidable*?
(ii) *Is* $\Sigma_2 - Th$(F-DistrLattices) *undecidable*?
(iii) *Are* $\Pi_3 - Th(E)$, $\Pi_3 - Th(\mathbf{R}_T)$,[1] $\Pi_3 - Th(\mathbf{R}_{tt})$[2] *and* $\Pi_3 - Th(\mathbf{R}_{btt})$ *undecidable*?

The class BP contrasts with the classes considered above by the fact that $Th(F - BP)$ is decidable. This was shown by Comer [Co69], using topological methods. In this final note, we give a more direct proof of this result. In the following, let $WMTh(\mathbf{N}, \leq)$ be the weak monadic theory of $(\mathbf{N}, \leq)$, i.e. the set of sentences with quantification over finite sets which hold in $(\mathbf{N}, \leq)$. This theory is decidable by [Bue60].

THEOREM 6.2. *$Th(F - BP)$ can be interpreted in $WMTh(\mathbf{N}, \leq)$.*

Proof. We first give a representation of finite Boolean pairs (B, U) by finite subsets of $\mathbf{N}$, using a finite set X as a parameter. Suppose $u_1, \ldots, u_k$ are the atoms of U. Choose closed intervals $I_1, \ldots, I_k$ of $(\mathbf{N}, \leq)$ such that $|I_r| = |\{b \leq u_r : b$ B-atom$\}|$ $(1 \leq r \leq k)$ and $1 + \max(I_r) < \min(I_{r+1})$ $(1 \leq r < k)$.

Let $X = I_1 \cup \cdots \cup I_k$, $B_X = P(X)$ and let U_X be the set of finite unions of intervals among $I_1, \ldots, I_k$. Clearly $(B_X, U_X) \cong (B, U)$. Moreover, each finite nonempty set X defines a finite Boolean pair in this way. We show that the relation "$Y \in U_X$" can be defined in the weak monadic language of $(\mathbf{N}, \leq)$. Let

$$\text{ClosedInterval}(Z) \equiv (\forall n)(\forall m)(\forall p)$$

$$[n \in Z \wedge m \in Z \wedge n \leq p \leq m \Rightarrow p \in Z].$$

[1]Added in proof: Lempp, Slaman and the author have recently given an affirmative answer for $\mathbf{R}_T$.
[2]Added in proof: the author can now show that $\Pi_3 - Th_{\wedge}(\{p_k : k \geq 1\})$ and hence $\Pi_3 - Th_{\vee}(\mathbf{R}_{tt})$ is h.u.

Then $Y \in U_X$ if

$$Y \subseteq X \wedge (\forall Z)[\text{ClosedInterval}(Z) \wedge Z \subseteq X \Rightarrow Z \subseteq Y \vee Z \cap Y = \emptyset].$$

Now for each formula φ in the language of Boolean pairs, we can effectively obtain a formula $\tilde{\varphi}(X)$ such that, for each nonempty finite $X \subseteq N$, $(B_X, U_X) \models \varphi \Leftrightarrow (N, \leq) \models \tilde{\varphi}(X)$. Hence

$$\varphi \in Th(F - BP) \quad \Leftrightarrow \quad (\forall X)[(\exists n)n \in X \Rightarrow \tilde{\varphi}(X)] \in WMTh(\mathbf{N}, \leq). \qquad \square$$

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