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Interpreting true arithmetic in the theory of the r.e. truth table degrees

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Abstract

We show that the elementary theory of the recursively enumerable tt-degrees has the same computational complexity as true first-order arithmetic. As auxiliary results, we prove theorems about exact pairs and initial segments in the tt-degrees.

0. Introduction

In the study of structures arising in recursion theory, determining the computational complexity of the elementary theory has often been a major point of interest. We are concerned here with this problem for the degree ordering of r.e. tt-degrees. Recall that the reducibilities $\leqslant_{wtt}$, $\leqslant_{tt}$ and $\leqslant_m$ are obtained from the general Turing-reducibility by restricting more and more the underlying concept of oracle computation. Given sets X, Y, X is weak truth-table reducible to Y (written $X \leqslant_{wtt} Y$) if $X = \{e\}^Y$ for some computation procedure $\{e\}$ such that the use function [19, p. 49] is recursively bounded. X is truth-table reducible to Y (written $X \leqslant_{tt} Y$) if $X = \{e\}^Y$ for some computation procedure $\{e\}$ which is total for every oracle (or, equivalently, $X \leqslant_{tt} Y$ if there is a recursive function g assigning to each input z a finite collection of oracle queries on Y and a Boolean function such that $z \in X$ iff the Boolean function applied to the answers on the queries yields 1). Finally, X is m-reducible to Y ($X \leqslant_m Y$) if $z \in X \Leftrightarrow f(z) \in Y$ for some recursive function f. Given a reducibility $\leqslant_r$ among the ones introduced above, the r-degree $\deg_r(X)$ of set X is $\{Y: Y \equiv_r X\}$, where we write $Y \equiv_r X$ for $X \leqslant_r Y \wedge Y \leqslant_r X$. $\mathbf{R}_r$ and $\mathbf{D}_r(\leqslant \emptyset')$ denote the partial orders of r-degrees of r.e. sets and of r-degrees of sets r-reducible of $\emptyset'$,

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respectively. Since $\mathbf{R}_m = \mathbf{D}_m(\leqslant \emptyset')$, seven structures arise in this way. It is easily verified that, for sets X, Y.

$$\sup(\deg_r(X), \deg_r(Y)) = \deg_r(X \oplus Y).$$

Hence all seven reducibility structures form upper semilattices (u.s.l.).

We review the known results about the complexity of the theories of these structures. The first undecidability proof for the elementary theory of $\mathbf{R}_T$ (briefly Th($\mathbf{R}_T$)) was announced in [7]; a simpler one is given in [3]. For the u.s.l. $\mathbf{R}_{tt}$, $\mathbf{D}_{tt}(\leqslant \emptyset')$ as well as $\mathbf{D}_{wtt}(\leqslant \emptyset')$, undecidability results have been obtained in [10] and in [9]. The proofs of these three results rely on similar methods.

The undecidability of Th($\mathbf{R}_{wtt}$) has been proved recently in [2]; this was done by showing that the structure E^3 of Σ^0_3-sets under inclusion is elementarily definable with parameters in $\mathbf{R}_{wtt}$. It is now clear that the same method can be applied to obtain undecidability proofs for $\mathbf{R}_m$ and $\mathbf{R}_{tt}$, as well as for r.e bounded truth-table degrees [11].

Since all the reducibility structures above are definable within $(\omega, +, \times)$, all their theories are (m-) reducible to true (first-order) arithmetic, i.e. to Th$(\omega, +, \times)$. After proving the undecidability of the theory of such structures, the next question is whether the theory is as complex as possible, namely whether true arithmetic is m-equivalent to it. For the r.e. Turing degrees, the question has been answered affirmatively in [8] and later, by a simpler proof, in [17]. For the Turing degrees below $\emptyset'$, the question has been answered affirmatively in [18]. The question remains open for $\mathbf{R}_m$ and $\mathbf{R}_{wtt}$.The difficulty is that $\mathbf{R}_m$ and $\mathbf{R}_{wtt}$ form distributive semilattices, a property which severely restricts the possibilities for coding. It is also unknown whether the question has an affirmative answer for the distributive lattices E^n, $n \geqslant 1$, of Σ^0_n-sets under inclusion.[2]

In this paper, we show that true arithmetic is m-equivalent to Th($\mathbf{R}_{tt}$). The coding scheme used is quite general and should be applicable to $\mathbf{D}_{tt}(\leqslant \emptyset')$ and $\mathbf{D}_{wtt}(\leqslant \emptyset')$ as well. The major remaining difficulty seems to be proving the exact pair theorem of Section 5 for these other degree structures.

1. Outline of the proof

We give an outline of the proof for the r.e. tt-degrees and introduce some notation. To show that true arithmetic is reducible to Th($\mathbf{R}_{tt}$), we carry out the following program (cf. [18]):

(a) define a standard model of arithmetic in $\mathbf{R}_{tt}$ with a special list of parameters, and

(b) obtain a formula $\psi(\bar{p})$ such that

[2] All these questions are solved now. See [12] for $\mathbf{R}_m$, [5] for E and [13] for $\mathbf{R}_{wtt}$ as well as a general survey.

(b1) for an arbitrary list of parameters, ψ *implies* that the formulas used in (a) define a standard model with these parameters, and

(b2) ψ is satisfied by the special parameter list in (a).

Note that (b) is stronger than (a). Thus (a) should be viewed as an intermediate goal, needed to obtain the special parameter list and formulas in the language of u.s.l. $\varphi_{\mathrm{num}}(x;\ \bar{p})$, $\varphi_{\oplus}(x, y, z; \bar{p})$ and $\varphi_{\otimes}(x, y, z; \bar{p})$ which define a standard model in $\mathbf{R}_{\mathrm{tt}}$ if this parameter list is substituted for $\bar{p}$.

For a sentence α in the language of arithmetic, let $\tilde{\alpha}(\bar{p})$ be the translation of α using these formulas. Then, by (b),

$$(\omega, +, \times) \models \alpha \iff \mathbf{R}_{\mathrm{tt}} \models (\exists \bar{p})[\psi(\bar{p}) \wedge \tilde{\alpha}(\bar{p})].$$

In this way, true arithmetic is m-reducible to $\mathrm{Th}(\mathbf{R}_{\mathrm{tt}})$.

Our proof is in three parts, corresponding to Sections 2–4. Section 2 contains definability results for equivalence relations, which, among others, make it possible to define a standard model of arithmetic in a certain lattice Π_∞ of equivalence relations with additional unary predicates. Section 3 provides theorems about r.e. tt-degrees which enable us, in Section 4, to carry over the results obtained in Section 2 to $\mathbf{R}_{\mathrm{tt}}$. Thus our approach is an indirect one, going through an interpretation of arithmetic in an auxiliary structure. The transfer of results about equivalence relations into results about r.e. tt-degrees is the central idea to meet (a) and (b2).

1.1. Notation and terminology for equivalence relations

Before we discuss the proof in more detail, we introduce some notation The unordered pair of distinct objects r, s is denoted by $[r, s]$. An equivalence relation X is a transitive set of unordered pairs. We write $r X s$ for $r = s \vee [r, s] \in X]$.

We say that an equivalence relation X is *on* a set Y, if, for $r \neq s$,

$$[r,s] \in X \rightarrow r \in Y \wedge s \in Y.$$

Let $\omega_+ = \omega - \{0\}$. If X is an equivalence relation on ω_+ and $S \subseteq \omega_+$, then S^X denotes the set $\{x \in \omega_+ \colon (\exists y \in S)[y X z]\}$. We say that S is *X-closed* if $S^X = S$. For $a \in \omega_+$, we write a^X instead of $\{a\}^X$. The sets a^X are called *equivalence classes of* X, or briefly *X-classes*.

Given equivalence relations X, Y on ω_+, we write $X \leqslant Y$ if $X \supseteq Y$. Intuitively speaking, X has less information than Y in the sense that we can distinguish fewer numbers via X than via Y. With this ordering, the equivalence relations on ω_+ form a lattice. Note that this lattice has a least element 0, which is the equivalence relation with only one equivalence class, that the supremum of X, Y is $X \cap Y$ and the infimum of X, Y is the transitive closure $\mathrm{tr\,cl}(X \cup Y)$.

Let $S \subseteq \omega_+$. By $S^{[2]}$, we denote the set $\{[x, y] \colon x, y \in S\}$. The equivalence relation $X \vee S^{[2]}$ can be interpreted as the restriction of X to S, in particular, $X \vee S^{[2]}$ is on S.

We let Π_∞ be the sublattice of equivalence relations with a cofinite equivalence class. Variables $P, Q, \ldots,$ range over Π_∞. Note that atoms in Π_∞ are the equivalence relations in Π_∞ with two equivalence classes. For $n \geqslant 1$, let E_n be the atom in Π_∞ which possesses the equivalence class $\{n\}$, and let $At_1 = \{E_n\colon n \geqslant 1\}$. A subset S of At_1 is called *recursive* if the set $\{n\colon E_n \in S\}$ is recursive. In a similar manner, we speak about recursive equivalence relations on At_1 etc.

We think of ω_+ and At_1 as conceptually different sets. Relations on ω_+ are called *level* 1, whereas relations on At_1 (and sometimes on the whole structure Π_∞) are *level* 2.

Once and for all, we fix a particular standard model of arithmetic: let

$$F = \{2n + 1\colon n \in \omega\},$$

and let

$$U = \{E_n\colon n \in F\}.$$

(Thus, F is level 1 and U is level 2). The ordering of F gives rise to a standard model $(U, \oplus, \otimes)$ in an obvious way.

1.2. How to satisfy (a)

The results that will follow in Section 3 allow the direct transfer of simple level 2 objects like the sets U and $\{P\colon P \leqslant R\}$ (R some fixed recursive equivalence relation). Therefore, in Section 2.1, we exhibit recursive equivalence relations $C_1, \ldots, C_6$ (not in Π_∞) such that $(U, \oplus, \otimes)$ is definable in the structure

$$(\Pi_\infty; 0, \leqslant, \vee, U, V, (\{P\colon P \leqslant C_i\})_{i=1,\ldots,6}), \tag{1.1}$$

where the auxiliary set V is defined by

$$V = U \cup \{E_{4n}\colon n \geqslant 1\}.$$

The first two theorems in Section 3 are also needed for (b1). We use the following terminology. An initial segment X of $\mathbf{R}_{tt}$ has *socle* $\mathbf{m}$ if $\mathbf{m}$ is the minimum element of $X - \{\mathbf{0}\}$. In Theorem 1, we show that, for each such initial segment, $X - \{\mathbf{0}\}$ lies within one single T-degree. In particular, since minimal tt-degrees are not high [14], all the tt-degrees in X are T-incomplete[3].

The next critical result is an exact pair theorem. First a definition. A set $S \subseteq \mathbf{R}_{tt}$ has a Σ^0_3-*representation* if the index set $\{e\colon \deg_{tt}(W_e) \in S\}$ is Σ^0_3. An ideal $\mathbf{I}$ of $\mathbf{R}_{tt}$ with a Σ^0_3-representation is called a Σ^0_3-*ideal*. We prove that, if a Σ^0_3-ideal contains only T-incomplete tt-degrees, then there exist r.e. tt-degrees $\mathbf{b}, \mathbf{c}$ such that $\mathbf{I} = \{x \in \mathbf{R}_{tt}\colon x \leqslant \mathbf{b}, \mathbf{c}\}$. We say that $\mathbf{b}, \mathbf{c}$ form an *exact pair* for I. In particular, every Σ^0_3-ideal with a socle possesses an exact pair.

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[3] This extends a result in [6]. There, Harrington and Haught show that if X is a finite initial segment of R_{tt} then X has a socle and X-[$\mathbf{0}$] lies within one T-degree.

The third theorem in Section 3 relies on methods in [10], where the undecidability of Th($\mathbf{R}_{tt}$) is proved. Given $n \geqslant 1$, let

$$\Pi_n = \{P: (\forall x)(\forall y)[x, y \geqslant n \rightarrow x\,P\,y]\}.$$

Π_n is a sublattice of Π_∞ and can be identified with the lattice of equivalence relations on $\{1, \dots, n\}$ (which is denoted by Π_n in [10]). They show the following: for each $n \geqslant 1$, there exist r.e. tt-degrees $\mathbf{a}_0$, $\mathbf{a}$ such that

$$(\Pi_n, \leqslant) \cong [\mathbf{a}_0, \mathbf{a}]. \tag{1.2}$$

Moreover, the set $[\mathbf{0}, \mathbf{a}]$ consists of r.e. tt-degrees only and has socle $\mathbf{a}_0$. By model theoretic considerations, this result implies that the theories of $\mathbf{R}_{tt}$ and $\mathbf{D}_{tt}(\leqslant \emptyset')$ are undecidable[4]. The possibility of transfering undecidability from equivalence relations to tt-degrees suggested by (1.2) inspired the approach used in our proof.

We derive a result similar to (1.2) for Π_∞: there exists a Σ^0_3-ideal $\mathbf{I}$ such that the partial orders Π_∞ and $\mathbf{I} - \{\mathbf{0}\}$ are isomorphic. Moreover, the ideal $\mathbf{I}$ is downward closed in the set of all tt-degrees and possesses a socle $\mathbf{a}_0$. Then, applying the exact pair theorem, we obtain r.e. tt-degrees $\mathbf{b}$, $\mathbf{c}$ such that

$$(\Pi_\infty, \leqslant) \cong \{\mathbf{x}: \mathbf{a}_0 \leqslant \mathbf{x} \leqslant \mathbf{b}, \mathbf{c}\}. \tag{1.3}$$

We briefly describe the proof of (1.2) as well as of our result. We use a partition of ω_+ into *blocks*

$$B_x = (d_x, d_{x+1}] \quad (x \geqslant 0), \tag{1.4}$$

where $(d_x)_{x \in \omega}$ is a recursive strictly increasing sequence of positive numbers (to be specified soon). Given an r.e. set A and a recursive equivalence relation R on ω_+, let

$$(A)_R = \{d_x + i: d_x + i \in B_x \wedge (\exists j)[d_x + j \in B_x \wedge i\,R\,j \wedge d_x + j \in A]\}.$$

$(A)_R$ can be viewed as the projection of A via R. Clearly, this projection is r.e. Moreover,

$$R \leqslant S \;\Rightarrow\; (A)_R \leqslant_{tt} (A)_S \tag{1.5}$$

(see Lemma 3.4). The proof of (1.2) in [10] uses blocks B_x defined by $d_x = nx$ ($x \in \omega$). An r.e. set A is constructed in such a way that

$$\{\deg_{tt}((A)_P): P \in \Pi_n\} \cup \{\mathbf{0}\} \tag{1.6}$$

[4] Using the terminology of [2], undecidability of Th(R_{tt}) and Th($\mathbf{D}_{tt}(\leqslant \emptyset')$) can be obtained from (1.2) in an alternative way as follows. The class of finite undirected graphs is elementarily definable with parameters (e.d.p.) in $\{\Pi_n: n \geqslant 1\}$ by methods in Section 2 of this paper; furthermore, the class $\{\Pi_n: n \geqslant 1\}$ is e.d.p in $\mathbf{R}_{tt}$ and in $\mathbf{D}_{tt}(\leqslant \emptyset)$ by (1.2). Undecidability now follows from the fact that the class of finite undirected graphs is hereditarily undecidable.

forms an initial segment and, for $P, Q \in \Pi_n$,

$$P \not\leqslant Q \Rightarrow (A)_P \not\leqslant_{tt} (A)_Q. \tag{1.7}$$

By (1.5), equivalence holds in (1.7), and $\mathbf{a}_0 = \deg_{tt}((\mathbf{A})_0)$ is the socle of the initial segment (1.6). Thus, (1.2) is satisfied via $\mathbf{a}_0$ and $\mathbf{a} = \deg_{tt}(\mathbf{A})$, where the isomorphism is given by $P \to \deg_{tt}((\mathbf{A})_P)$.

In the proof of our result, blocks are increasing in length. Define

$$d_0 = 0 \quad \text{and, for } x \geqslant 0,$$

$$d_x = d_{x-1} + x.$$

Then $|B_x| = x$ and $d_x + i \in B_x \Leftrightarrow 1 \leqslant i \leqslant x$. Let $(R_n)_{n \in \omega}$ be any sequence of uniformly recursive equivalence relations such that $\Pi_\infty \subseteq \{R_n: n \in \omega\}$. We construct an r.e. set A such that

$$\mathbf{I} = \{\deg_{tt}((A)_P): P \in \Pi_\infty\} \cup \{\mathbf{0}\} \text{ is downward closed in the set of all tt-degrees,} \tag{1.8}$$

$$(\forall n)(\forall m)[R_n \not\leqslant R_m \Rightarrow (A)_{R_n} \not\leqslant_{tt} (A)_{R_m}]. \tag{1.9}$$

Again by (1.5), equivalence holds in (1.9).

Define the map $\Phi: \Pi_\infty \to \mathbf{R}_{tt}$ by $\Phi(P) = \deg_{tt}((A)_P)$. If $R = R_n$ for some n and $\mathbf{r} = \deg_{tt}((A)_R)$, then the image of $\{P: P \leqslant R\}$ under Φ is $\{\mathbf{p} \in \mathbf{I} - \{\mathbf{0}\}: \mathbf{p} \leqslant \mathbf{r}\}$. Together with the exact pair theorem, this enables us not only to satisfy (1.3) via Φ, but even to define a copy of the structure (1.1) in $\mathbf{R}_{tt}$ with parameters (Section 4). This will be sufficient for (a). To define the copy of (1.1), we include the recursive equivalence relations $C_1, \ldots, C_6$ in our list (R_n) The special parameter list will consist of

$\mathbf{a}_0$	socle of the ideal $\mathrm{rg}(\Phi) \cup \{\mathbf{0}\}$,
$\mathbf{b}, \mathbf{c}$	exact pair for this ideal,
$\mathbf{u}_0, \mathbf{u}_1; \mathbf{v}_0, \mathbf{v}_1$	exact pairs for the ideals generated by $\Phi(U)$ and $\Phi(V)$,
$\mathbf{c}_i$ $(1 \leqslant i \leqslant 6)$	degrees defining the arithmetical operations; these tt-degrees are given by $\mathbf{c}_i = \deg_{tt}((A)_{C_i})$,
f, g, succ, id	four additional tt-degrees which will be needed for (b2). (1.10)

1.3. Satisfying (b)

We give conditions (CAr), (C1)–(C5) and (CSt) on an arbitrary list of parameters which can be expressed in the language of u.s.l. The formula $\psi(\bar{p})$ in (b) is the conjunction of formulas expressing these conditions. *To indicate the intended meanings of the parameters, we use the same symbols as in the special list* (1.10) *for the parameters in an arbitrary list* $\bar{p}$. The parameters **f, g, succ, id** occur in (C4) and (C5) only. Loosely speaking, these two conditions enable us to recover the standard part of the model of arithmetic we define using the other parameters.

We here state conditions (CAr) and (CSt) in their final form. The other conditions will be made precise only in Section 4.

(CAr) *the ternary relations defined by the formulas* $\varphi_{\oplus}$, $\varphi_{\otimes}$ *on* $\{\mathbf{c} \in \mathbf{R}_{tt}: \varphi_{num}(\mathbf{c})\}$ *give a model* $\mathbf{M} = (M, +, \times)$ *of Robinson arithmetic.*

(These formulas will be specified in Section 4.)

Condition (C1) says that the initial segment $\{\mathbf{x} \in \mathbf{R}_{tt}: \mathbf{x} \leqslant \mathbf{b}, \mathbf{c}\}$ has socle $\mathbf{a}_0$. Let $\dot{<}$ denote denote the ordering of $\mathbf{M}$. (C2) requires that, for each $\mathbf{k} \in M$, the supremum $s_{\mathbf{k}}$ of the (possibly infinite) set $\{\mathbf{l}: \mathbf{l} \dot{<} \mathbf{k}\}$ exists, and that $\mathbf{s}_{\mathbf{k}} \leqslant \mathbf{b}, \mathbf{c}$. Then, in particular, the set M is included in an initial segment which possesses a socle. Condition (C3) is

$$(\forall \mathbf{k} \in M)(\forall \mathbf{l} \in M)[\mathbf{l} \leqslant \mathbf{s}_{\mathbf{k}} \Rightarrow \mathbf{l} \dot{<} \mathbf{k}].$$

The conditions (C4) and (C5) are defined in such a way that

(C4) and (C5) together imply that the ideal $\mathbf{I}$ generated by the standard part S of $\mathbf{M}$ is a Σ_3^0-ideal, and (1.11)

with suitable definitions of **f, g, succ, id** in (1.10), (C4) and (C5) are satisfied. (1.12)

Let p, p' range over $\{\mathbf{x}: \mathbf{a}_0 \leqslant \mathbf{x} \leqslant \mathbf{b}, \mathbf{c}\}$). Regarding (1.11), note that

$$\mathbf{I} = \{\mathbf{p}: (\exists \mathbf{k} \in S)[\mathbf{p} \leqslant \mathbf{s}_{\mathbf{k}}]\} \cup \{\mathbf{0}\}.$$

One possibility to make $\mathbf{I}$ a Σ_3^0-ideal would be to (uniformly) obtain $\{\mathbf{p}': \mathbf{p}' \leqslant \mathbf{s}_{\mathbf{k}+1}\}$ from $\{\mathbf{p}: \mathbf{p} \leqslant \mathbf{s}_{\mathbf{k}}\}$ by an application of a second order $\exists$-formula $\varphi(X, x)$ without negation signs. Suppose we require that

$$\text{for each } \mathbf{k} \in S \text{ and each } \mathbf{p}', \mathbf{p}' \leqslant \mathbf{s}_{\mathbf{k}+1} \text{ iff } \mathbf{R}_{tt} \models \varphi(\{\mathbf{p}: \mathbf{p} \leqslant \mathbf{s}_{\mathbf{k}}\}, \mathbf{p}'). \quad (1.13)$$

Observe that $\{\langle i,j \rangle: W_i \leqslant_{tt} W_j\}$ is a Σ_3^0 relation and that the set of r.e. tt-degrees $\{\mathbf{x}: \mathbf{a}_0 \leqslant \mathbf{x} \leqslant \mathbf{b} \leqslant \mathbf{c}\}$ has a Σ_3^0 representation. Then, by the properties of $\varphi(X, x)$, for each $\mathbf{k} \in S$, the ideal $\{\mathbf{p}: \mathbf{p} \leqslant \mathbf{S}_{\mathbf{k}}\} \cup \{\mathbf{0}\}$ has a Σ_3^0-representation uniformly in $\mathbf{k}$. Hence $\mathbf{I}$ is a Σ_3^0-ideal.

In view of (1.12), we have to modify this scheme: we require (1.13) with $\mathbf{s}_{\mathbf{k}}$ and $\mathbf{s}_{\mathbf{k}+1}$ replaced by $\mathbf{s}_{\mathbf{k}} \vee \mathbf{f}$ and $\mathbf{s}_{\mathbf{k}+1} \vee \mathbf{f}$, respectively. This will lead to condition (C5). Condition (C4) makes it possible to recover $\mathbf{I}$ from the Σ_3^0-ideal

$$\{\mathbf{p}: (\exists \mathbf{k} \in S)[\mathbf{p} \leqslant \mathbf{s}_{\mathbf{k}} \vee \mathbf{f}] \cup \{\mathbf{0}\}$$

and shows that $\mathbf{I}$ is itself a Σ_3^0-ideal. Now, by the exact pair theorem and (C1), $\mathbf{I}$ possesses an exact pair $\mathbf{r}, \mathbf{s}$. By (C3), $S = \{\mathbf{k} \in M: \mathbf{k} \leqslant \mathbf{r}, \mathbf{s}\}$. Thus, for (b1) it is sufficient to require

(CSt) *for each pair* $\mathbf{r}, \mathbf{s}$, *if* $\{\mathbf{k} \in M: \mathbf{k} \leqslant \mathbf{r}, \mathbf{s}\}$ *is a proper nonempty initial segment of* $(M, \dot{<})$, *then this set possesses a maximum with respect to* $\dot{<}$.

To satisfy (1.12), we use the transfer idea once again. Thus, the four last tt-degrees in (1.10) will correspond to four recursive equivalence relations. In particular, **f** and **g** will correspond to $F^{[2]}$ and $(\omega_+ - F)^{[2]}$. We first formulate and prove (C4) and (C5) for equivalence relations (Section 2.2). We also add a transfer lemma in Section 3 and adjust the list (R_n). Then we are able to show that (C4) and (C5) hold for the special list (1.10).

Note that all conditions except (C4) and (C5) are immediate from the definitions of the tt-degrees in (1.10) and the fact that we are defining a standard model. In this way we satisfy (b2).

2. Equivalence relations

2.1. Defining a standard model of arithmetic in Π_∞

Recall that the variables P, Q range over Π_∞, that $F = \{2n+1\colon n \in \omega\}$, $U = \{E_n\colon n \in F\}$, and $V = U \cup \{E_{4n}\colon n \geqslant 1\}$. The results in Section 1 only use that U and V are recursive level 2 sets and that $\emptyset \subset_\infty U \subset_\infty V \subset_\infty \mathrm{At}_1$.

The standard model $(U, \oplus, \otimes)$ is determined by $(F, <)$ in an obvious way. Our goal in Section 2.1 is to define the arithmetical operations $\oplus$, $\otimes$ in a structure

$$(\Pi_\infty; 0, \leqslant, \vee, U, V, (\{P\colon P \leqslant C_i\})_{i=1,\dots,6}).$$

This is done in two steps. In Lemma 2.1 we show in a straightforward way that an arbitrary recursive relation $R \subseteq U^3$ can be defined by three recursive equivalence relations on V. It then remains to define a given level 2 recursive equivalence relation X on V in a structure

$$(\Pi_\infty; 0, \leqslant, \vee, V, \{P\colon P \leqslant C\}), \tag{2.1}$$

where C is an appropriately chosen level 1 recursive equivalence relation. To do so, we use cardinalities of equivalence classes. In Lemma 2.2, we verify that there exists a recursive equivalence relation C on ω_+ with finite classes only such that

$$X = \{[E_n, E_m]\colon |n^C| = |m^C| \wedge E_n \in V \wedge E_m \in V\}. \tag{2.2}$$

Then, in Lemma 2.3, we show the definability of the level 2 equivalence relation $\{[E_n, E_m]\colon |n^C| = |m^C|\}$ in $(\Pi_\infty; 0, \leqslant, \vee, \{P\colon P \leqslant C\})$. This is a purely algebraic result. In the course of the proof, we interpret equivalence relations P in Π_∞ with the property that all finite P-classes have cardinality 2 as bijections between finite sets.

Lemma 2.1. *There exist recursive equivalence relations $X_1, \dots, X_6$ on V such that $(U, \oplus, \otimes)$ is definable in the structure*

$$(V; U, X_1, \dots, X_6).$$

Proof. It is sufficient to show that any recursive ternary relation $R \subseteq U^3$ can be defined from U and recursive equivalence relations Y_1, Y_2, Y_3 on V. Fix a 1–1 recursive map $c: U^3 \to V - U$, and define

$$Y_i = \operatorname{tr\,cl}\{[x_i, c(x_1, x_2, x_3)]: Rx_1x_2x_3\} \quad (i = 1, 2, 3).$$

Clearly, the equivalence relations Y_i are recursive. Now

$$R = \{(x_1, x_2, x_3) \in U^3: (\exists z)[x_1\, Y_1\, z \wedge x_2\, Y_2\, z \wedge x_3\, Y_3\, z]\}.$$

This shows the definability of R. □

Lemma 2.2. *Let X be any recursive equivalence relation on V. Then there is a recursive equivalence relation C on ω_+ with finite equivalence classes only such that* (2.2) *holds.*

Proof. Let $\tilde{V} = \{n: E_n \in V\}$. Define a recursive function h by

$$h(n) = \min\{k: E_k\, X\, E_n\}.$$

To define C, attach to each $n \in \tilde{V}$ numbers in $\omega_+ - \tilde{V}$ in such a way that

$$|n^C| = 1 + h(n).$$

It is clear that C can be chosen recursive and that (2.2) holds. □

Lemma 2.3. *Let C be any equivalence relation on ω_+ with finite equivalence classes only. Then*

$$\{[E_n, E_m]: |n^C| = |m^C|\} \tag{2.3}$$

is definable in the structure $(\Pi_\infty; 0, \leqslant, \vee, \{P: P \leqslant C\})$.

Proof. We use the following notation. If the finite P-classes are $Z_1, \ldots, Z_n$, we write

$$P = Z_1| \ldots |Z_n|\!\!\longrightarrow.$$

Thus e.g. $E_n = n|\!\!\longrightarrow$ (omitting brackets). We use variables H, K, L for atoms in Π_∞. Note that, if $H = Z|\!\!\longrightarrow$ and R is any equivalence relation on ω_+, then

$$H \leqslant R \iff Z \text{ is an } R\text{-closed set.} \tag{2.4}$$

Let At_2 be the set of atoms H such that one H-class has cardinality two. If $H = n, m|\!\!\longrightarrow$ $(n \neq m)$, then H represents the unordered pair $[E_n, E_m]$ in the following sense: for each k, l,

$$H \leqslant E_k \vee E_l \iff [E_k, E_l] = [E_n, E_m].$$

This follows from (2.4), since $E_k \vee E_l = k|l|\!\!\longrightarrow$. We write Component$(K, H)$ if $K = E_n \;\vee\; K = E_m$.

In the following, the definability of At_1 in Π_∞ as well as of

$$\{n, m\vdash\!\!\!\text{——}: |n^C| = |m^C|\} \tag{2.5}$$

in $(\Pi_\infty; 0, \leqslant, \vee, \{P\colon P \leqslant C\})$ is shown. The definability of the relation (2.3) in this structure follows, since, for each k, l, $[E_k, E_l]$ is in the set (2.3) if and only if

$$(\exists H)[H \text{ is in } (2.5) \wedge H \leqslant E_k \vee E_l].$$

We prove a series of sublemmas, which introduce auxiliary predicates. In each sublemma,we show the equivalence of a semantical statement (i) about equivalence relations and a statement (ii) which is easily seen to be definable in Π_∞ or in a structure $(\Pi_\infty; 0, \leqslant, \vee, \{P\colon P \leqslant S\})$, where S is some fixed equivalence relation. In this way, the sublemma shows the definability of the auxiliary predicate given by (i).

Sublemma 1. *Let $K = X\vdash\!\!\!\text{——}$ and $L = Y\vdash\!\!\!\text{——}$. Then the following are equivalent:*

(i) $\emptyset \in \{X\cap Y, \bar{X}\cap Y, X\cap\bar{Y}, \bar{X}\cap\bar{Y}\}$.

(ii) *There are at most three atoms below $K \vee L$.*

In this case we write $\mathrm{Compat}(K, L)$.

Proof. (i) $\to$ (ii): If (i) holds, then $K \vee L$ possesses at most three equivalence classes. Now (ii) follows by (2.4).

(ii) $\to$ (i): If (i) fails, then each set among $X\cap Y, \bar{X}\cap Y, X\cap\bar{Y}, \bar{X}\cap\bar{Y}$ is an equivalence class of an atom below $K \vee L$. Then there exist four such atoms. □

Sublemma 2. *Let S be an equivalence relation on ω_+ with at most one infinite equivalence class, and let $H = X\vdash\!\!\!\text{——}$ be any atom. Then the following are equivalent.*

(i) *X or $\bar{X}$ is an S-class,*

(ii) $H \leqslant S \wedge (\forall K)[K \leqslant S \Rightarrow \mathrm{Compat}(H, K)]$.

In this case we write $\mathrm{Class}_S(H)$. *If $S \in \Pi_\infty$, we also write* $\mathrm{Class}(H, S)$.

Remark. This sublemma shows that the unary predicate Class_S is definable in $(\Pi_\infty; 0, \leqslant, \vee, \{P\colon P \leqslant S\})$ and that the binary predicate Class is definable in Π_∞.

Proof of Sublemma 2. (i) $\to$ (ii). Suppose that (i) holds. By (2.4), $H \leqslant S$. Now let $K = Y\vdash\!\!\!\text{——}$ be such that $K \leqslant S$. Then for each S-class Z, $Z \subseteq Y$ or $Z \subseteq \bar{Y}$. Hence X, Y satisfy (i) in Sublemma 1.

(ii) $\to$ (i): Suppose that neither X nor $\bar{X}$ is an S-class. Then we can choose finite S-classes Z_1, Z_2 such that $Z_1 \subset X$ and $Z_2 \subset \bar{X}$ (Z_2 exists since $\bar{X}$ is a union of S-classes and S possesses at most one infinite class). Let $K = Z_1 \cup Z_2\vdash\!\!\!\text{——}$. Then $K \leqslant S$ and $\neg\mathrm{Compat}(H, K)$. □

We are now able to show the definability in Π_∞ of At_1, At_2 and the binary predicate Component. For At_1, apply Sublemma 2 to the set $S = \emptyset$. Since $\emptyset$-classes are just the singletons, we can infer that, for each atom H,

$$H \in \mathrm{At}_1 \iff (\forall K)\,[\mathrm{Compat}(H, K)].$$

Hence At_1 is definable. To show the definability of At_2 and Component, observe that, for arbitrary P, K

$$P \in \mathrm{At}_2 \iff (\exists K_1)(\exists K_2)\,[K_1, K_2 \in \mathrm{At}_1 \wedge |\{P, K_1, K_2\}| = 3 \wedge P \leqslant K_1 \vee K_2], \tag{2.6}$$

$$\mathrm{Component}(K, P) \iff (\exists K_1)(\exists K_2)\,[(2.6)\ \textit{holds} \wedge K \in \{K_1, K_2\}].$$

Sublemma 3. *Let S be as in Sublemma 2 and $H = n, m|\!\!-\!\!-\!\!-$ ($n \neq m$). Then the following are equivalent.*

(i) $n\,S\,m$,

(ii) $(\forall K)\,[K \in \mathrm{Class}_S \Rightarrow \{\mathrm{Compat}(K, H) \wedge (K \in \mathrm{At}_1 \Rightarrow \neg\mathrm{Component}(H, K))\}]$.

In this case we write $\mathrm{Related}_S(H)$.

Proof. (i) $\to$ (ii): If $n\,S\,m$ and $K = X\vdash\!\!\!-\!\!\!- \in \mathrm{Class}_S$, then $n \in X \iff m \in X$ and hence $\mathrm{Compat}(K, H)$. Moreover, if X is the singleton $\{z\}$, then $K \in \mathrm{Class}_S$ implies that X is an S-class. Therefore $z \neq n, m$ and $\neg\mathrm{Component}(H, K)$.

(ii) $\to$ (i): Suppose that $\neg n\,S\,m$. If $|z^S| = 1$ for some $z \in \{n, m\}$, then (ii) fails for $K = z\vdash\!\!\!-\!\!\!-$ Otherwise, choose $x \neq n$ in n^S and $y \neq m$ in m^S. Let $X = \{n, m\}$, $Y = n^S$ and $K = Y\vdash\!\!\!-\!\!\!-$. Then $K \in \mathrm{Class}_S$. Since the sets $X \cap Y, \bar{X} \cap Y, X \cap \bar{Y}, \bar{X} \cap \bar{Y}$ in (i) of Sublemma 1 contain the numbers n, x, m, y, respectively, $\neg\mathrm{Compat}(K, H)$ and, again, (ii) fails. $\square$

If $u_1, v_1, \ldots, u_k, v_k \in \omega_+$ are pairwise distinct, then we use the equivalence relation

$$P = u_1, v_1 | \ldots | u_k, v_k \vdash\!\!\!-\!\!\!-$$

to represent the bijection $u_i \to v_i$ $(1 \leqslant i \leqslant k)$ between the sets $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$. Note that $P \in \mathrm{At}_2$ iff $k = 1$. First we show that the set of all P which represent bijections is definable. Using this, we derive the definability of the ternary relation

$$\{(P, X\vdash\!\!\!-\!\!\!-, Y\vdash\!\!\!-\!\!\!-) \colon P \textit{ represents a bijection between } X \textit{ and } Y\}.$$

Sublemma 4. *Let $P \in \Pi_\infty$ be arbitrary. Then the following are equivalent.*

(i) *each finite P-class has cardinality* 2,

(ii) $(\exists K)\,[\mathrm{Class}(K, P) \wedge (\forall H)\,[\mathrm{Class}(H, P) \wedge H \neq K \to H \in \mathrm{At}_2]]$.

In this case we write $\mathrm{Inj}(P)$.

Proof. (i) → (ii): Immediate by Sublemma 2.

(ii) → (i): Suppose that (ii) holds via $K = X\vdash\!\!\!\text{---}$. It is easy to see that the infinite P-class must be $\bar{X}$. If Y is any finite P-class, $Y \neq X$, then $H = Y\vdash\!\!\!\text{---}$ is in At_2, whence $|Y| = 2$. □

Definitions. The atom K in (ii) of Sublemma 4 is uniquely determined by P. Thus there is a unary function InfCl which is definable in in Π_∞ such that, if Inj(P), then (ii) holds via $K = \mathrm{InfCl}(P)$.

We write i_P for the map represented by P. Thus,

$$i_P(x) = y \Leftrightarrow \{x, y\} \text{ is a } P\text{-class.}$$

Sublemma 5. *Suppose that* Inj(P) *holds. Let* K, L *be atoms, and let* X, Y, Z *be the (finite) sets such that* $K = X\vdash\!\!\!\text{---}$, $L = Y\vdash\!\!\!\text{---}$ *and* $\mathrm{InfCl}(P) = Z\vdash\!\!\!\text{---}$. *Then the following are equivalent.*

(i) (i.1) *and* (i.2) *below hold*:

(i.1) $X \cap Y = \emptyset \wedge X \cup Y = Z$,

(i.2) *the map* i_p *defines a bijection between* X *and* Y (*Fig.* 1).

(ii) (ii.1) *and* (ii.2) *below hold*:

(ii.1) $\{H\colon H \leqslant K \vee L\} = \{K, L, \mathrm{InfCl}(P)\}$, *and this set has cardinality* 3,

(ii.2) $(\forall H)[H \in \mathrm{At}_2 \wedge \mathrm{Class}(H, P) \Rightarrow$
$(\neg\mathrm{Compat}(K, H) \wedge \neg\mathrm{Compat}(L, H)) \vee$
$(\mathrm{Component}(K, H) \wedge \mathrm{Component}(L, H))]$.

We write Bij(P, K, L) if Inj(P) and the equivalent conditions above hold.

Remarks. Note that (ii.1) implies Compat(K, L). From (ii.1) it follows that the atom InfCl(P) is below $\mathrm{K} \vee L$, a fact which will be used in the proof of (ii) → (i) to show that $Z = X \cup Y$. The two alternatives in (ii.2) correspond to the cases $P \notin \mathrm{At}_2$ and $P \in \mathrm{At}_2$, respectively.

Proof of Sublemma 5. (i) → (ii): Suppose that (i) holds. Then $K \vee L = X|Y\vdash\!\!\!\text{---}$, and (ii.1) follows. For (ii.2), let $H \in \mathrm{At}_2$ be arbitrary such that Class(H, P). If $P \notin \mathrm{At}_2$, then,

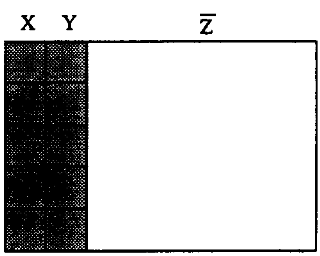

Fig. 1. The map i_P defines a bijection between X and Y.

by (i.2), the first alternative in (ii.2) holds. If $P \in \mathrm{At}_2$, then X and Y are singletons and $H = \mathrm{InfCl}(P)$. Hence the second alternative in (ii.2) holds.

(ii) → (i): Suppose that (ii) holds. Assume for a contradiction that $X \cap Y \neq \emptyset$. Since X, Y are finite and $\mathrm{Compat}(K, L)$, $X \subseteq Y$ or $Y \subseteq X$, say the first. Then, since $K \neq L$, X is strictly contained in Y, and $K \vee L = X \mid Y - X \vdash$——. We show $\bar{Z} = \bar{Y}$. Since the atom $\mathrm{InfCl}(P)$ is below $K \vee L$, Z or $\bar{Z}$ must be a $K \vee L$-class. By the finiteness of X, Y and Z, to show $\bar{Z} = \bar{Y}$ we only have to rule out the cases $Z = X$ and $Z = Y$. But if $Z = X$ or $Z = Y$, then some P-class $\{n, m\}$, $n \neq m$, is included in Y. Let $H = n, m \vdash$——. Then

$$\mathrm{Compat}(L, H) \wedge \neg\,\mathrm{Component}(L, H).$$

Hence (ii.2) is violated. This shows that $\bar{Z} = \bar{Y}$, contrary to (ii.1) Now, since $X \cap Y = \emptyset$, $K \vee L = X \mid Y \vdash$——. Using (ii.2) in the same way as above, we can infer that $X \cup Y = Z$. Moreover, again by (ii.2), the map i_p is a bijection between X and Y. □

We are now ready to establish the definability of (2.5).

Sublemma 6. *Let $H = n, m \vdash$—— $(n \neq m)$. Then the following are equivalent.*

(i) $|n^C| = |m^C|$

(ii) $\mathrm{Related}_C(H) \vee (\neg\,\mathrm{Related}_C(H)$
$\wedge (\exists P)[\mathrm{Class}(H, P) \wedge (\exists K)(\exists L)[\mathrm{Class}_C(K) \wedge \mathrm{Class}_C(L) \wedge \mathrm{Bij}(P, K, L)]])$. (2.7)

Remark. The significant condition in (ii) is Class (H, P): this implies that K, L represent the C-classes n^C, m^C (in some order). The condition $\neg\,\mathrm{Related}_C(H)$ in (ii) is in fact already implied by (2.7).

Proof of Sublemma 6. If $n\,C\,m$, then both (i) and (ii) are satisfied. Now assume that $\neg\, n\,C\,m$.

(i) → (ii): Suppose that (i) holds, and let $K = n^C \vdash$—— and $L = m^C \vdash$——. Choose P such that $\mathrm{Inj}(P)$ and $i_p(n) = m$. Then (2.7) holds via P, K and L.

(ii) → (i): Suppose that (2.7) holds via $P, K = X \vdash$—— and $L = Y \vdash$——. Since $\{n, m\}$ is a P-class and X, Y are C-classes, by (ii.2) in Sublemma 5, $\{X, Y\} = \{n^C, m^C\}$. Because i_p is a bijection between X and Y, this implies (i). □

As explained above, from Sublemma 6 we can infer the definability of (2.3). This concludes the proof of Lemma 2.3.

Summarizing the preceding three Lemmas, we obtain:

Theorem 2.4. *There are recursive equivalence relations $C_1, \ldots, C_6$ such that $(U, \oplus, \otimes)$ is first-order definable in the structure*

$$(\Pi_\infty;\, 0, \leqslant, \vee, U, V, (\{P\colon P \leqslant C_i\})_{i=1,\ldots,6}). \tag{2.8}$$

Proof. By Lemma 2.1, obtain recursive level 2 equivalence relations $X_1, \ldots, X_6$ such that the arithmetical operations $\oplus$, $\otimes$ are definable in the structure $(V; U, X_1, \ldots, X_6)$. Lemma 2.2 gives recursive level 1 equivalence relations $C_1, \ldots, C_6$ such that (2.2) holds for each $X_i\ C_i$. Now, using Lemma 2.3, for each i, the equivalence relation X_i is definable in $(\Pi_\infty; 0, \leqslant, \vee, V, \{P: P \leqslant C_i\})$. This shows the definability of $(U, \oplus, \otimes)$ in (2.8). □

2.2 The versions of the conditions (C4) *and* (C5) *for equivalence relations*

Notation. Let $S, T \subseteq \omega_+$. In lattice theoretic formulas, we write S instead of $S^{[2]}$. If $h: S \to T$ is a 1–1 map and R is an equivalence relation on S, then we define an equivalence relation on T by

$$h(R) = \{[h(x), h(y)]: [x, y] \in R\}.$$

Let $G = \omega_+ - F$. We view $(F, <)$ and $(G, <)$ as copies of $(\omega, <)$. In Section 2, variables k, l range over F. Given k, k' denotes the corresponding element of G, i.e. $k = 2n + 1 \leftrightarrow k' = 2n + 2$. For $k = 2n + 1$, let S_k be the supremum (in Π_∞) of the first n "numbers" in $(U, \oplus, \otimes)$, that is,

$$S_k = \sup\{E_l: l < k\}.$$

In the following Lemma we state and verify condition (C4) for equivalence relations.

Lemma 2.5. $(\forall k \in F)(\forall P)[P \leqslant S_k \Leftrightarrow P \leqslant (S_k \vee F) \ \wedge \ P \leqslant G]$.

Proof. Let $k \in F$ and $P \in \Pi_\infty$ be arbitrary. It suffices to show that

$$P \leqslant S_k \Leftrightarrow P \leqslant (S_k \vee F) \wedge G. \tag{2.9}$$

We use the following graphical representations of the equivalence relations in question.

	on F:	on G:
$F^{[2]}$	$=$ ————————	$\vert 0' \vert 1' \vert 2' \vert \ldots$
$G^{[2]}$	$= \vert 0 \vert 1 \vert 2 \vert \ldots$	————————
S_k	$= \vert 0 \vert 1 \vert \ldots \vert k\text{-}1 \vert$————	————————
$S_k \vee F$	$= \vert 0 \vert 1 \vert \ldots \vert k\text{-}1 \vert$————	$\vert 0' \vert 1' \vert 2' \vert \ldots$
$(S_k \vee F) \wedge G$	$= \vert 0 \vert 1 \vert \ldots \vert k\text{-}1 \vert$————	————————

Since there is only one infinite P-class, (2.9) is immediate from the diagram for $(S_k \vee F) \wedge G$. □

The version of condition (C5) for equivalence relations is treated in Lemma 6. We first describe the ideas behind (C5). Since we will use the idea to transfer the appropriate property from equivalence relations to tt-degrees, we aim at uniformly obtaining $\{P': P' \leqslant S_{k+1} \wedge F\}$ from $\{P: P \leqslant S_k \wedge F\}$. We again interpret certain equivalence relations, not in Π_∞ this time, as 1–1 maps. Define the maps f_{succ}: $F \to G$ and f_{id}: $G \to F$ by

$$f_{succ}(k) = (k+1)' \quad \text{and} \quad f_{id}(l') = l.$$

Then, for $k \in F$,

$$f_{id}(f_{succ}(S_k \vee F)) = S_{k+1} \vee F. \tag{2.10}$$

Now, let S and Id be the equivalence relations corresponding to these maps, that is

$$\begin{aligned} \mathrm{S} &= \{[k,(k+1)']: k \in F\}, \\ \mathrm{Id} &= \{[l',l]: l' \in G\}. \end{aligned} \tag{2.11}$$

Since $F \cap G = \emptyset$, for arbitrary equivalence relations X on F and Y on G,

$$f_{succ}(X) = (X \wedge \mathrm{S}) \vee G \text{ and } f_{id}(Y) = (Y \wedge \mathrm{Id}) \vee F. \tag{2.12}$$

Given $k \in F$, we let

$$H_k = (S_k \vee F) \wedge \mathrm{S},$$

$$L_k = (H_k \vee G) \wedge \mathrm{Id}.$$

In graphical representations,

$$\begin{aligned} H_k \quad &= |0'|01'| \dots |k-1\,k'|\!\!\longrightarrow, \\ &\quad \text{on } F: \qquad\quad \text{on } G: \\ H_k \vee G \ &= |0|1|2| \dots \quad |0'|1'| \dots |k'|\!\!\longrightarrow, \\ L_k \quad &= |00'|11'| \dots |kk'|\!\!\longrightarrow. \end{aligned}$$

We can now rewrite (2.10) in the following way:

$$\begin{aligned} S_{k+1} \vee F &= ((((S_k \vee F) \wedge \mathrm{S}) \vee G) \wedge \mathrm{Id}) \vee F \\ &= ((H_k \vee G) \wedge \mathrm{Id}) \vee F \\ &= L_k \vee F. \end{aligned} \tag{2.13}$$

We use (2.13) to obtain $\{P': P \leqslant S_{k+1} \vee F\}$ from $\{P: P \leqslant S_k \vee F\}$ in a uniform way. The key fact, which is immediate from the graphical representations, is that $H_k, L_k \in \Pi_\infty$.

Lemma 2.6. $(\forall k \in F)(\forall P)$

$$\begin{aligned} [P \leqslant (S_{k+1} \vee F) \Leftrightarrow (\exists Q_1)(\exists Q_2)\,[Q_1 \leqslant (S_k \vee F) \ &\wedge \ Q_1 \leqslant \mathrm{S} \\ \wedge \ Q_2 \leqslant (Q_1 \vee G) \ &\wedge \ Q_2 \leqslant \mathrm{Id} \\ \wedge \ P \leqslant Q_2 \vee F]]. \end{aligned} \tag{2.14}$$

Proof. Let $k \in F$ and $P \in \Pi_\infty$ be arbitrary. If $P \leqslant (S_{k+1} \vee F)$, then, since $S_{k+1} \vee F = L_k \vee F$, (2.14) holds via $Q_1 = H_k$ and $Q_2 = L_k$. For the converse implication, suppose that (2.14) holds via Q_1, Q_2. Then $Q_1 \leqslant H_k$, whence

$$Q_2 \leqslant (Q_1 \vee G) \wedge \mathrm{Id} \leqslant (H_k \vee G) \wedge \mathrm{Id} = L_k, \quad \text{and}$$

$$P \leqslant Q_2 \vee F \leqslant L_k \vee F = S_{k+1} \vee F. \quad \square$$

3. Auxiliary results on tt-degrees

Theorem 3.1. *Suppose that X is an initial segment of $\mathbf{R}_{tt}$ with socle $\mathbf{m} = \deg_{tt}(M)$. Let A be an r.e. set such that $\deg_{tt}(A) \in X - \{\mathbf{0}\}$. Then A is Turing-reducible to M. Moreover, A is T-incomplete.*

Proof. Choose a 1–1 recursive function f such that $A = \mathrm{rg}(f)$. Recall that the deficiency set D of A with respect to f is

$$[s : (\exists t > s)[f(t) < f(s)]\},$$

and that $A \leqslant_T D$. It is easily seen that $D \leqslant_{tt} A$. Then, by the hypothesis on M, $M \leqslant_{tt} D$. The set D is simple and semirecursive [15]. Hence, by a Theorem of Degtev [16], for each nonrecursive set S,

$$S \leqslant_{tt} D \;\Rightarrow\; D \leqslant_T S.$$

Therefore $D \leqslant_T M$ and $A \leqslant_T M$.

Degtev also has shown that minimal r.e. tt-degrees do not have high T-degree[5] [14]. Hence A is T-incomplete. $\square$

Theorem 3.2. *Let $\mathbf{I}$ be a Σ^0_3-ideal of $\mathbf{R}_{tt}$ consisting of T-incomplete tt-degrees only. Then $\mathbf{I}$ possesses an exact pair.*

Notation. As in [19], for $X \subseteq \omega$, $X^{[n]}$ denotes the set $\{\langle y,n\rangle : \langle y,n\rangle \in X\}$ and $X^{[\leqslant n]}$ denotes the set $X^{[0]} \cup \ldots \cup X^{[n]}$.

Main Lemma 3.3. *Let $(D_n)_{n\in\omega}$ be a uniformly r.e. sequence of sets and let*

$$D = \{\langle x, n\rangle : x \in D_n)\}.$$

Suppose that, for each e, the set $D^{[\leqslant e]}$ is T-incomplete. Then there exist r.e. sets B and C such that

$$(\forall n)[D_n \leqslant_{tt} B, C]$$

$$(\forall Z)[Z \leqslant_{tt} B, C \;\Rightarrow\; (\exists e)][Z \leqslant_{tt} D^{[\leqslant e]}]. \tag{3.1}$$

[5] Downey and Shore have extended this result: every r.e. set of minimal tt-degree is low_2.

Proof of Lemma 3.3. See Section 5. □

Proof of Theorem 3.2 (*using Lemma* 3.3). By a result of Yates [19, p. 253], there exists a u.r.e. sequence sequence $(D_n)_{n\in\omega}$ such that

$$\{W_e: \deg_{tt}(W_e) \in \mathbf{I}] = \{D_n: n \in \omega].$$

Apply Lemma 3 and let $\mathbf{b} = \deg_{tt}(B)$ and $\mathbf{c} = \deg_{tt}(C)$. Then $\mathbf{b},\mathbf{c}$ form an exact pair for $\mathbf{I}$. □

The strictly increasing sequence $(d_x)_{x\in\omega}$ was defined in Section 1.2, Recall that $B_x = (d_x, d_{x+1}]$.

Lemma 3.4 (Haught and Shore [10]). *Let R be any recursive equivalence relation. Then there exists a tt-reduction $[e]$ such that $(A)_R = [e]^{(A)_U}$ for each recursive equivalence relation $U, R \leqslant U$.*

Proof. For arbitrary numbers x, i, $1 \leqslant i \leqslant x$,

$$d_x + i \in (A)_R \iff (A)_U \models \bigvee_{1 \leqslant j \leqslant x \wedge iRj} d_x + j. \quad \square$$

We now introduce a property of an r.e. set A which can be viewed as a generalization of semirecursiveness. Let $(A_s)_{s\in\omega}$ be an enumeration of an r.e. set A. We say that A is *enumerated via blocks* if, for each s, either $A_{s+1} = A_s$ or, for some x such that $B_x \cap A_s = \emptyset$ and for some $z \in B_x$,

$$A_{s+1} = A_s \cup \{z\} \cup \bigcup \{B_y: x < y \wedge B_y \subseteq [0, s)\}.$$

Note that, for each x, $|B_x \cap A| \leqslant 1$ or $B_x \subseteq A$.

Lemma 3.5. *Let $(A_s)_{s\in\omega}$ be an enumeration of an r.e. set A via blocks. Then the following hold.*

(i) $A \leqslant_{wtt} (A)_0$
(ii) $\{x: B_x \subseteq A\} \leqslant_{tt} (A)_0$.

Proof (i) follows since $(A)_0 = \bigcup_s (A_s)_0$ and, for each y, s,

$$A_s | y \neq A_{s+1} | y \Rightarrow (A_s)_0 | y \neq (A_{s+1})_0 | y.$$

We show how to obtain a tt-reduction in (ii). Given x, w.l.o.g. assume that $|B_x| > 1$. Let s be the minimal number such that $B_x \subseteq [0, s)$. Then, since $(A)_0$ is the projection of A via the equivalence relation with only one equivalence class,

$$B_x \subseteq A \iff (\exists z)[z < x \wedge |B_z \cap A_s| = 0 \wedge d_{z+1} \in (A)_0]. \quad \square$$

The next lemma is needed for the transfer of the conditions derived in Section 2.2 to $\mathbf{R}_{tt}$.

Lemma 3.6. *Let $X \subseteq \omega_+$ be any infinite recursive set and let R be any recursive equivalence relation on ω_+. If a set A is enumerated via blocks, then*

$$(A)_{R \vee X^{(2)}} \equiv_{tt} (A)_R \oplus (A)_{X^{(2)}}.$$

Proof. By Lemma 3.4, $(A)_R \oplus (A)_{X^{(2)}} \leqslant_{tt} (A)_{R \vee X^{(2)}}$. We show how to obtain a tt-reduction of $(A)_{R \vee X^{(2)}}$ to $(A)_R \oplus (A)_{X^{(2)}}$. Given input $y = d_x + i$ $(1 \leqslant i \leqslant x)$, first ask whether $B_x \subseteq A$. By Lemma 5(ii) and since $(A)_0 \leqslant_{tt} (A)_R$, this question can be answered by applying a tt-reduction procedure to $(A)_R$ on input y. If the answer is yes, give 1 as output. Otherwise, $|B_x \cap A| \leqslant 1$. Distinguish two cases.

Case 1: $i \in \bar{X}$. Then

$$y \in (A)_{R \vee X^{(2)}} \Leftrightarrow y \in (A)_{X^{(2)}}.$$

Hence give 1 as output iff $y \in (A)_{X^{(2)}}$.

Case 2: $i \in X$.

Subcase 2.1: $y \notin (A)_R$. Then a fortiori $y \notin (A)_{R \vee X^{(2)}}$. Hence give 0 as output.

Subcase 2.2: $y \in (A)_R$. Since $|B_x \cap A| \leqslant 1$, there is a unique j_0, $0 \leqslant j_0 < x$, such that $i R j_0$ and $d_x + j_0 \in A$. Clearly, $y \in (A)_{R \vee X^{(2)}} \Leftrightarrow j_0 \in X$. But, since $y \in (A)_R, j_0$ is *not* in X iff

$$(\exists j)[0 \leqslant j < x \wedge i R j \wedge j \in \bar{X} \wedge d_x + j \in (A)_{X^{(2)}}] \tag{3.2}$$

(if $j \in \bar{X}$, then the last conjunct is equivalent to "$d_x + j \in A$"). Thus, ask (3.2) by applying a tt-reduction procedure to $(A)_{X^{(2)}}$ on input x and give 0 as output iff the answer is 1.

This describes a tt-reduction procedure as desired. □

Theorem 3.7. *Let $(R_n)_{n \in \omega}$ be a sequence of uniformly recursive equivalence relations such that $\Pi_\infty \subseteq \{R_n : n \in \omega\}$. Then there exists an r.e. set A which is enumerated via blocks such that*

(i) $\{\deg_{tt}((A)_P) : P \in \Pi_\infty\} \cup \{\mathbf{0}\}$ *is downward closed in the tt-degrees, and*

(ii) $(\forall n)(\forall m)[R_n \nleqslant R_m \Rightarrow (A)_{R_n} \nleqslant_{tt} (A)_{R_m}]$.

Proof. See Section 6. □

Corollary 3.8. *Let $(R_n)_{n \in \omega}$ be a sequence as above. Obtain the r.e. set A by the preceding Theorem, and let $\mathbf{a}_0 = \deg_{tt}((A)_0)$. Then there exist r.e. tt-degrees $\mathbf{b}, \mathbf{c}$ such that the map Φ defined by*

$$P \to \deg_{tt}((A)_P)$$

is an isomorphism between Π_∞ and $\{\mathbf{x} \in \mathbf{R}_{tt} : \mathbf{a}_0 \leqslant \mathbf{x} \leqslant \mathbf{b}, \mathbf{c}\}$. Moreover, the set

$$\{\mathbf{x} \in \mathbf{R}_{tt} : \mathbf{a}_0 \leqslant \mathbf{x} \leqslant \mathbf{b}, \mathbf{c}\} \cup \{\mathbf{0}\}$$

is downward closed in the set of all tt-degrees and has socle $\mathbf{a}_0$.

Proof. We first show that the set $X = \text{range}(\Phi) \cup \{\mathbf{0}\}$ has socle $\mathbf{a}_0$. By Lemma 4 and Theorem 3.7(ii), A is not recursive. Hence, by Lemma 5(i), $(A)_0$ is not recursive either. (In fact, we see that $X - \{0\}$ lies within one single wtt-degree. Compare this to Theorem 3.1!). Then, by Theorem 3.7(i) $\mathbf{a}_0$ is the socle of X.

Note that, for each P, Q, $(A)_P \oplus (A)_Q \leqslant_{tt} (A)_{P \vee Q}$. Hence, by Theorem 3.7(i), X is an ideal. Then X obviously is a Σ^0_3-ideal. Let $\mathbf{b}, \mathbf{c}$ be an exact pair for X. Then

$$X - \{\mathbf{0}\} = \{\mathbf{x} \in \mathbf{R}_{tt}: \mathbf{a}_0 \leqslant \mathbf{x} \leqslant \mathbf{b}, \mathbf{c}\}.$$

By Theorem 3.7(ii), the map Φ is an order embedding. □

Remark. In Section 2.1 it was shown that the set At_1 is definable in Π_∞. Then, since the range of Φ is definable with parameters in $\mathbf{R}_{tt}$, we can define the set $\Phi(\text{At}_1)$ in $\mathbf{R}_{tt}$ by a formula φ_{At_1}, using the parameters $\mathbf{a}_0$, $\mathbf{b}$ and $\mathbf{c}$.

4. The complexity of Th($\mathbf{R}_{tt}$)

Recall that, in Lemma 2.4, we obtained recursive equivalence relations $C_1, \ldots, C_6$ which enabled us to define the fixed standard model $(U, \oplus, \otimes)$ in the structure

$$(\Pi_\infty; 0, \leqslant, \vee, U, V, (\{P: P \leqslant C_i\})_{i=1,\ldots,6}). \tag{4.1}$$

Also recall that $G = \omega_+ - F$. The recursive equivalence relations S and Id were defined in (2.11). Let $(R_n)_{n\in\omega}$ be a list of uniformly recursive equivalence relations such that $\Pi_\infty \subseteq \{R_n: n \in \omega\}$. Moreover, suppose that the list includes $C_1, \ldots, C_6$, S, Id as well as, for each $P \in \Pi_\infty$, the recursive equivalence relations $P \vee F^{[2]}$ and $P \vee G^{(2]}$. Applying Theorem 3.7 to this list, we obtain an r.e. set A. This set will be kept fixed in Lemmas 4.1–4.3 below. As before, define the map $\Phi: \Pi_\infty \to \mathbf{R}_{tt}$ by $\Phi(P) = \deg_{tt}((A)_P)$. We use the following notational convention: if X denotes a recursive equivalence relation, then $\mathbf{x}$ denotes the tt-degree $\deg_{tt}((A)_X)$. Thus e.g. $\mathbf{e}_n = \deg_{tt}((A)_{E_n}) = \Phi(E_n)$. Note that, by this convention, we now have defined all the parameters in the special list (1.10) except for the three exact pairs. This will be done in Lemma 4.1. Lemma 4.2 satisfies (a) in Section 1, whereas Lemma 4.3 is concerned with (b2).

Lemma 4.1. *There exists r.e. tt-degrees* $\mathbf{b}, \mathbf{c}, \mathbf{u}_0, \mathbf{u}_1, \mathbf{v}_0, \mathbf{v}_1$ *such that the map* Φ *is an isomorphism between* (1) *and*

$$(\{\mathbf{x}: \mathbf{a}_0 \leqslant \mathbf{x} \leqslant \mathbf{b}, \mathbf{c}\}; \mathbf{0}, \leqslant, \vee, \{\mathbf{p}: \varphi_{\text{At}_1}(\mathbf{p}) \wedge \mathbf{p} \leqslant \mathbf{u}_0, \mathbf{u}_1\},$$
$$\{\mathbf{p}: \varphi_{\text{At}_1}(\mathbf{p}) \wedge \mathbf{p} \leqslant \mathbf{v}_0, \mathbf{v}_1\}, (\{\mathbf{p}: \mathbf{p} \leqslant \mathbf{c}_i\})_{i=1,\ldots,6}).$$

(*Here* $\mathbf{p}$ *ranges over* $\{\mathbf{x}: \mathbf{a}_0 \leqslant \mathbf{x} \leqslant \mathbf{b}, \mathbf{c}\}$. *The formula* φ_{At_1} *was defined after Corollary* 3.8.)

Proof. The ideal generated by $\Phi(U)$ possesses an exact pair $\mathbf{u}_0, \mathbf{u}_1$. Then, by the independence of the sequence $(\mathbf{e}_n)$, $\Phi(U) = \{\mathbf{p}: \varphi_{\mathrm{At}_1}(\mathbf{p}) \wedge \mathbf{p} \leqslant \mathbf{u}_0, \mathbf{u}_1\}$. In a similar way we obtain the tt-degrees $\mathbf{v}_0, \mathbf{v}_1$. □

We summarize the list of r.e. tt-degrees we are concerned with now:

$$\mathbf{a}_0, \mathbf{b}, \mathbf{c}, \mathbf{u}_0, \mathbf{u}_1, \mathbf{v}_0, \mathbf{v}_1,$$
$$\mathbf{c}_i (i = 1, \dots, 6), \tag{4.2}$$
$$\mathbf{f}, \mathbf{g}, \mathbf{s}, \mathbf{id}.$$

Lemma 4.2. *Three are formulas*

$$\varphi_{\mathrm{num}}(x; \bar{p}), \qquad \varphi_{\oplus}(x, y, z; \bar{p}) \quad \text{and} \quad \varphi_{\otimes}(x, y, z; \bar{p}) \tag{4.3}$$

in the language of u.s.l. which define a standard model of arithmetic in $\mathbf{R}_{\mathrm{tt}}$ *if the parameters* (4.2) *are substituted for* $\bar{p}$.

Proof. Immediate by Lemmas 2.4 and 4.1. Note that

$$\varphi_{\mathrm{num}}(x) \equiv \varphi_{\mathrm{At}_1}(x) \wedge x \leqslant \mathbf{u}_0, \mathbf{u}_1. \quad \square$$

In the following lemma, we give a precise formulation of the conditions (C1)–(C5) explained in Section 1. We verify these conditions for the special parameter list (4.2) The conditions (C4) and (C5) directly correspond to the Lemmas 2.5 and 2.6.

We let the variables $\mathbf{p}$, $\mathbf{q}_1$, $\mathbf{q}_2$ range over $\{\mathbf{x}: \mathbf{a}_0 \leqslant \mathbf{x} \leqslant \mathbf{b}, \mathbf{c}\}$ and the variables $\mathbf{k}, \mathbf{l}$ range over numbers in the standard model defined in Lemma 4.2. Moreover, the symbols $\dot{<}$ and $\dot{+}\,1$ denote the ordering of this standard model and its successor function.

Lemma 4.3. *The conditions* (C1)–(C5) *below hold for the special list* (4.2).

(C1) $\mathbf{0} < \mathbf{a}_0 \leqslant \mathbf{b}, \mathbf{c} \wedge (\forall \mathbf{x})[\mathbf{0} < \mathbf{x} \leqslant \mathbf{b}, \mathbf{c} \;\Rightarrow\; \mathbf{a}_0 \leqslant \mathbf{x}]$,

(C2) $(\forall \mathbf{k})[\mathbf{s}_\mathbf{k} = \sup\{\mathbf{l}: \mathbf{l} \,\dot{<}\, \mathbf{k}\}$ exists $\wedge\, \mathbf{s}_\mathbf{k} \leqslant \mathbf{b}, \mathbf{c}]$,

(C3) $(\forall \mathbf{k})(\forall \mathbf{l})[\mathbf{l} \leqslant \mathbf{s}_k \;\Rightarrow\; \mathbf{l} \,\dot{<}\, \mathbf{k}]$,

(C4) $(\forall \mathbf{k})(\forall \mathbf{p})[\mathbf{p} \leqslant \mathbf{s}_k \;\Leftrightarrow\; \mathbf{p} \leqslant (\mathbf{s}_k \vee \mathbf{f}) \wedge \mathbf{p} \leqslant \mathbf{g}]$,

(C5) $(\forall \mathbf{k})(\forall \mathbf{p})[\mathbf{p} \leqslant (\mathbf{s}_{k \dot{+} 1} \vee \mathbf{f}) \;\Leftrightarrow\; (\exists \mathbf{q}_1)(\exists \mathbf{q}_2)(\mathbf{q}_1 \leqslant (\mathbf{s}_k \vee \mathbf{f}) \;\wedge\; \mathbf{q}_1 \leqslant \mathbf{s}$
$\wedge\; \mathbf{q}_2 \leqslant (\mathbf{q}_1 \vee \mathbf{g}) \;\wedge\; \mathbf{q}_2 \leqslant \mathbf{id}$
$\wedge\; \mathbf{p} \leqslant \mathbf{q}_2 \wedge \mathbf{f}]$.

Proof. Condition (C1) is immediate from Corollary 3.8 and the definition of $\mathbf{b}, \mathbf{c}$. (C2) is trivial since the set $\{\mathbf{l}: \mathbf{l} \,\dot{<}\, \mathbf{k}\}$ is finite.

For (C3)–(C5), recall that k, l range over the standard model of arithmetic determined by the copy $(F, <)$ of $(\omega, <)$ (see Section 2). For the rest of this proof, we let

$\mathbf{k}, \mathbf{l}$ denote the tt-degrees corresponding to k, l. Since $\mathbf{l} = \Phi(E_l)$ for each l, by Corollary 3.8,

$$\mathbf{s}_k = \sup\{\Phi(E_l)\colon \mathbf{l} \dot{<} \mathbf{k}\} = \Phi(S_k).$$

To show (C3), suppose that $\mathbf{l} \leqslant \mathbf{s}_\mathbf{k}$. Since $\mathbf{l} = \Phi(E_l)$, this implies that $E_l \leqslant S_k$ and hence $\mathbf{l} \dot{\leqslant} \mathbf{k}$. Conditions (C4) and (C5) follow from the corresponding facts for equivalence relations by choice of the sequence (R_n) and Lemma 3.6. □

Theorem 4.4. *True first-order arithmetic is m-reducible to* $\mathrm{Th}(\mathbf{R}_{\mathrm{tt}})$.

Proof. Let $\psi(\bar{p})$ be the conjunction of (formulations in the language of u.s.l.) of the following conditions: (CAr) in Section 1, conditions (C1)–(C5) above, and (CSt) in Section 1. In (C1)–(C5), the variables $\mathbf{k}, \mathbf{l}$ now range over M, where $\mathbf{M}$ is the model of arithmetic defined in (CAr). The symbols $\dot{<}$ and $\dot{+}$ denote the ordering of M and its successor function. As explained in Section 1, we have to show that

(b.1) for an *arbitrary* list

$$\bar{\mathbf{p}} = \mathbf{a}_0,\ \mathbf{b},\ \mathbf{c},\ \mathbf{u}_0,\ \mathbf{u}_1,\ \mathbf{v}_0,\ \mathbf{v}_1;\ (\mathbf{c}_i)_{i=1\ \dots\ 6};\ \mathbf{f},\ \mathbf{g},\ \mathbf{s},\ \mathbf{id}.$$

$\psi(\bar{p})$ implies that the formulas (4.3) define a standard model of arithmetic in $\mathbf{R}_{\mathrm{tt}}$,
(b.2) ψ is satisfied by the special list of parameters (4.2).

Proof of (b.1). It is sufficient to verify in detail that the ideas used in Section 1 to motivate (C5) and (C4) work out. First suppose that some list $\bar{p}$ satisfies all the conditions except possibly (CSt). Let S be the standard part of $\mathbf{M}$ and let $\mathbf{I}$ be the ideal of $\mathbf{R}_{\mathrm{tt}}$ generated by S. We show that $\mathbf{I}$ satisfies the hypotheses of Theorem 3.2 and hence possesses an exact pair.

As before, the variable $\mathbf{p}$ ranges over $\{\mathbf{x}\colon \mathbf{a}_0 \leqslant \mathbf{x} \leqslant \mathbf{b}, \mathbf{c}\}$. Given $\mathbf{k} \in S$, let

$$\mathbf{I}_k = \{\mathbf{p}\colon \mathbf{p} \leqslant \mathbf{s}_\mathbf{k} \vee \mathbf{f}\} \cup \{\mathbf{0}\}.$$

Clearly, $\mathbf{I}_\mathbf{k}$ is an ideal. By (C5) and the fact that the set $\{\mathbf{x}\colon \mathbf{a}_0 \leqslant \mathbf{x} \leqslant \mathbf{b}, \mathbf{c}\}$ has a Σ^0_3-representation, the sequence of ideals $(\mathbf{I}_k)_{k \in S}$ is Σ^0_3 uniformly in k. Therefore the ideal

$$\bigcup\nolimits_{k \in S} \mathbf{I}_k$$

is Σ^0_3. By (C4),

$$\mathbf{I} = \{\mathbf{p}\colon (\exists \mathbf{k} \in S)[\mathbf{p} \leqslant \mathbf{s}_k]\} \cup \{\mathbf{0}\} = (\bigcup\nolimits_{k \in S} \mathbf{I}_k) \cap [\mathbf{0}, \mathbf{g}].$$

Hence $\mathbf{I}$ is a Σ^0_3-ideal as well. Moreover, by (C1) and (C2), $\mathbf{I}$ has socle $\mathbf{a}_0$. Thus all the hypotheses of Theorem 3.2 are satisfied.

Let $\mathbf{r}, \mathbf{s}$ be an exact pair for $\mathbf{I}$. Then $S = [\mathbf{l} \in M\colon \mathbf{l} \leqslant \mathbf{r}, \mathbf{s}\}$: if $\mathbf{l} \leqslant \mathbf{r}, \mathbf{s}$, then $\mathbf{l} \leqslant \mathbf{s}_\mathbf{k}$ for some $\mathbf{k} \in S$, whence $\mathbf{l} \in \mathbf{S}$ by (C3). Thus, if we add (CSt), the model M is standard. This shows (b.1).

Proof of (b.2). (CAr) and (CSt) are immediate. The other conditions were verified in Lemma 3.3. □

5. Proof of the Main Lemma 3.3

Main Lemma. *Let $(D_n)_{n\in\omega}$ be a uniformly r.e. sequence of sets and let*

$$D = \{\langle x, n\rangle : x \in D_n)\}.$$

Suppose that, for each e, the set $D^{[\leq e]}$ is T-incomplete. Then there exist r.e. sets B, C such that

$$(\forall n)[D_n \leqslant_{tt} B, C]$$

$$(\forall Z)[Z \leqslant_{tt} B, C \Rightarrow (\exists e)[Z \leqslant_{tt} D^{[\leq e]}]]. \tag{5.1}$$

We say that B, C form an exact pair for the sequence (D_n).

Notation. We define a uniform enumeration $([e]_{e\in\omega}$ of all (possibly partial) tt-reduction procedures: if $\{e\}(x)$ converges, $[e](x)$ denotes the result of this computation interpreted as a truth table, and $|[e](x)|$ is 1 + the maximum number occurring in this truth table. Moreover. $[e]^Y(x)$ is the Boolean value obtained from applying $[e](x)$ to the oracle Y. The approximation $[e]_s(x)$ is defined in the obvious way, and, if B is an r.e. set, we write $[e]^B(x)[s]$ for $[e]_s^{B_s}(x)$.

Proof. First recall the standard approach for exact pair constructions. This approach was used in exact pair constructions in the context of T-degrees (Spector/Kleene/Post, see [19]) and polynomial T-degrees [1], as well as to construct a particular sequence of r.e. tt-degrees with an exact pair [4] and to show that each Σ_3^0-ideal in the r.e. wtt-degrees possesses an exact pair [2].

One satisfies the infinitary coding requirements

$$P_n^X: \quad D^{[n]} =^* X^{[n]} \ (X = B, C)$$

and the requirements

$$Q_e: \quad Z = [e]^B = [e]^C \Rightarrow Z \leqslant_{tt} D^{[\leq e]}.$$

(rewritten appropriately when dealing with a different reducibility). The priority ordering of the requirements is $P_0^B > P_0^C > Q_0 > P_1^B > P_1^C > Q_1 > \dots$

Here, we will give a tree construction of an exact pair B, C for the sequence (D_n). This construction uses versions of the requirements above which are equipped with a guess at the outcome of the higher priority Q requirements. To understand the strategy for the Q requirements, as well as how the T-incompleteness of the sets $D^{[\leq e]}$ comes in and how the recovery of Z from $D^{[\leq e]}$ in (5.1) works, it is useful first to ignore the necessity of using a tree. Then we give the full tree construction which is needed to resolve the conflicts between a Q requirement and the lower priority requirements.

In satisfying a requirement Q_e, the basic idea (used in [4, 19] is to try to make the antecedent $[e]^B = [e]^C$ false. Given a stage number s, let

$$l(e, s) = \max [x\colon (\forall y < x)[[e]^B(y) = [e]^C(y)[s]]\}.$$

If $l(e, s) > x$ and there is some finite set F such that

$$[e]^{B \cup F}(x) \neq [e]^C(x) \quad \text{and} \quad F \cap \omega^{[\leqslant e]} = \emptyset, \tag{5.2}$$

then enumerating F into B creates a disagreement between $[e]^B = [e]^C$ and does not violate the higher priority coding requirements. If it is possible to restrain B and C below the use $|[e](x)|$ and we eventually enumerate F into B, then we have successfully diagonalized. Since we are considering tt-reductions, the existence of a set F satisfying (5.2) at a given stage can be effectively determined.

The strategy for the requirements P_n^X is standard: if $y \in D_{n,s}$ and y is not smaller than the restraint imposed by any higher priority Q requirement, then enumerate $\langle y, n\rangle$ into X. Because of the active strategy we pursue for the Q requirements, we can carry out the coding into B and into C at the same stage (unlike e.g. in the construction in [2]). Thus we rewrite the coding requirements as

$$P_n\colon D^{[n]} =^* B^{[n]} =^* C^{[n]};$$

the priority ordering is now $P_0 > Q_0 > P_1 > \ldots$

A Q_e requirement may be infinitary because infinitely often a diagonalization carried out by Q_e can be destroyed by a higher priority coding requirement (in this case we will be able to argue that $Z \leqslant_{tt} D^{[\leqslant e]}$). Therefore, to leave enough room for the lower priority coding requirements, we must take sure that $\underline{\lim}_s r(e, s)$ is finite, where $r(e, s)$ is the restraint imposed by Q_e at the end of stage s. To achieve this, we let Q_e appoint and cancel followers x_m^e $(m \geqslant 0)$. Q_e is allowed to diagonalize through a follower x_m^e only if the number m has been enumerated into the creative set K. Using the T-incompleteness of the set $D^{[\leqslant e]}$, it will then be possible to show that

$$\text{for some } m, \text{ the follower } x_m^e \text{ is undefined at the end of infinitely many stages.}^6 \tag{5.3}$$

Choosing a minimal such number m, we will be able to infer that $\underline{\lim}_s(e, s)$ is finite. We now describe the strategy for Q_e in more detail. Formally, we let $x_{-1}^e = 0$. With each follower $x = x_m^e$ of Q_e $(m \geqslant 0)$, we associate a restraint $|[e](x)|$ and a finite set F_m^e with the property (5.2). The restraint $r(e, s)$ imposed by Q_e at the end of stage s is the maximum restraint associated with all these followers. We write $x_m^e[s]\downarrow$ if x_m^e is defined at the end of stage s and $x_m^e[s]\uparrow$ otherwise. At stage s, the requirement Q_e can do the following two things:

[6] The idea to use T-incompleteness in this way is due to K. Ambos-Spies.

(a) either cancel followers or appoint a follower,

(b) act through a follower x^e_m via enumerating F^e_m into B.

Regarding (a). A follower is cancelled if the associated restraint has been violated since the stage when it was appointed.

We appoint x as x^e_m if m is minimal such that x^e_m is undefined, $l(e, s) > x$, $x > x^e_{m-1}$, and there is a set $F = F^e_m$ satisfying (5.2) as well as $\min(F) \geqslant |[e](x^e_{m-1})|$ (if $m \neq 0$). Thus we make sure that an action through x^e_m does not lead to the cancellation of any follower x^e_n, $n < m$.

Regarding (b). Q_e can act through x^e_m only if $m \in K_s$ and

$$Q_e \text{ has not acted through any value of } x^e_m \text{ at an earlier stage.} \tag{5.4}$$

We do not consider such an action as a violation of the restraint put on for the sake of x^e_m. The argument for (5.3) is as follows. Assume for a contradiction that, for each m,

$$(\exists s)(\forall t \geqslant s)[x^e_m[t]\downarrow]. \tag{5.5}$$

Then $\lim_s l(e, s) = \infty$, whence $[e]^B = [e]^C$. Assuming $D^{[n]} =^* B^{[n]} =^* C^{[n]}$ for each $n \leqslant e$, we show that $K \leqslant_T D^{[\leqslant e]}$. Given m, to determine whether $m \in K$, recursively in $D^{[e]}$ compute a stage number t such that $x = x^e_m[t]\downarrow$ and the sets $B^{[\leqslant e]}$ and $C^{[\leqslant e]}$ have settled down on the interval $[0, |[e](x)|)$. Then x^e_m remains uncancelled, and $m \in K \Leftrightarrow m \in K_t$: if m were enumerated into K after stage t, we would diagonalize successfully, contrary to the fact that $[e]^B = [e]^C$.

Let m be the minimal number such that (5.3) holds via m. We call m the *outcome* of requirement Q_e. By (5.3), $\lim_s r(e, s) = |[e](x^e_{m-1})|$.

We use (5.3) not only to show that $\lim_s r(e, s)$ is finite, but also for the recovery of Z from $D^{[\leqslant e]}$ in (5.1). Essentially, we show in Lemma 2 that

$$(\forall x > x^e_{m-1})[Z(x) = [e]^{\tilde{B}}(x)],$$

where $\tilde{B} = B \cup \omega^{[>e]}$ and $x^e_{m-1} = \lim_s x^e_{m-1}[s]$: if $x > x^e_{m-1}$ and $[e]^{\tilde{B}}(x) \neq [e]^C$ $(= Z(x))$, then at a stage when B, C have settled down on $[0, |[e](x)|)$ and x^e_m is undefined, the set $F = [0, |[e](x)|)^{[>e]}$ appears as a suitable choice for a set F^e_m associated with follower $x^e_m = x$. There is no reason to cancel x^e_m at any later stage, contrary to the definition of m.

We now consider the effect of a requirement Q_e on lower priority requirements. Since Q_e can act infinitely often, Q_e may violate a coding requirement P_i, $i > e$, infinitely often by enumerating elements into the set $B^{[i]}$. Also the argument to refute (5.5) ignores the influence of the higher priority Q requirements, which may prevent successful diagonalization by Q_e. To resolve these conflicts, we introduce versions P_γ, Q_γ of the requirements P_i, Q_i, where γ is on the tree $T = \omega^{[<\omega]}$ and $i = |\gamma|$. (We use standard notation for trees as in [19, p. 301]; variables α, β, γ range over strings in T.) If $e < i$ and $\beta = \gamma | e$, then the requirements P_γ, Q_γ guess that the outcome of the requirement Q_β is $\gamma(e)$. If Q_β wants to appoint a follower x^β_n, then the associated set F^β_n must not interfere with those requirements P_γ, Q_γ, $\beta \subset \gamma$, which assume that the outcome of Q_β is $< n$. By (5.4), after some stage, Q_β ceases to act through any follower x_k^β, $k < n$. Thus the requirements P_γ, Q_γ cannot be violated infinitely often by Q_β.

We fix some numbering $\gamma \to n(\gamma)$ of strings and write $X^{[\gamma]}$ instead of $X^{[n(\gamma)]}$. We also use the notation $X^{[<\gamma]}$, etc. in the obvious sense. The requirements P_γ now codes the set D_i, $i = |\gamma|$, into $B^{[\gamma]}$ and $C^{[\gamma]}$.

Stage s of the construction consists of substages $[s, i]$, $0 \leqslant i \leqslant s$. We write $[s, i] < [s', i']$ if the first substage is carried out before the second one.

We adapt the notation and terminology used above for the tree construction. Thus, for $\beta \in T$, requirement Q_β uses followers x_n^β. The value of x_n^β after substage $[s, i]$ is denoted $x_n^\beta[s, i]$. *The restraint* $r(\beta,[s, i])$ *imposed by* Q_β *at the end of substage* $[s, i]$ *is*

$$\max\{|[e](x_n^\beta[s, i])|: x_n^\beta[s, i]\downarrow\}.$$

For each γ, we define

$$R(\gamma,[s, i]) = \max\{r(\beta,[s, i]): \beta < \gamma\}.$$

We now give a rough description of stage s of the construction. In substage $[s, 0]$, we define $\delta_{[s,0]} = \lambda$. Now suppose that substage [s,e] has been carried out and $\beta := \delta_{[s,e]}$ has been defined. Stage s is called a β-stage.

In step 1 of stage $[s,e + 1]$, we let P_β do its coding. If we have been to the left of β since the last β-stage, in step 2 Q_β cancels all its followers. Otherwise, Q_β carries out (a) and (b) above in steps 2 and 3. The idea sketched above to avoid infinite injury of a requirement through Q_β is spelled out in this form: if Q_β appoints a follower $x = x_n^\beta$ and an associated set F, then it must be the case that

$$F \cap \omega^{[\gamma]} = \emptyset \quad \text{for each } \gamma < \beta * n,$$

$$\min(F) \geqslant R(\beta * n).$$

We then let $\delta_{s,e+1} = \delta_{s,e} * k$, where k is maximal such that x_{k-1}^β is now defined. If $e + 1 < s$, we go to the next substage. Otherwise we let $\delta_s = \delta_{s,e+1}$ and go to the next stage.

The verification proceeds in this way: in (i) and (ii) of Lemma 1, it is proved that, for each e, there exists a string α, $|\alpha| = e$ which is the $\leqslant_L$-least such that there are infinitely many α-stages. Thus there is a leftmost path f visited infinitely often. The string α is determined by the property that, for each $\gamma \subset \alpha$, $|\gamma| = i$, $\alpha(i)$ is the outcome of requirement Q_γ. The proof that a string with the latter property exists uses the argument for (5.3). As usual in the verification of tree constructions, after this it is shown that the requirements P_α and Q_α, where α is as in (i), are satisfied. This is carried out in (iv) of Lemma 5.1 and Lemma 5.2.

Construction

For each $\gamma \in T$, define $x_{-1}^\gamma = 0$.

Stage 0: Define $B_0 = C_0 = \emptyset$ and $\delta_0 = \lambda$ (the empty string).

Substage $[0, 0]$. For each $\gamma \in T$, define $x_{-1}^\gamma[0, 0] = 0$. (These 'followers" are never cancelled.)

Stage s ($s > 0$): Go through the substages $[s, i]$, $0 \leqslant i \leqslant s$.

Substage $[s, 0]$: Define $B_{[s,0]} = B_{s-1}$, $C_{[s,0]} = C_{s-1}$ and $\delta_{[s,0]} = \lambda$.

Substage $[s, e+1]$: Let $\beta = \delta_{s,e}$. Carry out the following four steps.

Step 1: (P_β *codes the set* D_e *into* B *and* C). For each $x < s$, if $x \geqslant R(\beta, [s, e])$ and $x = \langle z, n(\beta)\rangle$ for $z \in D_{e,s}$, then enumerate x into B and into C ($n(\beta)$ is the code number for string β).

Step 2 (Q_β *cancels followers*). Let s' be the greatest β-stage $< s$. If there is a stage t, $s' < t < s$ such that $\delta_t <_L \beta$, then cancel all followers of Q_β. Otherwise, if there is an m such that $x^\beta_m[s', e+1]$ is defined and, at some substage $[t, i+1]$ such that $[s', e+1] \leqslant [t, i+1] \leqslant [s, e]$, the restraint $r = |[e](x^\beta_m[s', e+1])|$ associated with x^β_m has been violated (i.e. numbers $< r$ have been enumerated into B or C as a result of an action which was not "Q_β acts through x^β_n"), then choose the minimal such m and cancel all followers x^β_n, $n \geqslant m$.

Step 3 (Q_β *appoints a new follower*). If some follower of Q_β has been cancelled during Step 2, do nothing. Otherwise let m be maximal such that $x^\beta_{m-1}[s, e]\downarrow$. If there exists an $x > x^\beta_{m-1}$ such that, for some set $F \subseteq [0, |[e](x)|)$,

$$l(e, s) > x, \tag{5.6}$$

$$[e]^{B \cup F}(x)[s, e] \neq [e]^C(x)[s, e], \tag{5.7}$$

$$\min(F) \geqslant R(\beta * m), \tag{5.8}$$

$$F \cap \omega^{[\gamma]} = \emptyset \quad \text{for each } \gamma < \beta * m, \tag{5.9}$$

then choose a minimal such x. (Actually, it suffices to consider the set $F = [R(\beta * m), s) \cap \omega^{[\geqslant \beta * m]}$, see Lemma 5.2). Appoint $x^\beta_m[s, e+1] := x$ as a new follower of Q_β and appoint a set F as above as $F^\beta_m[s, e+1]$.

Step 4 (Q_β *acts through a follower*). If there exists an $n \geqslant 0$ such that x^β_n is still defined, $n \in K_s$, and there was no action of Q_β through (any value of) x^β_n at any stage before, then choose a minimal such number n and let Q_β act through x^β_n by defining

$$B_{[s,e+1]} = B_{[s,e]} \cup F^\beta_n[s, e].$$

Let m be maximal such that x^β_{m-1} is now defined, and let $\delta_{s,e+1} = \delta_{s,e} * m$. If $e + 1 < s$, go to the next substage. Otherwise define $\delta_s = \delta_{s,e+1}$, $B_s = B_{s,e+1}$, $C_s = C_{s,e+1}$ and go to stage $s + 1$.

Verification

Lemma 5.1. *Let* $e \geqslant 0$.

(i) *There exists* $\alpha \in T$, $|\alpha| = e$, *such that*

$$(\forall i < e)[\alpha(i) = \max\{m: (\exists s)(\forall t \geqslant s)[x^{\alpha|i}_{m-1}[t, i+1]\downarrow]\}].$$

Let α *be as in* (i).

(ii) $\alpha = \underline{\lim}_s \delta_s | e$ *in the sense that*

$$(\text{a.e. } s)\,[\alpha \leqslant \delta_s] \wedge (\exists^\infty s)[\alpha \subseteq \delta_s].$$

In particular, α is unique.
Let s_α be the first α-stage with the property that

$$(\forall s \geqslant s_\alpha) \quad [\alpha \leqslant \delta_s \wedge \tag{5.10}$$

$$(\forall \gamma)(\forall k)[\gamma * k \subseteq \alpha \;\Rightarrow\; x^{\gamma}_{k-1}[s, |\gamma| + 1]\downarrow \wedge \tag{5.11}$$

$$Q_\gamma \text{ does not act at stage } s \text{ through any follower } x^{\gamma}_{n},\ n \leqslant k]]. \tag{5.12}$$

Recall that $r(\beta, [s, i]) = \max(|[e](x^{\beta}_{n}[s, i])|\colon x^{\beta}_{n}[s, i]\downarrow\}$ *and* $R(\gamma, [s, i]) = \max\{r(\beta, [s, i])\colon \beta < \gamma\}$. *In the following let*

$$R(\gamma) = \underline{\lim}_{[s,i]} R(\gamma, [s, i]).$$

(iii) *Let s be any α-stage $\geqslant s_\alpha$. Then*

$$R(\alpha) = R(\alpha, [s, e + 1]). \tag{5.13}$$

Moreover, if $e > 0$ and s' is the first $\alpha|(e - 1)$-stage $> s$, then

$$R(\alpha) = R(\alpha, [s', e]). \tag{5.14}$$

(iv) *If $\gamma \subseteq \alpha$ and $i = |\gamma|$, then for $X = B, C$,*

$$X^{[\gamma]} = X^{[\gamma]}_{s_\alpha} \cup \{\langle z, n(\gamma)\rangle\colon z \in D_i \wedge \langle z, n(\gamma)\rangle \geqslant R(\gamma)\}.. \tag{5.15}$$

Remarks. 1. The stage s_α defined in (5.10) exists by (i), (ii) and the fact that a requirement Q_γ acts at most once through a follower x^{γ}_{n} (no matter what its value is).

2. Since we do not cancel and redefine a follower at the same stage, for each string $\gamma * k$ in (5.10), $x^{\gamma}_{k-1}[s_\alpha, |\gamma| + 1]$ is the final value of this follower. We denote this final value by x^{γ}_{k-1}.

3. We need (5.13) for the proof of (iv), namely to show that the coding of D_{e+1} into $B^{[\alpha]}$ and $C^{[\alpha]}$ (carried out at substage $[s, e + 1]$ of α-stages s) works. (5.14) is needed to give requirement $Q_{\alpha|e-1}$ sufficiently many chances to appoint follower $x^{\alpha|e-1}_{m}$.

Proof of Lemma 5.1. The proof is by induction on e. If $e = 0$, we let α be the empty string. Then (i) and (ii) hold vacuously. Moreover, for each substage $[s, i]$, by definition $R(\alpha, [s, i]) = 0$. This implies (iii) and (iv).

Now suppose the lemma is true for e via the (uniquely determined) string β. Let s_β be the β-stage defined as in (5.10). We show the Lemma for $e + 1$.

Proof of (i). By the inductive hypothesis for (i) and uniqueness of β, the desired string α satisfying (i) must have the form $\beta * m$ for some m. Thus, assume for a contradiction that for each $m \geqslant 0$, $\beta * m$ fails to meet (*i*). Then, for each m, there is a stage s such that

$$(\forall t \geqslant s)[x^{\beta}_{m-1}[t, e + 1]\downarrow = x^{\beta}_{m-1}[s, e + 1]].$$

Hence, by (5.6), $\lim_s l(e, s) = \infty$ and therefore

$$[e]^B = [e]^C. \tag{5.16}$$

Recursively in $D^{[\leqslant e]}$, we will compute β-stages

$$s_\beta = t_0 < t_1 < \ldots$$

with the property that

$$m \in K \iff m \in K_{t_{m-1}}. \tag{5.17}$$

This immediately implies that K is T-reducible to $D^{[\leqslant e]}$, contrary to our assumption on the sequence $(D_i)_{i\in\omega}$.

Technically, we define the sequence $(t_m)_{m\in\omega}$ in such a way that, for each m,

for every $t \geqslant t_m$, $x^\beta_m[t, e+1]\downarrow = x^\beta_m[t_m, e+1]$ (in particular, the restraint $r = |[e](x^\beta_m)[t_m, e+1]|$ imposed for the sake of x^β_m is not violated at any stage $t \geqslant t_m$), and (5.18)

the requirement Q_β does not act through x^β_m at any stage $t \geqslant t_m$. (5.19)

Clearly, (5.18) and (5.19) imply (5.17): if $t \geqslant t_m$ and $m \in K_t - K_{t-1}$, then, by (5.18) and definition of s_β, $x^\beta_m[t_m, e+1] = x^\beta_m[t', e]$, where t' is the first β-stage $\geqslant t$. Hence Q_β acts at stage t' through some follower x^β_n, $n \leqslant m$, contrary to (5.19) for n.

Let $t_0 = s_\beta$, and, for $m > 0$ let t_m be the first β-stage $t > t_{m-1}$ such that

$$\begin{aligned} &x := x^\beta_m[t, e+1] \text{ is defined and, where } r = |[e](x)|, \text{ for } X = B, C \\ &(\forall \gamma \subseteq \beta)(\forall y < r)[y \in \omega^{[\gamma]} \Rightarrow [y \in X \iff y \in X_{s_\beta} \vee (y \geqslant R(\gamma) \\ &\wedge y \in \{\langle z, n(\gamma)\rangle : z \in D_{|\gamma|, t}\})]]. \end{aligned} \tag{5.20}$$

Such a stage t exists by the inductive hypothesis for (iv) the definition of s_β and the fact that x^β_m reaches a final value. Clearly, the sequence (t_m) defined in that way is recursive in $D^{[\leqslant e]}$.

By induction on m, we now prove (5.18) and (5.19). Suppose these two conditions hold for all $n < m$. First, assume for a contradiction that (5.18) fails. Then there is a substage $[t, i+1] \geqslant [t_m, e+1]$ such that

$x = x^\beta_m(t, i]\downarrow = x^\beta_m[t_m, e+1]$ and at $[t, i+1]$ for the first time a requirement P_γ or Q_γ acts and causes a change of B or C below $r = |[e](x)|$, where this action is not an action of Q_β through x^β_m.

Since $\beta \leqslant \delta_t$ and δ_t is a γ-stage, it is not the case that $\gamma <_L \beta$. Thus

$$\gamma \subset \beta \ \vee \ \gamma = \beta \ \vee \ \beta < \gamma. \tag{5.21}$$

First we show that the restraint r cannot be violated by an action of P_γ after substage $[t_m, e+1]$. If $\gamma \subseteq \beta$, then this is implied by (5.20); if $\beta < \gamma$, then an action of P_γ does not violate the restraint by construction (see Step 1).

Thus Q_γ acts at $[t, i+1]$ through some follower x^γ_k, thereby violating the restraint r. By the definition of s_β,

$$\gamma \subset \beta \Rightarrow k > \beta(i) \quad (\text{where } i = |\gamma|). \tag{5.22}$$

Together with (5.21), this implies that $\beta < \gamma * k$. Thus, by (5.8), when $x_k^\gamma(t, i+1]$ was appointed at a substage $[t_\gamma, i+1]$, the restraint imposed by β at that substage was respected (i.e. $\min(F_k^\gamma[t_\gamma, i+1]) \geqslant r(\beta, [t_\gamma, i+1])$). It now suffices to show that the current value of x_m^β already existed at $[t_\gamma, i+1]$, namely that

$$x_m^\beta[t_\gamma, i+1]\downarrow = x; \tag{5.23}$$

then $\min(F_k^\gamma[t_\gamma, i+1]) \geqslant r$ and the action of Q_γ does not violate the restraint r, contradiction.

Assume that (5.23) fails. Then x_m^β is undefined at the end of some substage $[t_\beta, e+1]$, where $[t_\gamma, i] \leqslant [t_\beta, e+1] < [t_m, e+1]$. Thus t_β is a $\beta * n$-stage for some $n \leqslant m$. We show that

$$x_k^\gamma[t_\beta, i+1] \text{ is undefined,} \tag{5.24}$$

contrary to the choice of t_γ. We distinguish four cases.

Case 1: $\gamma \subset \beta$. Then (5.24) follows from (5.22).

Case 2: $\gamma = \beta$. Since $t_m > t_{m-1}$, by the inductive hypothesis for (5.19), $k \geqslant m$. Moreover, $k \neq m$ since Q_γ does not act through x_m^β. Now (5.24) follows from the fact that t_β is a $\beta * n$-stage for some $n \leqslant m$.

Case 3: $\beta <_L \gamma$. Then (5.24) is immediate, since in Step 2 of substage $[t_\beta, i+1]$, all the followers of Q_γ are cancelled.

Case 4: $\beta \subset \gamma$. Since $x_m^\beta[t, i]$ is defined (by minimality of $[t, i]$), $\beta * n \subseteq \gamma$ for some $n \geqslant m$. Therefore, at stage t_β we are to the left of γ and, again, all the followers of Q_γ are cancelled.

This proves (5.24), (5.23) and hence (5.18) for m.

Now assume that (5.19) fails for m. Then Q_β acts through x_m^β at some β-stage $> t_m$. By (5.18), we diagonalize successfully, whence $[e]^B \neq [e]^C$. This contradicts (5.16).

For the remainder of the proof of Lemma 1, let m be the minimal number such that $(\exists^\infty s)[x_m^\beta[s, e+1]\uparrow]$, and let $\alpha = \beta * m$. It is now immediate that (i) holds via α.

Proof of (ii). By definition of m, we can choose $s_0 \geqslant s_\beta$ such that

$$(\forall s \geqslant s_0)[x_{m-1}^\beta \downarrow [s, e+1]].$$

Then $(\forall s \geqslant s_0)[\alpha \leqslant \delta_s]$. Moreover, at each β-stage $s \geqslant s_0$ where $x_m^\beta[s, e+1]$ is undefined, we have $\alpha \subseteq \delta_s$. This proves (ii).

The stage s_α is defined in (5.10).

Proof of (iii). Let M be the maximum restraint imposed by any requirement Q_γ, $\gamma <_L \alpha$, and let R be the maximum of M and all the restraints $r(\gamma, [s_\alpha, e+1])$, where $\gamma \subset \alpha$. By definition of s_α, all the followers of any requirement Q_γ, $\gamma <_L \alpha$ or $\gamma \subset \alpha$ present at $[s_\alpha, e+1]$ remain uncancelled. Hence $R(\alpha, [s, i+1]) \geqslant R$ for each substage $[s, i+1] \geqslant [s_\alpha, e+1]$. Moreover, if s is an α-stage, then at the end of substage $[s, e+1]$ there are no followers of such a requirement Q_γ besides the followers present at $[s_\alpha, e+1]$. Therefore $R(\alpha, [s, e+1]) = R$. Since there are infinitely many α-stages, this shows that $R(\alpha) = R$, whence (5.13).

If $e > 0$ and s' is the first $\alpha|(e-1)$ stage following an α-stage $s \geqslant s_\alpha$, then the requirement Q_β possesses the same followers at the end of substages $[s', e]$ and $[s, e+1]$, since the first opportunity for this requirement to cancel or appoint followers is at substage $[s', e+1]$. Hence

$$R(\alpha, [s', e]) = R(\alpha, [s, e+1]) = R.$$

This proves (5.14).

Proof of (iv). By inductive hypothesis, we only have to prove (5.15) for α. To do so, it suffices to show that for each α-stage $s \geqslant s_\alpha$ and for $X = B, C$

$$X_s^{[\alpha]} = X_{s_\alpha}^{[\alpha]} \cup \{\langle z, n(\alpha)\rangle : z \in D_{e+1,s} \wedge \langle z, n(\alpha)\rangle \geqslant R(\alpha)\}. \tag{5.25}$$

Clearly, (5.25) holds for $s = s_\alpha$. Now, by (5.12) in the definition of s_α, if a requirement Q_γ acts through x_k^γ at any substage $[t, i+1] \geqslant [s_\alpha, e+1]$, then $\alpha < \gamma * k$. Hence, by (5.9), the action of Q_γ does not result in an enumeration of numbers into the set $X^{[\alpha]}$. Therefore, only the coding of requirement P_α can cause numbers to be enumerated into $X^{[\alpha]}$. If

$$x = \langle z, n(\alpha)\rangle < R(\alpha),$$

then x will not be enumerated into B or C via coding of D_{e+1} after stage s_α. Thus, (5.25) for $s = s_\alpha$ implies (5.25) for each α-stage $s \geqslant s_\alpha$. □

Lemma 5.2. *Suppose that $Z = [e]^B = [e]^C$. Then Z is tt-reducible to $D^{[\leqslant e]}$.*

Proof. Let $\alpha = \beta * m$ be the unique string of length $e+1$ satisfying (i) of Lemma 5.1, and define s_α as in (5.10). By Lemma 5.1(i), whenever a follower x_m^β of Q_β is appointed at some β-stage $\geqslant s_\alpha$, then it must be cancelled at some later stage. We use this fact to show that

$$(\forall x > x_{m-1}^\beta)[Z(x) = [e]^{\tilde{B}}(x)], \tag{5.26}$$

where the set $\tilde{B}$ is defined by

$$\tilde{B} = (\omega^{[<\alpha]} \cap B) \cup (\omega^{[\geqslant \alpha]} \cap (B_{s_0} \cup [R(\alpha), \infty))),$$

and $s_0 \geqslant s_\alpha$ is a stage s such that B has settled down on $[0, R(\alpha))$ at the end of stage s. Note that $\tilde{B} \leqslant_m D^{[\geqslant e]}$, since $\tilde{B}^{[\geqslant \alpha]}$ is recursive, $\tilde{B}^{[\gamma]}$ is finite for each $\gamma <_L \alpha$, and $\{z : \langle z, \gamma\rangle \in \tilde{B}^{[\gamma]}\} =^* D_{|\gamma|}$ for each $\gamma \subset \alpha$. Therefore (26) implies that Z is tt-reducible to $D^{[\leqslant e]}$.

To show (5.26), let $x > x_{m-1}^\beta$ be arbitrary and choose $s_1 \geqslant s_0$ such that

$$(\forall s \geqslant s_1)[l(e, s) > x]$$

and, for $X = B, C$

$$X_{s_1} \cap [0, |[e](x)|) = X \cap [0, |[e](x)|)].$$

Then X has settled down on each interval $[0, |[e](z)|)$, $z \leqslant x$. After stage s_1 we never appoint any z, $s^{\beta}_{m-1} < z \leqslant x$ as a follower x^{β}_m of Q_{β}: otherwise there would be no reason to cancel x^{β}_m at any later stage, contrary to (i) of Lemma 5.1.

Let $F = [0, |[e](y)| \cap \{z \in \omega^{[\geqslant \alpha]}: z \geqslant R(\alpha)\}$. By choice of s_0,

$$(B \cup F) \cap [0, |[e](z)|) = \tilde{B} \cap [0, |[e](z)|).$$

Thus, for (5.26), it suffices to show that

$$Z(x) = [e]^{B \cup F}(x). \tag{5.27}$$

Let s be any α-stage $> s_1$ and let s' be the least β-stage $> s$. We demonstrate that, unless (5.27) holds, at stage s', x and F would be a suitable choice as a follower x^{β}_m of Q^{β}_m of Q_{β} and an associated set F^{β}_m. First (5.6) and (5.9) are satisfied by choice of s_1 and F. Second (5.8) holds since by (5.14) in (iii) of Lemma 5.1, $R(\alpha, [s', e]) = R(\alpha)$. Finally, if (5.27) fails, then, since $l(e, s) > x$.

$$[e]^{B \cup F}(x)[s', e] \neq [e]^{C}(x)[s', e].$$

Hence (5.7) is also satisfied and we appoint some z, $x^{\beta}_{m-1} < z \leqslant x$ as new follower x^{β}_m of Q_{β}. As mentioned above, this is impossible. Hence (5.27) holds, which shows (5.26). This concludes the proof of Lemma 5.2 and of the Main Lemma 3.3. □

6. Proof of Theorem 3.7

Notation and Definitions. Strings $\sigma, \tau \in 2^{<\omega}$ are *compatible* if $\sigma \subseteq \tau$ of $\tau \subseteq \sigma$. For $X \subseteq \omega$, we say that X extends σ, written $\sigma \subseteq X$, if $\sigma = X \restriction n$, where $n = |\sigma|$. If R is a recursive equivalence relation R on ω_+ and $|\sigma| = d_z + 1$ for some z, define a string $(\sigma)_R$ of the same length (the projection of σ via R) in a way corresponding to the definition of $(A)_R$ in Section 1.

Let $e > 0$. As in [10] we define a string $[e]^{\sigma}$ of length $\leqslant |\sigma|$ as follows: $[e]^{\sigma}(x)$ is defined if and only if

$$x < |\sigma| \wedge \{e\}_{|\sigma|}(x) \text{ converges } \wedge \ (\forall x' \leqslant x)|[e](x')| < |\sigma|].$$

In this case, we let $[e]^{\sigma}(x)[e]^{\{n:\, \sigma(n) = 1\}}(x)$. By this definition,

$$|\sigma| = |\sigma'| \ \Rightarrow \ |[e]^{\sigma}| = |[e]^{\sigma'}|.$$

Moreover,

$$\sigma \subseteq \tau \ \Rightarrow \ [e]^{\sigma} \subseteq [e]^{\tau} \quad \text{and} \quad \sigma \subseteq X \ \Rightarrow \ [e]^{\sigma} \subseteq [e]^{X}.$$

Let $[e]$ be any tt-reduction. We say that strings σ, τ $[e]$-*split* if $[e]^\sigma$ and $[e]^\tau$ are not compatible.

A *tree* is a total function

$$T: \{0^m: m \geq 0\} \cup \{0^m * j: 1 \leq j \leq m + n(T)\} \to 2^{<\omega}$$

(where $n(T)$ is a fixed positive integer) such that, for σ, $\sigma' \in \mathrm{dom}(T)$,

$$\sigma \subset \sigma' \;\Leftrightarrow\; T(\sigma) \subset T(\sigma').$$

We call the strings $T(0^m * j)$, $0 \leq j \leq m + n(T)$ the *branches on level m* of T.

Let T_1, T_2 be trees. T_2 is a *subtree* of T_1 if

$T_1(0) \subseteq T_2(\lambda)$ (where λ is the empty string) and

$$(\forall k)(\exists k')[T_2(0^k) = T_1(0^{k'}) \wedge k + n(T_2) \leq k' + n(T_1) \wedge$$
$$(\forall j)[1 \leq j \leq k + n(T_2) \to T_2(0^k * j) = T_1(0^{k'} * j)]]. \tag{6.1}$$

A subtree T_2 of T_1 is obviously determined by $n(T_2)$ and the values of T_2 on the set $\{0^m: m \geq 0\}$.

We restate Theorem 3.7.

Theorem. *Let $(R_n)_{n\in\omega}$ be a sequence of uniformly recursive equivalence relations on ω_+ such that $\Pi_\infty \subseteq \{R_n : n \in \omega\}$. Then there exists an r.e. set A which is enumerated via blocks such that*

(i) $\{\deg_{tt}((A)_P): P \in \Pi_\infty\} \cup \{0\}$ *is downward closed in the tt-degrees, and*

(ii) $(\forall n)(\forall m)[R_n \not\leq R_m \;\Rightarrow\; (A)_{R_n} \not\leq_{tt} (A)_{R_m}]$.

Proof. The set A is constructed in stages. As in [10], at Stage $s > 0$ of the construction, we define A_s and a sequence of trees $T_{e,s}$, $e = 0, \ldots, s$.

Stage s consists of three phases. In Phase 1, we define the initial tree at stage s, $T_{0,s}$. This tree is defined in such a way that, for each k, A_{s-1} extends $T_{0,s}(0^k * 0)$. The other branches on level k represent alternative versions of A. These alternative versions will enable us to satisfy (ii).

In phase 2 we work for (i). We define the trees $T_{e,s}$, $e = 1, \ldots s$, where $T_{e,s}$ is a subtree of $T_{e-1,s}$ and $n(T_{e,s}) = e$. Note that A_{s-1} lies on the leftmost path of each tree $T_{e,s}$. In Phase 3 we work for (ii). Carrying out a diagonalization in order to satisfy (ii) for a pair R_n, R_m causes A_s to extend a branch $T_{e,s}(j)$ for some $e, j > 0$.

We now discuss the strategies for (i) and (ii), introduce some more notation and describe the different phases of a stage in more detail.

6.1. The definition of the initial tree in Phase 1

For $s > 0$, let $x(0, s) < x(1, s) < \ldots$ be the enumeration in order of magnitude of the set $\{x: B_x \cap A_{s-1} = \emptyset\}$. Thus the mth block which is disjoint from A_{s-1} is $(d_{x(m,s)}, d_{x(m,s)+1}]$. $T_{0,s}$ is defined in such a way that, for arbitrary j, $1 \leq j \leq k$, enumerating the

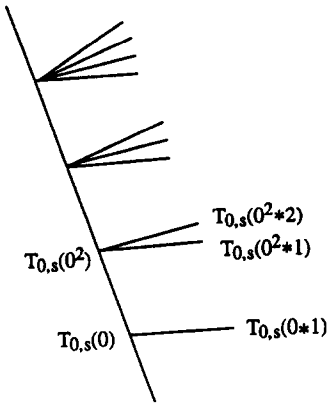

Fig. 2. The initial tree at stage s.

```
                    Bx(m,s)                       By  s
------------------- (------------] ------------------ (------|-----]---
      T0,s(0^m)     0...010...0 111...             100...
```

Fig. 3. The format of a string $T_{0,s}(0^m*j)$ of length $\leqslant s$.

number $d_{x(m,s)}+j$ and all the blocks B_y such that $x(m,s)<y$ and $B_y \subseteq [0,s)$ into A_s causes A_s to extend $T_{0,s}(0^m*j)$. This will be the only way to enumerate elements into A; in particular, A is enumerated via blocks.

Let $n(T_{0,s})=0$, and let B_y be the block containing s. Define, for each $m \geqslant 0$,

$$T_{0,s}(0^m) = A_{s-1}\,|d_{x(m,s)}+1.$$

For each j, $1 \leqslant j \leqslant m$, let

$$T_{0,s}(0^m*j) = T_{0,s}(0^m)*\eta_j*1^u, \tag{6.2}$$

where η_j is the string η of length $|B_{x(m,s)}|$ such that $\eta(j-1)=1$ and $\eta(j'-1)=0$ for $j' \neq j$, and $u \geqslant 0$ is chosen minimal such that (6.2) gives a string of length $\geqslant d_y+1$ (see Figs. 1 and 2). Note that $|T_{0,s}(0^m*j)| = d_z+1$ for some z.

The only branches which will matter in the construction are the ones of length $\leqslant s$ (Fig. 3). The other branches come in only to avoid the use of partial trees.

6.2. *Phase 2 and the strategy for (i)*

The condition (i) can be reformulated as follows: for each $P \in \Pi_\infty$ and each $i>0$,

$$[i] \text{ total } \Rightarrow [i]^{(A)_P} \text{ recursive } \vee (\exists Q \in \Pi_\infty)[[i]^{(A)_P} \equiv_{tt} (A)_Q]. \tag{6.3}$$

However, for technical reasons, we satisfy a modified version of (6.3). First some notation. For $i>0$, let $\langle i\rangle$ be a tt-reduction such that,. for every oracle Z, if $[i]$ is total, then

$$\langle i\rangle^X = [i]^{(X)_S}, \text{ where } S = E_1 \vee \ldots \vee E_{i-1} \tag{6.4}$$

(recall that E_i is the element of Π_∞ with two equivalence classes, one of them being $\{i\}$) The tt-reduction $\langle i\rangle$ can be obtained from $[i]$ in a uniform way as follows: if the truth table $\{i\}(x)$ is defined, replace queries of the form $d_z + k$, where $i \leqslant k \leqslant z$, by the disjunction

$$d_z + i \vee \ldots \vee d_{z+1} - 1.$$

This gives the truth table for $\langle i\rangle(x)$. Note that, if $[i]$ is total, then so is $\langle i\rangle$. Instead of (6.3), we satisfy the following condition: for each $e > 0$,

$$\langle e\rangle \textit{ total } \Rightarrow \langle e\rangle^A \textit{ recursive} \vee (\exists Q \in \Pi_e)[\langle e\rangle^A \equiv_{tt} (A)_Q]. \tag{6.5}$$

To show that (6.5) implies (6.3), suppose that $[i]$ is total and let $Y = [i]^{(A)_P}$. Choose k such that $P \leqslant E_1 \vee \ldots \vee E_{k-1}$. By Lemma 3.4, obtain a tt-reduction $[j]$ such that $(A)_P = [j]^{(A)_U}$ for each recursive equivalence relation $U \geqslant P$. Let $[e]$ be the composition of the two tt-reductions $[i]$, $[j]$, i.e., for each oracle Z,

$$[e]^Z = [i]^{([j])^Z}.$$

Then $Y = [e]^{(A)_U}$ for each recursive U such that $P \leqslant U$.

W.l.o.g. suppose that $e \geqslant k$. Then $P \leqslant S := E_1 \vee \ldots \vee E_{e-1}$ and hence, by (6.4),

$$Y = [e]^{(A)_S} = \langle e\rangle^A.$$

In this way, from (6.5) we can infer (6.3).

We now describe how to take subtrees in Phase 2 in order to obtain (6.5). Consider a tree $T_{e,s}$, $e > 0$. As in [10], to measure the distribution of $\langle i\rangle$-splittings ($1 \leqslant i \leqslant e$) among the branches on a level m of this tree, we assign an e-state to $T_{e,s}(0^m)$. In taking subtrees, we maximize this e-state. Consider a single i, $1 \leqslant i \leqslant e$. The ordering of e-states will be defined in such a way that the following happens: first we try to get an $\langle i\rangle$-splitting between $T_{e,s}(0^m * 0)$ and each branch $T_{e,s}(0^m * j)$, $1 \leqslant j \leqslant i$. If we have succeeded, we try to get many $\langle i\rangle$-splits among the branches $T_{e,s}(0^m * j)$, $1 \leqslant j \leqslant i$ (see Fig. 4). To formalize this, define a relation P on ω_+ by

$$u P v \Leftrightarrow T_{e,s}(0^m * \min(u, m+e)),\ T_{e,s}(0^m * \min(v, m+e)) \text{ do not } \langle i\rangle\text{-split}. \tag{6.6}$$

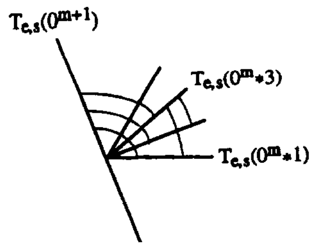

Fig. 4. The $\langle e\rangle$-splittings (indicated by arcs) on level m, if Case 2 holds, where $e = 3$ and $P = \{[1, 2]\}$.

By the definition of the strings $[e]^\tau$, P is an equivalence relation. Then, by the definition of $\langle i \rangle$, $P \in \Pi_i$. Fix a map $\text{code}_i\colon \Pi_i \to [i+1, i+|\Pi_i|]$ such that

$$Q_1 \leqslant Q_2 \;\Rightarrow\; \text{code}_i(Q_1) \leqslant \text{code}_i(Q_2).$$

To get many $\langle i \rangle$-splits, we want P to contain few pairs, or, in other words, we maximize $\text{code}_i(P)$.

Formally, an e-state is a string $(n_1, \dots, n_e)$, where $1 \leqslant n_i \leqslant i + |\Pi_i|$ $(1 \leqslant i \leqslant e)$. To assign an e-state to $T_{e,s}(0^m)$, for each such i we distinguish two cases.

Case 1: There exists, j, $1 \leqslant j \leqslant i$, such that $T_{e,s}(0^m{*}0)$ does not $\langle i \rangle$-split with $T_{e,s}(0^m{*}j)$. Then let n_i be the minimal such j.

Case 2: *Otherwise*. Define an equivalence relation $P \in \Pi_i$ by (6.6) and let $n_i = \text{code}_i(P)$.

As usual, e-states are ordered by lexicographical order. Keep in mind that more splittings always means a higher e-state. We write e-state $(T_{e,s}(0^m))$ for the e-state of $T_{e,s}(0^m)$.

Suppose that $T_{e-1,s}$ has been defined. To determine the subtree $T_{e,s}$ of $T_{e-1,s}$, for each $m \geqslant 0$ we have to make a choice

$$T_{e,s}(0^m) = T_{e-1,s}(0^k),$$

which yields the maximal accessible e-state. Since the e-state of $T_{e,s}(0^m)$ depends on $T_{e,s}(0^{m+1})$, together with defining $T_{e,s}(0^m)$ we also temporarily define $T_{e,s}(0^{m+1})$; this temporary value is denoted by $T_{e,s-1/2}(0^{m+1})$. We then require that $T_{e,s}(0^{m+1})$ extends $T_{e,s-1/2}(0^{m+1})$; any such extension is as good for $T_{e,s}(0^m)$ in terms of splittings as the temporary value.

6.3. *The strategy for (ii) and Phase 3*

For (ii), it is sufficient to satisfy the diagonalization requirements

$$(A)_{R_n} \neq [e]^{(A)_{R_m}} \quad (R_n \not\leqslant R_m) \tag{6.7}$$

We make an effective list (P_k) of these requirements. This is possible since the equivalence relations R_n are recursive uniformly in n. We also require that, if P_k is one of the requirements (6.7), then already for some pair j, j' such that $1 \leqslant j, j' \leqslant k$,

$$\neg j\, R_n\, j' \wedge j\, R_m\, j'.$$

Let $d_x + 1 = |T_{k,s}(\lambda)|$. Then $x \geqslant k$ by (6.1). Let j, j' be as above, and let $u = d_x + j$. We have the opportunity to let A_s extend $T_{k,s}(j)$ or $T_{k,s}(j')$. The two possible choices for A_s result in the same value for $[e]^{(A_s)_{R_m}}(u)$, but in a different value for $(A)_{R_n}(u)$. Thus, for some $i \in \{j, j'\}$, if in Phase 3 we cause A_s to extend the branch $T_{k,s}(i)$, then we have diagonalized for P_k. This leads to the following definition of requiring attention: let $1 \leqslant i \leqslant k$. At stage s, P_k *requires attention via* i if $|T_{k,s}(i)| \leqslant s$,

$(A_{s-1})_{R_n}(y) = [e]^{(A_{s-1})_{R_m}}(u)[s]$ for each $y < |T_{k,s}(i)|$ such that $\{e\}_s(y)\downarrow$, and there is $y < s$ such that $(T_{k,s}(i))_{R_n}(y) \neq [e]^{(T_{k,s}(i))_{R_m}}(y)$.

6.4. The main ideas in the verification

The verification will proceed as follows. First we show that, for each e and each m, $\lim_s T_{e,s}(0^m)$ exists, using the facts that the requirements P_k are finitary and that (for $e > 0$) there are only finitely many e-states. We next verify that there exists an e-state σ and a number m such that

$$(\forall n \geqslant m)(\text{a.e. } s)[e\text{-state}(T_{e,s}(0^n)) = \sigma].$$

We then prove (6.5). Suppose that $\langle e \rangle$ is total. First, if $\sigma(e) \leqslant e$, then $\langle e \rangle^A$ is recursive. The argument is as follows: let $j = \sigma(e)$, and s_1 be a stage number such that $T_{e,s}(0^{m+1})$ has reached its limit, and from s_1 on, $T_{e,s}(0^m)$ permanently has e-state σ. To recursively compute $\langle e \rangle^A(y)$, let $s \geqslant s_1$ be a stage number such that (among others) $|\langle e \rangle|(y) < s$, and let

$$\eta = T_{e,s}(0^m{*}j).$$

Then $\langle e \rangle^A(y) = \langle e \rangle^\eta(y)$, otherwise at some $t \geqslant s$ we could introduce an $\langle e \rangle$-splitting between $T_{e,t}(^{0m}{*}0)$ and $T_{e,t}(0^m{*}j)$, which contradicts $j = \sigma(e)$ and the choice of s_1.

Now suppose that $\sigma(e) = \text{code}_e(P)$. We show that, for arbitrary given $M \geqslant 0$, it is possible to compute a finite set G of strings which have length $\geqslant M$ such that A extends exactly one of them and, for $\eta_1, \eta_2 \in G$,

$$\eta_1, \eta_2 \text{ do not } \langle e \rangle\text{-split} \;\Leftrightarrow\; \text{the strings } (\eta_1)_P, (\eta_2)_P \text{ are compatible.} \tag{6.8}$$

This makes it possible to give tt-reductions of $\langle e \rangle^A$ to $(A)_P$ $[(A)_P$ to $\langle e \rangle^A]$. On input y, determine M such that $\langle e \rangle^{A|M}(y) = \langle e \rangle^A(y)[A|M)_P(y) = (A)_P(y)]$, and compute a set G as above. Using $(A)_P$ $[\langle e \rangle^A]$ as an oracle, it is possible to approximate the unknown string $\eta' \in G$ such that $\eta' \subseteq A$ by a string $\eta \in G$. By (6.8), this approximation will suffice to correctly compute the value $\langle e \rangle^{\eta'}(y)$ $[(\eta')_P(y)]$.

6.5. The construction

Stage 0: Let $A_0 = \emptyset$.

Stage s, $s > 0$.

Phase 1: Define $T_{0,s}$ as in Section 6.1.

Phase 2: Inductively, for $1 \leqslant e \leqslant s$ define $T_{e,s}$ as a subtree of $T_{e-1,s}$. Let $n(T_{e,s}) = e$. It suffices to define $T_{e,s}(0^m)$ for each $m \geqslant 0$. Let $T_{e,s-1/2}(\lambda) = T_{e-1,s}(0)$. Now suppose that $\gamma := T_{e,s-1/2}(0^m)$ has been defined, and let k be the number such that $\gamma = T_{e-1,s}(0^k)$.

Case 1: $|T_{e-1,s}(0^{k+1})| > s$. Define

$$T_{e,s}(0^m) = \gamma \quad\text{and}\quad T_{e,s-1/2}(0^m{*}0) = T_{e-1,s}(0^{k+1}).$$

Case 2: *Otherwise.* Consider all the possible choices for the values $T_{e,s}(0^m)$ and $T_{e,s-1/2}(0^m*0)$

$$T_{e,s}(0^m) = T_{e-1,s}(0^{k'}),$$
$$T_{e,s-1/2}(0^m*0) = T_{e-1,s}(0^{k''}), \tag{6.9}$$

where $k'' > k'$, $k' > k$ (because $T_{e,s}(0^m)$ has to extend γ) and $|T_{e-1,s}(0^{k''})| \leqslant s$. Since $|T_{e-1,s}(0^{k+1})| \leqslant s$, there is such a choice, namely the one given by $k' = k$, $k'' = k + 1$. Each choice determines an e-state of $T_{e,s}(0^m)$; let σ be the maximal among these e-states. There may be several choices which yield e-state σ. We try to conserve the situation at stage $s - 1$ (this will be necessary for the verification of (6.5)): if there is a such a choice where $T_{e,s}(0^m)$ and $T_{e,s-1/2}(0^m*0)$ retain the values they had at stage $s - 1$, we take it. Otherwise, choose $k' \geqslant k$ minimal and $k'' \geqslant k'$ minimal for k' such that the e-state we obtain is σ, and define $T_{e,s}(0^m)$ and $T_{e,s-1/2}(0^m*0)$ accordingly.

Phase 3: Choose k minimal such that requirement P_k requires attention via some i. Let

$$A_s = A_{s-1} \cup \{z \leqslant s\colon T_{k,s}(i)(z) = 1\}.$$

In this case, we say that P_k *acts.* If k fails to exist, let $A_s = A_{s-1}$.

Remarks. 1. Let $n, n' \geqslant 0$. Then

$$n \leqslant n' \wedge |T_{e,s}(0^{n'})| \leqslant s \;\Rightarrow\; e\text{-state}(T_{e,s}(0^n)) \geqslant e\text{-state}(T_{e,s}(0^{n'})). \tag{6.10}$$

Otherwise e-state$(T_{e,s}(0^n))$ would not be optimal.

2. Suppose that $1 \leqslant i \leqslant e$, $T_{e,s}(0^n) = T_{i,s}(0^r)$ and $|T_{e,s}(0^n)| \leqslant s$. Then

$$i\text{-state}(T_{i,s}(0^r)) = e\text{-state}(T_{e,s}(0^n))\restriction i.$$

This is obvious since $T_{e,s-1/2}(0^n*0)$ is a possible choice for $T_{i,s-1/2}(0^r*0)$,

6.6 *The verification*

Lemma 6.1. *Each requirement P_k, $k \geqslant 0$, requires attention only finitely often.*

Proof. As in [10, Proof of Lemma 1]. □

Lemma 6.2. *For each $m \geqslant 0$, $T_0(0^m) = \lim_s T_{0,s}(0^m)$ exists.*

Proof. It suffices to show that $\lim_s x(m, s)$ exists. This is done as in [10, Proof of Lemma 2]. □

Lemma 6.3. *Let $e \geqslant 0$. Then, for each $m \geqslant 0$,* (ia), (ib) *and* (ii) *below hold.*

(ia) *If $e \neq 0$, then $\lim_s T_{e,s-1/2}(0^m)$ exists.*

(ib) *$T_e(0^m) = \lim_s T_{e,s}(0^m)$ exists.*

(ii) *If* $e \neq 0$, *then there exists an e-state* σ_m *such that*

$$(\text{a.e. } s)\,[e\text{-state}(T_{e,s}(0^m)) = \sigma_m].$$

Notation. We write $T_{e,t}(0^m)\downarrow$ if $|T_{e,t}(0^m)| \leqslant t$ and

$$(\forall s \geqslant t)\,[T_{e,s}(0^m) = T_e(0^m) \wedge e\text{-state}(T_{e,s}(0^m)) = \sigma_m].$$

Remark. Let $d_z = |T_e(0^m)| + 1$. It is immediate by (ib) and the definition of $T_{0,s}(0^m * j)$ in Phase 1 that, for all j, $1 \leqslant j \leqslant e + m$, $\lim_s(T_{e,s}(0^m * j))|d_{z+1} + 1$ exists.

Proof (*By induction over e and over m*). The case $e = 0$ is handled in Lemma 2. Suppose that $e > 0$ and the Lemma holds for $e - 1$. We now use induction on m; thus suppose that (ia), (ib) and (ii) hold for e and all $m' < m$. Let s_0 be a stage number such that no requirement P_k, $k \leqslant e + m$, requires attention at any stage $s \geqslant s_0$ and (*if* $m \neq 0$) $T_{e,s_0}(0^{m-1})\downarrow$. If $m = 0$, then (ia) is immediate, since we define $T_{e,s-1/2}(\lambda) = T_{e-1,s}(0)$. Now suppose that $m > 0$. By the remark above and by the inductive hypothesis on $e - 1$, all the necessary splittings on level $m - 1$ of the tree $T_{e,s}$ are given by an initial segment of the leftmost branch of T_{e-1}. To be precise, there exists a number of $\tilde{k}$ such that, for some stage number $s_1 \geqslant s_0$, $T_{e-1,s_1}(0^{\tilde{k}})\downarrow$ and, if we define,

$$T_{e,s-1/2}(0^m) = T_{e-1}(0^{\tilde{k}}),$$

then $T_{e,t}(0^{m-1})$ has e-state σ_{m-1}. Let $\tilde{k}$ be minimal such. If we have to change the value $T_{e,(s)-1/2}(0^m)$ at a stage $s \geqslant s_1$, then, since k'' in (6.9) is minimal, from stage s on, we actually make this choice, hence $\lim_s T_{e,s-1/2}(0^m)$ exists.

For (ib), we show that for some stage $s_2 > s_1$ (to be defined later), if $t > s_2$ and

$$T_{e,t}(0^m) \neq T_{e,t-1}(0^m), \tag{6.11}$$

then

$$e\text{-state}(T_{e,t}(0^m)) > e\text{-state}(T_{e,t-1}(0^m)). \tag{6.12}$$

Since there are only finitely many e-states, this will establish (ib).

By choice of s_0, (6.11) cannot be caused by a diagonalization in Phase 3. Thus, a branch $T_{i,t-1}(0^r)$, $1 \leqslant i \leqslant e$, increased its i-state at stage t, here r is small enough so that this change effects $T_{e,t-1}(0^m)$. We will choose s_2 large enough so that, for $t \geqslant s_2$, it is possible to hand down this increased i-state to $T_{e,t}$. This will establish (6.12).

We define a sequence of numbers k_j, for j, $0 \leqslant j \leqslant e$ in decreasing order, and let

$$\gamma_j = \lim_s T_{j,s-1/2}(0^{k_j}).$$

Let $k_e = m$. If $0 \leqslant j \leqslant e$ and k_{j+1} is defined, let k_j be the minimal k such that

$$|T_j(0^k)| \geqslant |\gamma_{j+1}|.$$

Now let $s_2 \geqslant s_1$ be a stage number such that, for each $j < e$, $T_{j,s-1/2}(0^{k_j})$ has reached its limit at s_2 and $(\forall k \leqslant k_j)[T_{j,s_2}(0^k)\downarrow]$. Suppose that $t \geqslant s_2$ is a stage number such that $T_{e,t}(0^m) \neq T_{e,t-1}(0^m)$. There exist $i \leqslant e$, r, r' such that $r \leqslant r'$,

$T_{e,t-1}(0^m) = T_{i,t-1}(0^{r'})$ and

$$v := i\text{-state}(T_{i,t}(0^r)) > i\text{-state}(T_{i,t-1}(0^r)). \tag{6.13}$$

Note that, by (6.10)

$$i\text{-state}(T_{i,t-1}(0^r)) \geqslant i\text{-state}(T_{i,t-1}(0^{r'})) = e\text{-state}(T_{e,t}(0^m))|i. \tag{6.14}$$

Inductively, we will show that for $j = i, \ldots, e$,

$$j\text{-state}(T_{j,t}(0^{k_j}))|i \geqslant v. \tag{6.15}$$

Then, by (6.13) and (6.14),

$$e\text{-state}(T_{e,t}(0^m))|i > e\text{-state}(T_{e,t-1}(0^m))|i,$$

which establishes (6.12).

First suppose that $j = i$. By choice of s_2, $k_j < r$. By (6.10), this implies (6.15).

Now suppose that $i \leqslant j < e$ and (6.15) holds for j. By definition of s_2 and k_j.

$$|T_{j,t}(0^{k_j})| = |T_j(0^{k_j})| \geqslant \gamma_{j+1}.$$

Therefore, in Phase 2 of stage t we have the opportunity to define

$$T_{j+1,t}(0^{k_{j+1}}) = T_{j,t}(0^{k_j}),$$

$$T_{j+1,t-1/2}(0^{k_{j+1}}*0) = \text{some extension of } T_{j,t-1/2}(0^{k_j}*0).$$

This shows (6.15) for $j + 1$. In this way we establish (ib).

Let $s_3 \geqslant s_2$ be a stage number such that, for $s \geqslant s_3$, $T_{e,s}(0^m) = T_e(0^m)$. For (ii), it suffices to show that, if $s > s_3$, then

$$e\text{-state}(T_{e,s}(0^m)) \geqslant e\text{-state}(T_{e,s-1}(0^m)). \tag{6.16}$$

Note that, for each j, $1 \leqslant j \leqslant e + m$, $T_{e,s}(0^m*j)$ extends $T_{e,s-1}(0^m*j)$. Moreover, it is possible to choose $T_{e,s-1/2}(0^m*0)$ in such way that this string extends the corresponding value at stage $s - 1$, $T_{e,(s-1)-1/2}(0^m*0)$. Thus, at stage s we have a choice where all the $\langle i \rangle$-splittings, $1 \leqslant i \leqslant e$ between the branches on level m remain. This shows (6.16) and completes the Proof of Lemma 3.

Lemma 6.4. *Let $e > 0$. There exists an e-state σ, called the final e-state of T_e, and a number m such that*

$$(\forall n \geqslant m)(\text{a.e. } s)[e\text{-state}(T_{e,s}(0^n)) = \sigma].$$

Proof. Suppose that $n \leqslant n'$, and let $\sigma_n, \sigma_{n'}$ be as in (ii) of Lemma 3. With an appropriate stage number, (6.10) implies $\sigma_n \geqslant \sigma_{n'}$. Since there are only finitely many e-states, this establishes Lemma 4. □

Lemma 6.5. *For each $i \geqslant 0$, the requirement R_i is satisfied.*

Proof. Clear from Lemma 1 and the discussion in Section 3. Also see [10, Proof of Lemma 4].

The last two Lemmas will prove (6.5). Let $e > 0$ be arbitrary. We use the following definitions.

σ, m	as in Lemma 6.4,
$m(i)$ $(1 \leqslant i \leqslant e)$	the numbers such that $T_e(0^m) = T_i(0^{m(i)})$,
s_0	a stage number such that $T_{i,s_0}(0^{m(i)})\downarrow$ $(1 \leqslant i \leqslant e)$ and no requirement R_j, $j \leqslant e$, is active at any stage $s \geqslant s_0$.

Lemma 6.6. *Let $k > m$ be arbitrary and let $s \geqslant s_0$ be any stage number such that $T_{e,s}(0^k)$ has e-state σ and $|T_{e,s}(0^k)| \leqslant s$. Consider the following set of branches of $T_{e,s}$:*

$$G = \{T_{e,s}(0^k)\} \cup \{T_{e,s}(0^n * j)\colon m \leqslant n < k \ \wedge\ 1 \leqslant j \leqslant e + n\}.$$

Then (i) and (ii) below hold.

(i) $(\exists \eta \in G)[\eta \subseteq A]$.

(ii) *Suppose that $\sigma(e) > e$. Let $\sigma(e) = \mathrm{code}_e(P)(P \in \Pi_e)$ and let $\eta_1, \eta_2 \in G$ be arbitrary. Then*

$$\eta_1, \eta_2 \textit{ do not } \langle e\rangle\textit{-split} \ \Leftrightarrow\ \textit{the strings } (\eta_1)_P, (\eta_2)_P \textit{ are compatible.} \qquad (6.17)$$

Remark. Note that all the strings in G (Fig. 5) have a length of at least $|T_{e,s}(0^k)|$.

Proof. (i) Suppose that $T_{e,s}(0^k) \not\subseteq A$. Let $n < k$ be the maximal number such that $v := T_{e,s}(0^n) \subseteq A$. Then $m \leqslant n$ and $A_s \,|\, |v| = A \,|\, |v|$. For i, $1 \leqslant i \leqslant e$, let $n(i)$ be the number

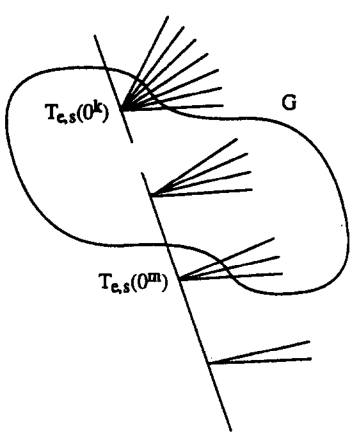

Fig. 5. The set G.

such that $T_{e,s}(0^n) = T_{i,s}(0)^{n(i)}$). We claim that, from stage s on, each tree T_i is stable up to level $n(i)$, that is, for each $t \geqslant s$ and each i, $1 \leqslant i \leqslant e$,

$$(\forall r \leqslant n(i))[T_{i,t}(0^r) = T_{i,s}(0^r)]. \tag{6.18}$$

Suppose otherwise. Choose $t \geqslant s$, then i, then r minimal such that $T_{i,t-1}(0^r) \neq T_{i,t}(0^r)$. Since $A_s \| v| = A \| v|$ and $|T_{i,t-1}(0^r)| \leqslant |v|$, the change cannot be caused by a diagonalization in Phase 3. Note that $r > m(i)$ by choice of s_0. Hence, for each $s' \geqslant s$.

$$i\text{-state}(T_{i,s'}(0^r)) \leqslant i\text{-state}(T_{i,s'}(0^{m(i)})) = \sigma | i.$$

Since $T_{i,t-1}(0^r) = T_{i,s}(0^r)$ possesses the optimal i-state $\sigma | i$ and since we are as conservative as possible in Phase 2 of stage t, $T_{i,t}(0^r) = T_{i,t-1}(0^r)$, a contradiction. This shows (6.18).

By definition of n, it must be the case that some requirement $P_{e'}$, $e' > e$, acts at a stage $t \geqslant s$ and causes A_t to extend some branch $T_{e',t}(j) = T_{e,t}(0^n * j)$ $(1 \leqslant j \leqslant e + n)$. Thus all the blocks B_x such that $z < x$ and $B_x \subseteq [0, t)$ are enumerated into A_t. By (6.18),

$$\eta := T_{e,s}(0^m * j) \subseteq T_{e,t}(0^n * j) \subseteq A.$$

This shows (i).

(ii) Suppose that η_1, η_2, P are as in (ii), and w.l.o.g. suppose that $\eta_1 \neq \eta_2$.

Case 1: For some n, $m \leqslant n < k$, and some j_1, j_2,

$$\eta_h = T_{e,s}(0^n * j_h) \quad (h = 1, 2).$$

Since $|\eta_1| = |\eta_2|$, $|\langle e\rangle^{\eta_1}(y)| = |\langle e\rangle^{\eta_2}(y)|$. Then, since the e-state of $T_{e,s}(0^n)$ is σ,

$$\eta_1, \eta_2 \text{ do not } \langle e\rangle\text{-split} \Leftrightarrow \langle e\rangle^{\eta_1} = \langle e\rangle^{\eta_2} \Leftrightarrow j_1 P j_2$$
$$\Leftrightarrow (\eta_1)_P, (\eta_2)_P \text{ are compatible.}$$

Case 2: *Otherwise, i.e. η_1 and η_2 are on different levels of the tree $T_{e,s}$.* Then, by the definition of $T_{0,s}$ $(\eta_1)_P$ and $(\eta_2)_P$ are not compatible. Moreover, since $\sigma(e) > e$, η_1 and $\eta_2 \langle e\rangle$-split.

This shows (6.17). □

Lemma 6.7. *Suppose that the tt-reduction $\langle e\rangle$ is total.*

(i) *If $\sigma(e) = \text{code}_e(P)$ (where $P \in \Pi_e$), then $\langle e\rangle^A \equiv_{tt} (A)_P$.*

(ii) *If $\sigma(e) \leqslant e$, then $\langle e\rangle^A$ is recursive.*

Proof. (i) We give reduction procedures of $\langle e\rangle^A$ to $(A)_P$ and of $(A)_P$ to $\langle e\rangle^A$ which are total for every oracle, i.e., which are tt-reductions.

To compute $\langle e\rangle^A$ from $(A)_P$: Given input y, let $M = |\langle e\rangle(y)|$. If $M \leqslant |T_e(0^m)|$, then let the output be $\langle e\rangle^{\{z : T_e(0^m)(z) = 1\}}(y)$. Otherwise, compute $k > m$ and $s > s_0$ such that $M \leqslant |T_{e,s}(0^k)| \leqslant s$, e-state$(T_{e,s}(0^k)) = \sigma$ and $\{e\}_{s'}(y)\downarrow$, where $s' = |T_{e,s}(0^k)|$. Define

G as in Lemma 6.6. Then $\eta' \subseteq A$ for some $\eta' \in G$. As explained above, using $(A)_P$ as an oracle, it is possible to approximate η', namely to find $\eta \in G$ such that

$$(\eta)_P \subseteq (A)_P.$$

By definition of k and s, $|\eta|,|\eta'| \geqslant M$. Hence, by Lemma 6.6(ii)

$$\langle e\rangle^A(y) = \langle e\rangle^{\eta'}(y) = \langle e\rangle^{\eta}(y).$$

Thus we give $\langle e\rangle^{\eta}(y)$ as output.

To compute $(A)_P$ *from* $\langle e\rangle^A$: For input y, let z be the number such that $y \in B_z$, and let $M = d_{z+1} + 1$. If $M \leqslant |T_e(0^m)|$, then give $T_e(0^m)_P(y)$ as output. Now suppose otherwise. Compute $k > m$ and $s > s_0$ such that $M \leqslant |T_{e,s}(0^k)| < s$ and $T_{e,s}(0^k)$ has e-state σ. Define G as in Lemma 6.6. We again approximate the string $\eta' \in G$ such that $\eta' \subseteq A$: using the set $\langle e\rangle^A$ as an oracle, find $\eta \in G$ such that

$$\langle e\rangle^{\eta} \subseteq \langle e\rangle^A.$$

Then, η, η' do not $\langle e\rangle$-split. Hence, by Lemma 6.6(ii), $(\eta)_P$ and $(\eta')_P$ are compatible. Since $|\eta|, |\eta'| \geqslant M$, $(A)_P(y) = (\eta')_P(y) = (\eta)_P(y)$. Thus we give $(\eta)_P(y)$ as output.

Both reduction procedures can be made total for every oracle: the search through the finite set G for a string η with the desired property terminates, no matter what the oracle is; if it does not terminate successfully, just stop and arbitrarily give 0 as output.

(ii) Suppose that $1 \leqslant \sigma(e) \leqslant e$. To show that $\langle e\rangle^A$ is recursive, let $j = \sigma(e)$, and let $s_1 \geqslant s_0$ be stage number such that for every $s \geqslant s_1$, $T_{e,s}(0^{m+1}) = T_e(0^{m+1})$. Given input y, let z be the number such that $|\langle e\rangle(y)| \in B_z$. Compute $s > s_1$ such that $B_z \subseteq [0, s)$, $\{e\}_s(y)\downarrow$ and $|T_e(0^{m+1})| \leqslant s$, and let $\eta = T_{e,s}(0^m * j)$. Then $|\langle e\rangle(y)| \leqslant |\eta|$. Evaluate $\langle e\rangle^{\eta}(y)$; we claim that this is the correct answer for $\langle e\rangle^A(x)$. Otherwise, for some $t \geqslant s$ and $\tilde{k} > m$, the following hold:

$$\begin{aligned} &T_e(0^{m+1}) \subseteq T_{e-1,t}(0^{\tilde{k}}) \subseteq A,\ |T_{e-1,t}(0^{\tilde{k}})| \leqslant t \text{ and}, \quad \text{since } \eta \subseteq T_{e,t}(0^m * j), \\ &T_{e-1,t}(0^{\tilde{k}}),\ T_{e,t}(0^m * j)\ \ \langle e\rangle\text{-split}. \end{aligned} \tag{6.19}$$

In phase 2 of stage t, we make a choice

$$\begin{aligned} T_{e,t}(0^m) &= T_{e-1,t}(0^{k'}), \\ T_{e,t-1/2}(0^m * 0) &= T_{e-1,t}(0^{k''}), \end{aligned}$$

which, by the definition of s_0, yields e-state σ. Since $t \geqslant s_1$,

$$T_{e-1,t}(0^{k'}) = T_{e,s_0}(0^m) \quad \text{and} \quad T_{e-1,t}(0^{k''}) \subseteq T_e(0^{m+1}) \subseteq T_{e-1,t}(0^{\tilde{k}}).$$

Hence, by choosing $T_{e-1,t}(0^{\tilde{k}})$, instead of $T_{e-1,t}(0^{\tilde{k}})$, all the $\langle i\rangle$-splittings, $1 \leqslant i \leqslant e$, between branches on level m remain, and in addition, by (6.19), we get an $\langle e\rangle$-splitting between $T_{e,t}(0^m * j)$ and $T_{e,t}(0^m * 0)$. Thus our choice is not the best possible, contradiction.

This completes the proof of Lemma 6.7 and of Theorem 3.7. □

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