Model theory of the computably enumerable many-one degrees

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Abstract

We investigate model theoretic properties of $\mathcal{R}_m$, the partial order of computably enumerable many-one degrees. We prove that all nontrivial final segments and the set of minimal degrees are automorphism bases, and that some proper half open initial segment is an elementary substructure of $\mathcal{R}_m - \{1\}$. This shows that $\mathcal{R}_m$ is not a minimal model.

In an appendix, we show that the many-one degree of an r-maximal set is join irreducible.

Keywords: Model theory, many-one degrees

This article is dedicated to Mari Santos.

1 Introduction

Many-one reducibility, introduced by Post [9], is a rather fine way to measure the relative complexity of subsets of ω: X is many-one reducible to Y, written $X \leq_m Y$, if $X = f^{-1}(Y)$ for a computable function f (we assume that f can also assume the values TRUE and FALSE to avoid trivialities). However, it appears naturally in a wide variety of contexts, for instance interpretability of theories and word problems of subgroups. Let $\mathcal{D}_m$ and $\mathcal{R}_m$ denote the upper semilattices of many-one degrees of all sets, and of the computably enumerable (c.e.) sets, respectively. Both upper semilattices are *distributive*, namely

$$\forall \boldsymbol{x} \forall \boldsymbol{a} \forall \boldsymbol{b} [\boldsymbol{x} \leq \boldsymbol{a} \vee \boldsymbol{b} \ \Rightarrow\ \exists \boldsymbol{a}_0 \leq \boldsymbol{a} \exists \boldsymbol{b}_0 \leq \boldsymbol{b}\ \boldsymbol{x} = \boldsymbol{a}_0 \vee \boldsymbol{b}_0]. \tag{1.1}$$

$\mathcal{R}_m$ is the only c.e. degree structure known to permit a characterization [2], which is an effectivization of a purely algebraic characterization of $\mathcal{D}_m$ due to Ershov. As a consequence, both structures have the maximum possible number of automorphisms, 2^{2^ω} for $\mathcal{D}_m$ and 2^ω for $\mathcal{R}_m$. The author proved in [5] that a copy of $(\boldsymbol{N}, +, \times)$ can be coded in $\mathcal{R}_m$ using first-order formulas without parameters. In particular, $\mathrm{Th}(\mathcal{R}_m) \equiv_m \mathrm{Th}(\boldsymbol{N}, +, \times)$. In the main part of this paper we obtain further results of a model theoretic flavor, mainly on $\mathcal{R}_m$, and indicate what the situation is for $\mathcal{D}_m$. All the results clarify how parts of the structure (like nontrivial initial and final segments) interact with the structure as a whole.

A subset X of a structure $\boldsymbol{A}$ is an *automorphism base* if the only automorphism of $\boldsymbol{A}$ fixing X pointwise is the identity. Ambos-Spies (to appear) proves that each nontivial initial segment of the c.e. Turing degrees is an automorphism base. We obtain a dual result for final segments of $\mathcal{R}_m$. Moreover, we show that the set of minimal degrees forms an automorphism base. Observe that this set is small in the sense that it forms a single orbit under the action of the automorphism group (the last fact follows from [2]).

Let $\mathcal{R}_m^- = \mathcal{R}_m - \{\mathbf{1}\}$. We prove that there is an incomplete $\boldsymbol{e} \in \mathcal{R}_m$ such that $[\mathbf{0}, \boldsymbol{e})$ is an elementary submodel of $\mathcal{R}_m^-$ via the inclusion embedding. In particular, $\mathcal{R}_m^-$ (and hence $\mathcal{R}_m$, since $[\mathbf{0}, \boldsymbol{e}) \cup \{\mathbf{1}\} \prec \mathcal{R}_m$) has a proper elementary submodel, i.e. is not a minimal model over the empty set.

We remark that, for the study of enumerable sets, many-one- reducibility is interesting partially because it is closely related to structural properties of an enumerable set. For instance, a maximal enumerable set must have minimal many-one degree (see [7] for a proof). In an appendix, we prove a similar fact for this larger class of r-maximal sets: the many-one degree of an r-maximal set is join-irreducible. (Recall that a c.e. set A is r-maximal iff ω^* is join-irreducible in $\mathcal{L}^*(A)$, the lattice of c.e. supersets of A modulo finite differences.)

2 Automorphism bases

We first introduce some auxiliary notions. An *ideal* of an upper semilattice is a nonempty subset which is closed downward and under supremum. If I is an ideal in $\mathcal{R}_m$, we will say that $\boldsymbol{b}$ is a *strong minimal cover (s.m.c.) of* I if $I = [\mathbf{0}, \boldsymbol{b})$. In the special case that $I = [\mathbf{0}, \boldsymbol{c}]$, we also say that $\boldsymbol{b}$ is a s.m.c. of $\boldsymbol{c}$. We state a Lemma which is a special case of Theorem 3.1 in Ershov and Lavrov [3] (where a completely different notation is used). Inspection of their proof shows that the strong minimal cover is obtained in an effective way.

LEMMA 2.1 ([3])
Suppose that $I \subseteq \mathcal{R}_m$ is a proper Σ_3^0-ideal and $\boldsymbol{a} < \mathbf{1}$. Then one can effectively in indices for I and $\boldsymbol{a}$ obtain a strong minimal cover $\boldsymbol{b}$ of I such that $\boldsymbol{b} \not\leq \boldsymbol{a}$.

THEOREM 2.2
For each $\boldsymbol{a} < \mathbf{1}$, the interval $[\boldsymbol{a}, \mathbf{1}]$ is an automorphism base for $\mathcal{R}_m$.

Proof. Recall that, if $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}_m$, $\boldsymbol{b}$ is called a *strong minimal cover* of $\boldsymbol{a}$ if $[\mathbf{0}, \boldsymbol{b}) = [\mathbf{0}, \boldsymbol{a}]$. We consider the following map from $\mathcal{R}_m$ to a set of arithmetical objects constructed from $[\boldsymbol{a}, \mathbf{1}]$: let

$$H(\boldsymbol{x}) = (\boldsymbol{x} \vee \boldsymbol{a}, G(\boldsymbol{x})),$$

where $G(\boldsymbol{x}) = \{\boldsymbol{s} \vee \boldsymbol{a} : \boldsymbol{s} \not\leq \boldsymbol{a} \ \&\ \exists \boldsymbol{u} \leq \boldsymbol{x}, \boldsymbol{a}\ [\boldsymbol{s} \textit{ s.m.c. of } \boldsymbol{u}]\}$. We will prove that H is 1-1. Since H is definable (in the appropriate sense) with parameter $\boldsymbol{a}$, this suffices: suppose that the automorphism Φ fixes $[\boldsymbol{a}, \mathbf{1}]$ pointwise. Then $H(\Phi(\boldsymbol{x})) = \Phi(H(\boldsymbol{x})) = H(\boldsymbol{x})$ for each $\boldsymbol{x}$ (where, for any pair $(\boldsymbol{z}, W)$ of a degree and a set of degrees, we define $\Phi(\boldsymbol{z}, W) = (\Phi(\boldsymbol{z}), \Phi(W))$. Hence Φ is the identity.

Let $R(\boldsymbol{u}, \boldsymbol{v}) = \{\boldsymbol{z} : \boldsymbol{z} \leq \boldsymbol{u}, \boldsymbol{v}\}$. To prove that H is 1-1, we first show that $G(\boldsymbol{x}) = G(\boldsymbol{y})$ implies $R(\boldsymbol{x}, \boldsymbol{a}) = R(\boldsymbol{y}, \boldsymbol{a})$. Suppose that the second equality fails. Then we can choose, say, a $\boldsymbol{u} \in R(\boldsymbol{x}, \boldsymbol{a}) - R(\boldsymbol{y}, \boldsymbol{a})$. For $\boldsymbol{z} \leq \boldsymbol{a}$, let

$$F(\boldsymbol{z}) = \{\boldsymbol{s} \vee \boldsymbol{a} : \boldsymbol{s} \not\leq \boldsymbol{a} \ \&\ \boldsymbol{s} \text{ s.m.c. of } \boldsymbol{z}\}$$

(so that $G(\boldsymbol{x}) = \bigcup_{\boldsymbol{z} \in R(\boldsymbol{x}, \boldsymbol{a})} F(\boldsymbol{z})$). We claim that for each $\boldsymbol{v} \in R(\boldsymbol{y}, \boldsymbol{a})$, $F(\boldsymbol{u}) \cap F(\boldsymbol{v}) = \emptyset$, which clearly implies that $G(\boldsymbol{x}) \neq G(\boldsymbol{y})$. Notice that $\boldsymbol{v} \not\geq \boldsymbol{u}$. Let $\boldsymbol{s}, \boldsymbol{t} \not\leq \boldsymbol{a}$ be strong minimal covers of $\boldsymbol{u}, \boldsymbol{v}$, respectively. Then $\boldsymbol{t} \not\geq \boldsymbol{s}$: clearly $\boldsymbol{t} \neq \boldsymbol{s}$. But if $\boldsymbol{t} > \boldsymbol{s}$, then $\boldsymbol{v} \geq \boldsymbol{s} > \boldsymbol{u}$. Now suppose that also $\boldsymbol{s} \vee \boldsymbol{a} = \boldsymbol{t} \vee \boldsymbol{a}$. By distributivity, $\boldsymbol{t} = \boldsymbol{t}_0 \vee \boldsymbol{t}_1$, where $\boldsymbol{t}_0 \leq s$ and $\boldsymbol{t}_1 \leq \boldsymbol{a}$. Since $\boldsymbol{t} \not\geq \boldsymbol{s}$, $\boldsymbol{t}_0 < \boldsymbol{s}$, so $\boldsymbol{t}_0 \leq \boldsymbol{v} \leq \boldsymbol{a}$. Hence $\boldsymbol{t} \leq \boldsymbol{a}$, contrary to our assumption on t.

Now suppose that $H(\boldsymbol{x}) = H(\boldsymbol{y})$. Since $\boldsymbol{x} \leq \boldsymbol{y} \vee \boldsymbol{a}$, $\boldsymbol{x} = \boldsymbol{x}_0 \vee \boldsymbol{x}_1$ where $\boldsymbol{x}_0 \leq \boldsymbol{y}$ & $\boldsymbol{x}_1 \leq \boldsymbol{a}$. But then $\boldsymbol{x}_1 \in R(\boldsymbol{x}, \boldsymbol{a}) = R(\boldsymbol{y}, \boldsymbol{a})$, so $\boldsymbol{x} \leq \boldsymbol{y}$. Similarly, $\boldsymbol{y} \leq \boldsymbol{x}$. ∎

Note that the same proof works to show the analogous result for $\mathcal{D}_m$.

In the rest of the paper we will use the terminology and techniques of Denisov [2] (see also Odifreddi [8]), which we review first. A main concept is the notion of an L-semilattice, which is a type of distributive upper semilattice with $0, 1$ that comes with an enumeration, so that certain effectivity conditions are satisfied. For instance, one requires that the supremum is given by a computable binary function. Lachlan [4] proved that up to isomorphism the L-semilattices are the initial intervals of $\mathcal{R}_m$. By his proof, each initial interval $[\mathbf{o}, \boldsymbol{z}]$ of $\mathcal{R}_m$ can be turned into an (enumerated) L-semilattice in a canonical way.

We also need the following tool for the characterization of $\mathcal{R}_m$ from Denisov [2], a saturation property of $\mathcal{R}_m$. Recall that $\mathcal{R}_m^- = \mathcal{R}_m - \{\mathbf{1}\}$.

THEOREM 2.3 (Denisov [2])
For enumerated L-semilattices U_0, U and effective embeddings $g : U_0 \mapsto \mathcal{R}_m, h : U_0 \mapsto U$ as initial intervals, $\mathbf{1} \notin \text{rg}(g)$, there is an effective embedding as an initial interval $f : U \mapsto \mathcal{R}_m$ such that $g = f \circ h$.

REMARK 2.4

1. The proof in [2] shows that an index for f is obtained in an effective way. However, the procedure only gives a useful output if the hypotheses, for instance $\mathbf{1} \notin \text{rg}(g)$, are satisfied.
2. Denisov notices that if $\boldsymbol{x}, \boldsymbol{y}$ denote the largest elements of the ranges of g and f, respectively, (so $\boldsymbol{x} \leq \boldsymbol{y}$) and $\boldsymbol{z}$ is given such that $\boldsymbol{z} \not\leq \boldsymbol{x}$, one can achieve that $\boldsymbol{z} \not\leq \boldsymbol{y}$.
3. As an application of the Theorem, if $\boldsymbol{a}, \boldsymbol{b} < \mathbf{1}$ and $f : [\mathbf{o}, \boldsymbol{a}] \mapsto [\mathbf{o}, \boldsymbol{b}]$ is effective (relative to the canonical enumerations of the L-semilattices mentioned above), then, using the forth- and back method, f can be extended to an automorphism of $\mathcal{R}_m$. Thus, for instance all the minimal degrees are automorphic.

THEOREM 2.5
The set of minimal degrees forms an automorphism base for $\mathcal{R}_m$.

Proof. We begin with a lemma:

LEMMA 2.6
Suppose that the L-semilattices $[0, \boldsymbol{e}]$ and $\mathcal{R}_m$ are effectively isomorphic. Then the following property holds for $\boldsymbol{e}$:

$$\gamma(\boldsymbol{y}) \equiv \forall \boldsymbol{q}[\boldsymbol{y} \not\leq \boldsymbol{q} \Rightarrow \exists \boldsymbol{m} \text{ minimal } (\boldsymbol{m} \leq \boldsymbol{y} \ \& \ \boldsymbol{m} \not\leq \boldsymbol{q})]. \tag{2.1}$$

Proof. Suppose $\boldsymbol{e} \not\leq \boldsymbol{q}$. Since $[0, \boldsymbol{e}]$ and $\mathcal{R}_m$ are effectively isomorphic, $I = [\mathbf{o}, \boldsymbol{e}] \cap [\mathbf{o}, \boldsymbol{q}]$ is a proper Σ_3^0-ideal of $[\mathbf{o}, \boldsymbol{e}]$. There is a minimal degree $\boldsymbol{m} \leq \boldsymbol{e}$ which is not in I: let $\boldsymbol{a} < \boldsymbol{e}$ be an upper bound of I (which exists e.g. by Lemma 2.2 in [5]). Apply Lemma 2.1 to $[0, \boldsymbol{e}]$ in place of $\mathcal{R}_m$, with the ideal $\mathbf{o}$ and $\boldsymbol{a}$ in order to obtain $m \not\leq \boldsymbol{a}$ as a s.m.c. of $\mathbf{o}$. ∎

We now define a map H as in the proof of Theorem 2.2: let

$$H(\boldsymbol{x}) = \{[\mathbf{o}, \boldsymbol{y}] \cap M : \gamma(\boldsymbol{y}) \ \& \ \boldsymbol{y} \geq \boldsymbol{x}\},$$

where $M \subseteq \mathcal{R}_m$ denotes the set of minimal degrees. Since H is definable, it is sufficient to prove that H is 1-1. Suppose that $\boldsymbol{z} \not\leq \boldsymbol{x}$, but $H(\boldsymbol{z}) = H(\boldsymbol{x})$. Apply Theorem 2.3 with $U_0 = [\mathbf{o}, \boldsymbol{x}], U = \mathcal{R}_m$ (and the canonical embeddings). By the second remark following the Theorem, we can obtain f such that $\boldsymbol{y} = f(1_U)$ is not above $\boldsymbol{z}$. By the previous Lemma, $\gamma(\boldsymbol{y})$ holds. Since $H(\boldsymbol{z}) = H(\boldsymbol{x})$, there must be $\boldsymbol{u} \geq \boldsymbol{z}$ satisfying $\gamma(\boldsymbol{u})$ so that $[\mathbf{o}, \boldsymbol{y}] \cap M = [\mathbf{o}, \boldsymbol{u}] \cap M$. This contradicts $\boldsymbol{u} \not\leq \boldsymbol{y}$. ∎

REMARK 2.7
The following related results will appear in [6].

1. Actually, each $\emptyset$-definable subset $D \not\subseteq \{\mathbf{o}, \mathbf{1}\}$ of $\mathcal{R}_m$ is an automorphism base. Thus, for instance, for each $n \geq 2$ the set of degrees $\boldsymbol{a}$ such that $[\mathbf{o}, \boldsymbol{a}]$ is a chain of length n is an automorphism base.
2. For $\mathcal{D}_m$, we have proved that the minimal degrees do *not* form an automorphism base. It follows that the group $\mathrm{Aut}(\mathcal{D}_m)$ is not simple.

3 Elementary substructures

We show the existence (above any given incomplete $\boldsymbol{a}$) of an incomplete $\boldsymbol{e} \in \mathcal{R}_m$ such that $[\mathbf{o}, \boldsymbol{e})$ is an elementary submodel of $\mathcal{R}_m^-$ via inclusion. We make use of the elementary chain principle from model theory. We write $\boldsymbol{A} \prec_k \boldsymbol{B}$ if $\boldsymbol{A}$ is a submodel of $\boldsymbol{B}$ and the inclusion map and its inverse preserve the truth of Σ_k-properties of elements of $\boldsymbol{A}$.

LEMMA 3.1 (Chang, Keisler [1])
If $\boldsymbol{A}_0 \prec_k \boldsymbol{A}_1 \prec_k \ldots$ is a Σ_k-elementary chain and $\boldsymbol{A}_\omega = \bigcup_{i \in \omega} \boldsymbol{A}_i$, then $\boldsymbol{A}_i \prec_k \boldsymbol{A}_\omega$ for each i. Moreover, if $\boldsymbol{A}_i \prec_k \boldsymbol{B}$ for each i, then $\boldsymbol{A}_\omega \prec_k \boldsymbol{B}$. ∎

THEOREM 3.2
For each $\boldsymbol{a} < \mathbf{1}$ there is an $\boldsymbol{e} < \mathbf{1}$ such that $\boldsymbol{a} \leq \boldsymbol{e}$ and $[\mathbf{o}, \boldsymbol{e})$ is an elementary submodel of $\mathcal{R}_m^-$.

COROLLARY 3.3
$\mathcal{R}_m$ is not a minimal model.

Proof. Clearly, $[\mathbf{o}, \boldsymbol{e}) \cup \mathbf{1}$ is a proper elementary submodel of $\mathcal{R}_m$. ∎
Proof of the Theorem. For each $\boldsymbol{x}$, one can effectively obtain $\boldsymbol{y}$ such that

1. $\boldsymbol{x} < \mathbf{1} \Rightarrow \boldsymbol{x} < \boldsymbol{y} < \mathbf{1}$, and
2. $[\mathbf{o}, \boldsymbol{y}] \cong \mathcal{R}_m$ via an effective isomorphism which acts as the identity on $[\mathbf{o}, \boldsymbol{x}]$.

To see this, consider the (effective) inclusion embedding h of the L-semilattice $U_0 = [\mathbf{o}, \boldsymbol{x}]$ into the L-semilattice $U = \mathcal{R}_m \cup \{\boldsymbol{t}\}$, where $\boldsymbol{t}$ is a new largest element. Applying Theorem 2.3 with $g : U_0 \mapsto \mathcal{R}_m$ being the inclusion map, obtain an effective $f : U \to \mathcal{R}_m$ which is the identity on $[\mathbf{o}, \boldsymbol{x}]$, and let $\boldsymbol{y}$ as the image of $\mathbf{1} \in U$.

Let us write $\boldsymbol{y} = F_0(\boldsymbol{x})$. F_0 is an effective map on indices for c.e. m-degrees. Thus $F_0(\boldsymbol{x})$ actually depends on the index via which $\boldsymbol{x}$ is given. Iterating F_0 we obtain, by the effectivity of Denisov's construction, for any $\boldsymbol{x} < \mathbf{1}$ a u.c.e. chain

$$\boldsymbol{x} < F_0(\boldsymbol{x}) < F_0(F_0(\boldsymbol{x})) < \dots .$$

In a sense we will obtain $\boldsymbol{e}$ by iterating F_0 on $\boldsymbol{a}$ ω^ω many times. The iteration up to ω^k gives a degree $F_k(\boldsymbol{a})$ such that $[\mathbf{o}, F_k(\boldsymbol{a})) \prec_k \mathcal{R}_m^-$. The construction bears some resemblance to the reflection theorems from set theory.

Let $F_1(\boldsymbol{x})$ be a degree $\boldsymbol{y}$ such that $[\mathbf{o}, \boldsymbol{y}) = \bigcup_i [\mathbf{o}, F_0^{(i)}(\boldsymbol{x}))$. Such an $\boldsymbol{y}$ can be obtained effectively in $\boldsymbol{x}$ by applying Lemma 2.1 with $\boldsymbol{a} = \mathbf{o}$, since we effectively obtain an index for the Σ_3^0- ideal $\bigcup_i [\mathbf{o}, F_0^{(i)}(\boldsymbol{x}))$. Moreover, $\boldsymbol{x} < F_1(\boldsymbol{x})$.

In general, suppose that the function $F_k(\boldsymbol{x})$ has been defined for all $\boldsymbol{x}$. F_k is effective on indices and $F_k(\boldsymbol{x}) > \boldsymbol{x}$ for $\boldsymbol{x} < \mathbf{1}$. Now let $F_{k+1}(\boldsymbol{x})$ be a degree $\boldsymbol{y}$ such that $[\mathbf{o}, \boldsymbol{y}) = \bigcup_i [\mathbf{o}, F_k^{(i)}(\boldsymbol{x}))$. Then F_{k+1} is a function on indices with the same two properties.

Claim 3.4
For $\boldsymbol{x} < \mathbf{1}$, $k \geq 0$, $[0, F_k(\boldsymbol{x})) \prec_k \mathcal{R}_m^-$.

Proof of the Claim. By induction on k. For $k = 0$, we assert that $[\mathbf{o}, F_0(\boldsymbol{x}))$ is embedded as an ordering into $\mathcal{R}_m^-$, which is correct. To prove the statement for $k+1$, let $\boldsymbol{z} = F_{k+1}(\boldsymbol{x}), \boldsymbol{z}_j = F_k^{(j)}(\boldsymbol{x}) (j \geq 0)$. By the inductive hypothesis, $[\mathbf{o}, \boldsymbol{z}_j) \prec_k \mathcal{R}_m^-$, so the elementary chain principle implies that

$$[\mathbf{o}, \boldsymbol{z}) \prec_k \mathcal{R}_m^- \text{ and } \forall j\ [\mathbf{o}, \boldsymbol{z}_j) \prec_k [\mathbf{o}, \boldsymbol{z}). \tag{3.1}$$

Suppose $\boldsymbol{b}_0, \dots, \boldsymbol{b}_{r-1} < \boldsymbol{z}$, and consider the formula

$$\varphi(\overline{\boldsymbol{b}}) = \exists \widetilde{y} \psi(\overline{\boldsymbol{b}}, \widetilde{y}),$$

where ψ is a boolean combination of Σ_k- formulas and $\widetilde{y}$ is a tuple of variables of a certain length. We have to show that

$$[\mathbf{o}, \boldsymbol{z}) \models \varphi(\overline{\boldsymbol{b}}) \Leftrightarrow \mathcal{R}_m^- \models \varphi(\overline{\boldsymbol{b}}).$$

Choose an i such that $\boldsymbol{b}_0, \dots, \boldsymbol{b}_{r-1} < \boldsymbol{z}_i$.

1. First suppose that $[\mathbf{o}, \boldsymbol{z}) \models \varphi(\overline{\boldsymbol{b}})$. Choose $j > i$ and a tuple $\widetilde{\boldsymbol{c}}$ of elements in $[\mathbf{o}, \boldsymbol{z}_j)$ such that $[\mathbf{o}, \boldsymbol{z}) \models \psi(\overline{\boldsymbol{b}}, \widetilde{\boldsymbol{c}})$. By (3.1), $[\mathbf{o}, \boldsymbol{z}_j) \models \psi(\overline{\boldsymbol{b}}, \widetilde{\boldsymbol{c}})$. Then, because $[\mathbf{o}, \boldsymbol{z}_j) \prec_k \mathcal{R}_m^-$, $\mathcal{R}_m^- \models \psi(\overline{\boldsymbol{b}}, \widetilde{\boldsymbol{c}})$.
2. Now suppose that $\mathcal{R}_m^- \models \varphi(\overline{\boldsymbol{b}})$. Pick a witness $\widetilde{\boldsymbol{c}}$ such that $\mathcal{R}_m^- \models \psi(\overline{\boldsymbol{b}}, \widetilde{\boldsymbol{c}})$. We will find a similar tuple witnesses $\widetilde{e}$ below $F_0(\boldsymbol{z}_i) < \boldsymbol{z}_{i+1}$. Apply Theorem 2.3 with $[\mathbf{o}, F_0(\boldsymbol{z}_i)]$ in place of $\mathcal{R}_m$, where $U_0 = [\mathbf{o}, \boldsymbol{z}_i]$, $g : U_0 \mapsto [\mathbf{o}, F_0(\boldsymbol{z}_i)]$ is the inclusion map, $U = [\mathbf{o}, \boldsymbol{d}]$, where $\boldsymbol{d} = \sup(\boldsymbol{z}_i, \widetilde{\boldsymbol{c}})$, and $h : U_0 \mapsto U$ is the inclusion map. The map $f : U \mapsto [\mathbf{o}, F_0(\boldsymbol{z}_i)]$ obtained in this way is an effective isomorphism between $[\mathbf{o}, \boldsymbol{d}]$ and $[\mathbf{o}, f(\boldsymbol{d})]$, which, by remark 3, can be extended to an automorphism of $\mathcal{R}_m$ acting as the identity on $[\mathbf{o}, \boldsymbol{z}_i]$. Let $\widetilde{\boldsymbol{e}} = f(\widetilde{\boldsymbol{c}})$. It follows that $\mathcal{R}_m^- \models \psi(\overline{\boldsymbol{b}}, \widetilde{\boldsymbol{e}})$, so we have found a tuple of witnesses $\widetilde{\boldsymbol{e}}$ as desired.
   Now, by the inductive hypothesis $[\mathbf{o}, \boldsymbol{z}_{i+1}) \prec_k \mathcal{R}_m^-$, therefore $[\mathbf{o}, \boldsymbol{z}_{i+1}) \models \psi(\overline{\boldsymbol{b}}, \widetilde{\boldsymbol{e}})$. By (3.1), $[\mathbf{o}, \boldsymbol{z}) \models \psi(\overline{\boldsymbol{b}}, \widetilde{\boldsymbol{e}})$ and therefore $[\mathbf{o}, \boldsymbol{z}) \models \varphi(\overline{\boldsymbol{b}})$.

Finally, let $\boldsymbol{e} > \boldsymbol{a}$ be such that $[\mathbf{o}, \boldsymbol{e}) = \bigcup_{k \geq 0}[\mathbf{o}, F_k(\boldsymbol{a}))$. Since $[\mathbf{o}, F_l(\boldsymbol{a})) \prec_k \mathcal{R}_m^-$ for all $l \geq k$, we conclude that $[\mathbf{o}, \boldsymbol{e}) \prec \mathcal{R}_m^-$ by the elementary chain principle. ∎

Notice that in fact $[\mathbf{o}, \boldsymbol{e}) \cong \mathcal{R}_m^-$, because $[\mathbf{o}, \boldsymbol{e})$ satisfies the characterization of $\mathcal{R}_m^-$ given in [2]. However, the isomorphism cannot be Δ_3^0 (let alone effective) when viewed as an embbeding into $\mathcal{R}_m$, because by construction of $\boldsymbol{e}$ we have a u.c.e. chain $(F_k(\boldsymbol{a}))$ such that $\boldsymbol{x} < \boldsymbol{e} \;\Leftrightarrow\; \exists k \; \boldsymbol{x} \leq F_k(\boldsymbol{a})$. Such a chain converging to $\mathbf{1}$ does not exist, because $\{i : W_i \equiv_m K\}$ is Σ_3^0-complete. The new presentation of $\mathcal{R}_m^- \cong [\mathbf{o}, \boldsymbol{e})$ might be useful because it allows for a theory of effective automorphisms.

Theorem 3.2 can be obtained for $\mathcal{D}_m$ by simply taking a closure of $[\mathbf{o}, \boldsymbol{a}]$ under Skolem functions ω many times. This uses the fact that each initial interval of $\mathcal{D}_m$ is countable.

4 Appendix

THEOREM 4.1
Suppose that the c.e. set A is r-maximal and let $\boldsymbol{a}$ be its many-one degree. Then $\boldsymbol{a}$ is join-irreducible.

Proof. Recall the Lachlan map $\Psi : \mathcal{L}(A) \mapsto [\mathbf{o}, \boldsymbol{a}]$ introduced in [4]: $\Psi(X)$ is well-defined as the many-one degree of $f^{-1}(A)$, where f is any computable function with range X. Lachlan proves that Ψ is an onto upper semilattice homomorphism. Clearly $\Psi(A) = \mathbf{o}, \Psi(\omega) = \boldsymbol{a}$ and Ψ does not depend on finite differences.

Since ω^* is join irreducible in $\mathcal{L}^*(A)$, it is sufficient to prove that $\psi(B) < \boldsymbol{a}$ for any coinfinite $B \in \mathcal{L}(A)$. Suppose otherwise. Choose a computable function f with range B. Then $A \leq_m f^{-1}(A)$ via a computable function h. Let F, H denote the maps $X \mapsto f^{-1}(X)$ and $Y \mapsto h^{-1}(Y)$, respectively $(X, Y \subseteq \omega)$. Thus $A = H(F(A))$, and the maps restricted to c.e. sets $F : [A, B] \mapsto [F(A), \omega]$ and $H : [F(A), \omega] \mapsto [A, \omega]$ are distributive lattice embeddings preserving the least and greatest elements. But A is a major subset of B, so, by the Owings Splitting Theorem (see [10]) $[A, B]$ has nontrivial complemented elements, while $[A, \omega]$ does not, a contradiction. ∎

Clearly, $\boldsymbol{a}$ it is not minimal unless A is maximal. It would be interesting to see to what extent $[\mathbf{o}, \boldsymbol{a}]$ determines $\mathcal{L}(A)$.

References

[1] C. C. Chang and H. J. Keisler. *Model Theory.* North–Holland Publishing Co., Amsterdam, 1973.

[2] S. D. Denisov. The structure of the uppersemilattice of recursively enumerable m-degrees and related questions, I. *Algebra and Logic*, 17:418–443, 1972.

[3] Y. Ershov and L. Lavrov. The upper semilattice L(γ). Alg. Log. (transl), 12:93–106, 1973.

[4] A. H. Lachlan. Initial segments of many-one degrees. *Canad. J. Math.*, 22:75–85, 1970.

[5] A. Nies. The last question on recursively enumerable many-one degrees. *Algebra i Logika*, 33(5):550–563, 1995.

[6] A. Nies. Model theoretic properties of degree structures. Proc. of the 11th Conference on Logic, Methodology and Philosophy of Science, to appear.

[7] P. Odifreddi. *Classical Recursion Theory (Volume I).* North–Holland Publishing Co., Amsterdam. 1989.

[8] P. Odifreddi. *Classical Recursion Theory (Volume II).* North–Holland Publishing Co., Amsterdam. 1999.

[9] E. L. Post. Recursively enumerable sets of positive integers and their decision problems. *Bull. Amer. Math. Soc.*, 50:284–316, 1944.

[10] R. Soare. Recursively Enumerable Sets and Degrees. *Perspectives in Mathematical Logic, Omega Series*, Heidelberg, 1987.

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