INTERVALS OF THE LATTICE OF COMPUTABLY ENUMERABLE SETS AND EFFECTIVE BOOLEAN ALGEBRAS

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Abstract

We prove that each interval of the lattice $\mathscr{E}$ of c.e. sets under inclusion is either a boolean algebra or has an undecidable theory. This solves an open problem of Maass and Stob [**11**]. We develop a method to prove undecidability by interpreting ideal lattices, which can also be applied to degree structures from complexity theory. We also answer a question left open in [**7**] by giving an example of a non-definable subclass of $\mathscr{E}^*$ which has an arithmetical index set and is invariant under automorphisms.

1. *Introduction*

Intervals play an important role in the study of the lattice $\mathscr{E}$ of computably enumerable (c.e.) sets under inclusion. Several interesting properties of a c.e. set can be given alternative definitions in terms of the structure of $\mathscr{L}(A)$, the lattice of c.e. supersets of A. For instance, hyperhypersimplicity of a coinfinite c.e. set A is equivalent to $\mathscr{L}(A)$ being a boolean algebra, and A is r-maximal if and only if $\mathscr{L}(A)$ has no non-trivial complemented elements.

A further type of interval is obtained by considering the major subset relation: for $A, B \in \mathscr{E}$,

$$A \subset_m B \Leftrightarrow A \subset_\infty B \wedge (\forall W \text{ c.e.})[B \cup W = \mathbb{N} \Rightarrow A \cup W =^* \mathbb{N}].$$

Maass and Stob [**11**] proved that for each pair A, B such that $A \subset_m B$, up to isomorphism one obtains the same lattice $[A, B]_{\mathscr{E}}$. This structure is denoted by $\mathscr{M}$. For any lattice $\mathscr{X}$ of sets considered here, $\mathscr{X}^*$ will denote the quotient structure of $\mathscr{X}$ modulo finite differences. From the Maass–Stob result, it follows that $\mathscr{M}^*$ is a distributive lattice with strong homogeneity properties: all non-trivial closed intervals are isomorphic to the whole structure, and all non-trivial complemented elements are automorphic within $\mathscr{M}^*$. However, $\mathscr{M}^*$ is not a boolean algebra.

A natural question to ask is which intervals $[A, B]_{\mathscr{E}}$ have an undecidable theory. For instance, Maass and Stob pose this question for $\mathscr{M}$, as a part of a programme to analyse the structure of $\mathscr{M}$. Recall that boolean algebras have a decidable theory by a result due to Ershov (see [**2**]). It is known that Th($\mathscr{E}$) is undecidable [**8**] and has, in fact, the same computational complexity as true first-order arithmetic [**7**]. However, so far intervals of $\mathscr{E}$ isomorphic to the whole structure have been the only case where such an undecidability result was known. Our principal result is that each interval which is not a boolean algebra has, in fact, an undecidable theory. In particular, this proves that Th($\mathscr{M}$) is undecidable.

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The possible structure of intervals of $\mathcal{E}$ is still not very well understood. However, Lachlan [**10**] shows that the boolean algebras which can be represented as $\mathcal{L}(A)^*$, A hyperhypersimple, are precisely the Σ^0_3 boolean algebras. The class of r-maximal sets is much more elusive. However, recently P. Cholak and the author [**3**] have shown that infinitely many non-isomorphic lattices $\mathcal{L}^*(A)$, A r-maximal, exist.

We now explain our methods. Many proofs that a problem is undecidable are indirect: one gives a reduction of a problem which is already known to be undecidable to the problem in question. A theory is a consistent set of first-order sentences in some language which is closed under logical inference. For theories of structures or of classes of structures, a particular type of reduction based on the notion of interpretations of structures is used. It makes use of the following stronger notion of undecidability: call a theory T in a first-order language L *hereditarily undecidable* (h.u.) if each set $X \subseteq T$ which contains the valid L-sentences (that is, the sentences which can be inferred from $\emptyset$) is undecidable. The transfer principle states that if $\mathbf{A}$ is an L_1-structure, $\mathbf{B}$ is an L_2-structure, and $\mathbf{A}$ can be interpreted in $\mathbf{B}$ with parameters, then

$$\text{Th}(\mathbf{A}) \text{ h.u.} \quad \Rightarrow \quad \text{Th}(\mathbf{B}) \text{ h.u.} \tag{1}$$

See [**9**, Chapter 5] for a definition of the concept of interpretations of structures. An interpretation with parameters of $\mathbf{A}$ in $\mathbf{B}$ is an interpretation of $\mathbf{A}$ in a structure obtained from $\mathbf{B}$ by adding finitely many constants. The transfer principle was obtained by Burris and McKenzie (see [**1**]) and holds, in fact, for the theories of classes of structures as well.

Interpretability is a transitive relation. Therefore, using the transfer principle, one can proceed to show that more and more theories are h.u. For instance, to prove that Th($\mathcal{E}$) is h.u., one begins with the initial class $\mathbf{C}$ of finite structures for an appropriate finite relational signature. Hereditary undecidability of Th($\mathbf{C}$) can be shown directly by viewing such structures as terminating computations (Lavrov; see, for example, [**13**]). Now $\mathbf{C}$ can be interpreted in the class of finite symmetric graphs, which in turn can be interpreted in $\mathcal{E}$ (see [**7**]). In the original proof of undecidability of Th($\mathcal{E}$) due to Herrmann [**8**], a further intermediate class was used. For $k \geq 1$, let $\mathcal{E}^k = (\Sigma^0_k, \subseteq)$. Relativization of any of the proofs mentioned to $\emptyset^{(k-1)}$ gives hereditary undecidability of Th($\mathcal{E}^k$) for any $k \geq 1$, a fact which will be used to obtain our result: we develop a method to interpret $\mathcal{E}^k$ with parameters, which is general enough to work in other similar settings where no direct interpretation of a sufficiently rich class of finite structures is apparent. Indeed, in [**4**], R. Downey and the author use the method to show the undecidability of the theory of various low-level complexity degree structures, for instance of the polynomial time T-degrees of exponential time computable sets.

Our method works by using interpretations of $\mathcal{E}^k$ in the lattice of c.e. ideals of c.e. boolean algebras as an intermediate step. Thus the results are of interest also in the theory of effective boolean algebras (see, for instance, [**12**]). Recall that a boolean algebra is c.e. if $\mathcal{B} = D/I$ for a c.e. ideal I of the computable dense boolean algebra D. We call a c.e. boolean algebra $\mathcal{B}$ effectively dense if for each element x of $\mathcal{B}$, we can effectively find an element $y \leq x$ such that $x \neq 0$ implies $0 < y < x$. Thus, for example, the recursive dense boolean algebra is effectively dense, but in fact many other c.e. presentations of the countable dense boolean algebra are as well. For instance, consider the Lindenbaum algebra of sentences over Peano arithmetic.

This c.e. boolean algebra is effectively dense by Rosser's theorem, a refinement of Gödel's second incompleteness theorem (see, for example, [**6**]).

Let $\mathcal{I}(\mathcal{B})$ be the lattice of c.e. ideals of an effectively dense boolean algebra $\mathcal{B}$; if we work with relativizations to $\varnothing^{(k-1)}$, $\mathcal{I}(\mathcal{B})$ denotes the lattice of Σ^0_k ideals of $\mathcal{B}$. We prove that

$$\mathrm{Th}(\mathcal{I}(\mathcal{B})) \text{ is hereditarily undecidable} \tag{2}$$

by giving an interpretation of $\mathcal{E}^3$ in $\mathcal{I}(\mathcal{B})$. Now, in several situations where no direct coding of a sufficiently complex class of finite structures in **A** is apparent, there is a natural way to interpret $\mathcal{I}(\mathcal{B})$ (in fact, the two-sorted structure $(\mathcal{B}, \mathcal{I}(\mathcal{B}))$ for an appropriate Σ^0_3 or Σ^0_2 boolean algebra which is effectively dense relative to $\varnothing''$ ($\varnothing'$, respectively). Then relativizations of the fact (2) are used to obtain undecidability of Th(**A**).

We shall formulate our results for intervals of $\mathcal{E}^*$; the case of $\mathcal{E}$ follows as an easy corollary. As an example of an application of our method, consider the case of intervals $[D^*, A^*]$ of $\mathcal{E}^*$, and assume that $D \subset_m A$ (we shall see that this is no essential restriction). The Σ^0_3 boolean algebra $\mathcal{B}$ of complemented elements in $[D^*, A^*]$ is $\varnothing''$-effectively dense by the Owings Splitting Theorem [**15**]. Moreover, we show that, for each Σ^0_3 ideal I of $\mathcal{B}$, there exists a c.e. set C_I such that $D \subseteq C_I \subseteq A$ and

$$I = \{X^* \in \mathcal{B} : X \cap C_I =^* D\}. \tag{3}$$

Conversely, each ideal of that form must be a Σ^0_3 ideal. Now, for the desired interpretation, we represent ideals I ambiguously by elements $c = C_I^*$. Inclusion of Σ^0_3 ideals can be defined within $[D^*, A^*]$ using the formula

$$\varphi_{\leq}(c_1, c_2) \equiv \forall x (x \text{ complemented in } [d, a] \Rightarrow (x \wedge c_1 = d \Rightarrow x \wedge c_2 = d)), \tag{4}$$

where $d = D^*$, etc. Thus $\mathcal{I}(\mathcal{B})$ can be interpreted in $[D^*, A^*]$, and $\mathrm{Th}([D^*, A^*])$ is hereditarily undecidable. In the case of low-complexity degree structures, an $\varnothing'$-effectively dense Σ^0_2 boolean algebra $\mathcal{B}$ is found such that $(\mathcal{B}, \mathcal{I}(\mathcal{B}))$ can be interpreted in a natural way.

In an appendix to the paper, we continue the study of non-definability in $\mathcal{E}^*$ which was begun in [**7**]. There, an example was given of a binary relation on $\mathcal{E}^*$ which satisfies the conditions (necessary for definability without parameters) of being arithmetical and invariant under automorphisms, but the relation is, in fact, not definable. Here we obtain an example of such a relation which is, in fact, a subclass of $\mathcal{E}^*$. Recall that a c.e. set A is *quasimaximal* if $\mathcal{L}^*(A)$ is a finite boolean algebra. In this case, let

$$n(A) = \text{number of atoms in } \mathcal{L}^*(A). \tag{5}$$

In [**7**] it was shown that quasimaximality is first-order definable in $\mathcal{E}^*$. However, we prove that the class $\{A^* : n(A) \geq 2 \ \wedge \ n(A) \text{ is a power of } 2\}$ is not definable.

2. *Computably enumerable boolean algebras*

In this section we obtain the necessary results about c.e. boolean algebras. First we give detailed definitions of the concepts used in the Introduction. We specify the notion of c.e. boolean algebra as follows. A c.e. boolean algebra is represented by a model

$$(\mathbb{N}, \preceq, \vee, \wedge)$$

such that $\preceq$ is a c.e. relation which is a preordering, $\vee, \wedge$ are total computable binary functions, and the quotient structure

$$\mathscr{B} = (\mathbb{N}, \preceq, \vee, \wedge)/\equiv$$

is a boolean algebra (where $n \equiv m \Leftrightarrow n \preceq m \wedge m \preceq n$).

We require that 0 is an index for the least element of $\mathscr{B}$, and 1 is an index for the greatest element. Then $0 \not\equiv 1$ by the definition of boolean algebras. Note that, in an effective way, for each b_n we can find an index for a complement of $b_n/\equiv$ in $\mathscr{B}$, denoted by $\mathrm{Cpl}(b_n)$. At stage s of the algorithm, see if there is $b \leq s$ such that $b_n \wedge b \equiv 0$ and $b_n \vee b \equiv 1$, and these equivalences can be verified in $\leq s$ steps. If so, return b as an output. We write $b - c$ for $b \wedge \mathrm{Cpl}(c)$, and $b \prec c$ for $b \preceq c \wedge c \not\preceq b$. In general, '$b \prec c$' is not decidable.

A c.e. boolean algebra $\mathscr{B}$ is *effectively dense* if there is a computable function F such that

$$x \not\equiv 0 \quad \Rightarrow \quad 0 \prec F(x) \prec x. \tag{6}$$

We can assume that $\forall x(F(x) \preceq x)$: else replace F by the computable function $F(x) \wedge x$.

In fact, we apply relativizations of our results to some oracle set X, usually $X = \emptyset''$ or $X = \emptyset'$ (for the polynomial time degrees). In such relativizations, all effectivity notions have to be replaced by the corresponding notions relative to X; for instance 'c.e.' becomes 'c.e. in X', and the functions $\vee, \wedge$ are computable in X. Thus, in the relativized case, our notion is a little more general than requiring that $\mathscr{B} = \mathbf{D}/I$ for an X-c.e. ideal I of the computable dense boolean algebra $\mathbf{D}$ (where $\vee, \wedge$ would still be computable).

We shall identify subsets of $\mathscr{B}$ with the corresponding index sets, which we always require to be closed under the equivalence relation $\equiv$. Thus an ideal of $\mathscr{B}$ is called c.e. if the corresponding set of indices is c.e. The c.e. ideals form a sublattice $\mathscr{I}(\mathscr{B})$ of the distributive lattice of all ideals, because, for c.e. ideals I, J, the infimum $I \cap J$ and the supremum $I \vee J = \{b \vee c : b \in I \wedge c \in J\}^{\equiv}$ are c.e. again. In what follows, we shall actually give interpretations in the two-sorted structure $(\mathscr{B}, \mathscr{I}(\mathscr{B}))$. But this structure can be interpreted in the lattice $\mathscr{I}(\mathscr{B})$ in a natural way: represent $b \in \mathscr{B}$ by the principal ideal $\hat{b} = [0, b]_{\mathscr{B}}$. Since the principal ideals are just the complemented elements in $\mathscr{I}(\mathscr{B})$, the set of ideals in $\mathscr{I}(\mathscr{B})$ representing elements of $\mathscr{B}$ is definable in $\mathscr{I}(\mathscr{B})$ without parameters. Moreover, the membership relation '$b \in I$' can be translated into '$\hat{b} \subseteq I$'.

THEOREM 2.1. *Suppose that $\mathscr{B}$ is a c.e. boolean algebra which is effectively dense. Then $\mathscr{I}(\mathscr{B})$ has a hereditarily undecidable theory.*

REMARKS. (1) The proof of the theorem will relativize to any oracle set X. Thus if $\mathscr{B}$ is X-c.e. and (6) holds via an X-computable function F, then $\mathscr{I}(\mathscr{B})$ has an h.u. theory as well.

(2) The hypothesis that $\mathscr{B}$ be *effectively* dense is needed: it is possible to construct a c.e. dense boolean algebra such that each c.e. ideal is principal, thus $\mathscr{I}(\mathscr{B}) \simeq \mathscr{B}$ has a decidable theory. To construct such a $\mathscr{B}$, one adapts the construction of a maximal c.e. set due to Friedberg (see [**15**]), replacing numbers by strings in $2^{<\omega}$ and c.e. sets by c.e. subsets of $2^{<\omega}$ which are closed under taking extensions of strings.

The main component of the proof is a uniform definability lemma for the Σ^0_3 ideals of $\mathscr{B}$ which contain a certain 'separating' c.e. ideal I_0, where $|\mathscr{B}/I_0| = \infty$. This proof uses some ideas from **[7]** in the more general context of c.e. boolean algebras. We shall find a formula with parameters $\varphi(x; L, I_0)$ such that, as L varies over c.e. ideals, $\{x : (\mathscr{B}, \mathscr{I}(\mathscr{B})) \models \varphi(x; L, I_0)\}$ will range over the Σ^0_3 ideals of $\mathscr{B}$ containing I_0. Then, intuitively speaking because Σ^0_3 is far beyond the level of complexity of the c.e. structure $\mathscr{B}$ itself, it will be possible to give an interpretation of $\mathscr{E}^3$ in $\mathscr{I}(\mathscr{B})$, using I_0 as a parameter.

We say that a c.e. ideal I_0 of $\mathscr{B}$ is *separating* if the following holds in $\mathscr{B}$:

$$\forall x \exists y \leq x \ y \in I_0 \wedge (x \notin I_0 \Rightarrow y \not\equiv 0) \tag{7}$$

and, moreover, y can be determined effectively in x. The intuition is that a separating ideal non-trivially meets all the principal ideals $\neq \{0\}$, in an effective way.

LEMMA 2.2. *$\mathscr{B}$ possesses a separating c.e. ideal I_0 such that the boolean algebra $\mathscr{B}/I_0$ is infinite.*

Proof. We write b_n instead of n if we think of the number n as determining an element of the boolean algebra under consideration, and call b_n an *index* for the element $b_n/\equiv$. First we consider the easier problem of how to build a separating ideal I_0 such that $\mathscr{B}/I_0$ has at least two elements. Recall that F is the function from (6). Let $y_0 = F(b_0)$ (so $y_0 \not\equiv 1$). If $y_0, \ldots, y_n$ have already been defined, then let $y_{n+1} = y_n \vee F(b_{n+1} - y_n)$. Let I_0 be the ideal generated by $\{y_i : i \in \mathbb{N}\}$. Then I_0 is c.e. and separating, because $b_{n+1} - y_n \not\equiv 0$ if $b_{n+1} \notin I_0$. Also $I_0 \neq \mathscr{B}$: otherwise, suppose that n is the least number such that $y_{n+1} \equiv 1$. Then $F(b_{n+1} - y_n) \geq \mathrm{Cpl}(y_n)$, which is impossible by our hypothesis on F and since $\mathrm{Cpl}(y_n) \not\equiv 0$.

We now refine the construction in order to make $\mathscr{B}/I_0$ infinite. To this end, we also define elements $z_0 < z_1 < \ldots$ of $\mathscr{B}$ such that $(z_n/I_0)_{n \in \mathbb{N}}$ is a strictly ascending chain in $\mathscr{B}/I_0$. As above, let $y_0 = F(b_0)$, and let $z_0 = 0$. Now, if $y_0, \ldots, y_n$ and $z_0 < \ldots < z_n$ have already been defined, then consider the 'partition'

$$p_0 = z_1 - z_0, \quad \ldots, \quad p_{n-1} = z_n - z_{n-1}, \quad p_n = \mathrm{Cpl}(z_n).$$

Our intention is never to put so much into I_0 that one of the components of the partition goes completely into I_0. Let $c_{n+1} = b_{n+1} - y_n$. Note that if $c_{n+1} \not\equiv 0$, then the same must hold for $c_{n+1} \wedge p_i$ for some i. Thus if we let

$$y_{n+1} = y_n \vee \bigvee_{i \leq n} F(c_{n+1} \wedge p_i),$$

we make sure that (7) is satisfied for $x = b_{n+1}$.

To extend the ascending sequence, also let

$$z_{n+1} = F(\mathrm{Cpl}(y_{n+1})) \vee z_n. \tag{8}$$

Again, let I_0 be the ideal generated by $\{y_i : i \in \mathbb{N}\}$. We verify that I_0 has the required properties. Since the sequence (y_n) is effective, I_0 is c.e. Moreover, I_0 is separating, because if $b_{n+1} \notin I_0$, then $c_{n+1} \not\equiv 0$, and therefore $y_{n+1} \not\equiv 0$. Furthermore, y_{n+1} was determined effectively from b_{n+1}. If n is least such that $y_{n+1} \equiv 1$, then

$$\bigvee_{i \leq n} (F(b_{n+1}) \wedge \mathrm{Cpl}(y_n) \wedge p_i) \geq \mathrm{Cpl}(y_n),$$

contrary to the fact that $F(b_{n+1}) \wedge \mathrm{Cpl}(y_n) \wedge p_i < b_{n+1} \wedge \mathrm{Cpl}(y_n) \wedge p_i$ for some i. Thus $y_n \not\equiv 1$ for each n.

Fix n. We now show that $d = z_{n+1} - z_n \notin I_0$. Since $y_{n+1} \not\equiv 1$, $d \not\equiv 0$, so $d \not\preceq y_0$. Suppose that k is least such that $d \preceq y_{k+1}$. Then $k > n$, because $d \preceq \mathrm{Cpl}(y_{n+1})$, but $y_{k+1} \preceq y_{n+1}$ for $k \leq n$. We now argue as above, but restricted to the interval $[0, d]$. By the minimality of k, $d \wedge y_k < d$, so

$$0 < d - y_k \preceq \bigvee_{i \leq k} F(c_{k+1} \wedge p_i).$$

Since the $(p_i)_{i \leq k}$ form a partition and $d = p_n$, in the supremum above only the term $F(c_{k+1} \wedge p_n)$ matters. Thus (recall that $c_{k+1} = b_{k+1} - y_k$)

$$d - y_k \preceq F(c_{k+1} \wedge d) = F((d - y_k) \wedge b_{k+1}).$$

Since $d - y_k \not\equiv 0$, this contradicts the properties of F.

LEMMA 2.3. *Suppose that I_0 is a c.e. separating ideal. For each Σ^0_3 ideal J, $I_0 \subseteq J$, there is a c.e. ideal $L \subseteq I_0$ such that*

$$x \in J \Leftrightarrow \exists r \in I_0 \; \forall s \in I_0 \; (s \wedge r \equiv 0 \Rightarrow x \wedge s \in L). \tag{9}$$

Proof. Choose a computable function $x \mapsto y(x)$ such that given x, $y(x)$ is a witness for (7). We first define a computable sequence (s_n) which generates I_0 as an ideal and has further useful properties. To start with, since I_0 is c.e., there is some computable sequence (y_i) generating I_0. Let $\mathscr{B}_{\leq e}$ be a finite set of indices for the boolean algebra generated by $\{0, \dots, e\}$ ($\mathscr{B}_{\leq e}$ can be obtained effectively from e). Moreover, let $s_0 = y_0$ and

$$s_{n+1} = (y_{n+1} - \hat{s}_n) \vee \bigvee \{y(z - \hat{s}_n) : z \in \mathscr{B}_{\leq n}\}, \tag{10}$$

where $\hat{s}_n = s_0 \vee \dots \vee s_n$. Clearly $s_i \wedge s_j \equiv 0$ for $i \neq j$.

We make use of a lemma from [7] (Lemma 4 in the appendix): if P is a Σ^0_3 set, then there is a uniformly c.e. sequence (Z_i), $Z_i \subseteq \{0, \dots, i-1\}$, such that $e \in P \Rightarrow$ a.e. i $[e \in Z_i]$ and $\exists^\infty i$ $[Z_i \subseteq P]$. Applying this to $P = J$ (viewed as an index set), we obtain

$$e \in J \Rightarrow \text{a.e. } i \; [e \in Z_i],$$

$$\exists^\infty i \; [Z_i \subseteq J].$$

Let L be the ideal generated by

$$\{e \wedge s_i : e \in Z_i\}.$$

Clearly $L \subseteq I_0$ and L is c.e. We now verify (9).

'$\Rightarrow$' Suppose that $x \in J$. Choose $\tilde{i}$ such that $\forall i > \tilde{i}$ $(x \in Z_i)$, and let $r = s_0 \vee \dots \vee s_{\tilde{i}}$. If $s \in I_0$ and $s \wedge r = 0$, then, for some $j > \tilde{i}$, $s \preceq s_{\tilde{i}+1} \vee \dots \vee s_j$. But $x \wedge s_k \in L$ for all $k > \tilde{i}$. Therefore $x \wedge s \in L$.

'$\Leftarrow$' Now suppose that $x \notin J$. Given $r \in I_0$, choose k such that $r \preceq \hat{s}_k$. Choose $i > k$ such that $Z_i \subseteq J$ and also $i > x$. We show that the witness s_i is a counterexample to (9), that is, $x \wedge s_i \notin L$.

Let $v = x - \bigvee_{e \in Z_i} e - \hat{s}_{i-1}$. Then $v \notin I_0$: else, since $\hat{s}_{i-1} \in I_0 \subseteq J$ and $\bigvee_{e \in Z_i} e \in J$, we could infer that $x \in J$. Therefore $y(v) \not\equiv 0$. Also, $z = x - \bigvee_{e \in Z_i} e \in \mathscr{B}_{\leq i-1}$, so $v = z - \hat{s}_{i-1}$ occurs in the disjunction (10) where s_i is defined. Hence $y(v) \preceq s_i \wedge v$, and therefore $s_i \wedge x - \bigvee_{e \in Z_i} e \not\equiv 0$. But this implies that s_i is a counterexample: if

$x \wedge s_i \in L$, then by the fact that the (s_k) are pairwise disjoint, $x \wedge s_i \preceq \bigvee_{e \in Z_i} e \wedge s_i$. This means that $s_i \wedge (x - \bigvee_{e \in Z_i} e) \equiv 0$, a contradiction.

LEMMA 2.4. *$\mathscr{E}^3 = (\Sigma^0_3, \subseteq)$ can be interpreted in the two-sorted structure $(\mathscr{B}, \mathscr{I}(\mathscr{B}))$.*

Proof. By Lemma 2.2, fix a separating ideal I_0 of $\mathscr{B}$ such that $\mathscr{B}/I_0$ is infinite. By the previous lemma, the lattice $\mathbf{L}$ of Σ^0_3 ideals of $\mathscr{B}$ which contain I_0 can be interpreted in $(\mathscr{B}, \mathscr{I}(\mathscr{B}))$, using I_0 as a parameter. So it is sufficient to show that $(\Sigma^0_3, \subseteq) \simeq [C, D]_{\mathbf{L}}$ for some $C, D \in \mathbf{L}$. We distinguish two cases.

Case A: *$\mathscr{B}/I_0$ has infinitely many atoms.* Let $C = I_0$ and let D be the ideal generated by I_0 and the preimages in $\mathscr{B}$ of atoms of $\mathscr{B}/I_0$. Notice that 'x/I_0 is an atom of $\mathscr{B}/I_0$' is a Π^0_2 property of indices, so there is a function $f \leq_T \varnothing''$ such that $(f(n)/I_0)_{n \in \mathbb{N}}$ is an enumeration of the atoms of $\mathscr{B}/I_0$ without repetition. This implies that D is a Σ^0_3 ideal and, moreover,

$$J \mapsto \{n \in \mathbb{N} : f(n) \in J\}$$

is an isomorphism between $[C, D]_{\mathbf{L}}$ and $(\Sigma^0_3, \subseteq)$.

Case B: *$\mathscr{B}/I_0$ has only finitely many atoms.* If $\mathscr{B}/I_0$ has only finitely many atoms, let b be a preimage in $\mathscr{B}$ of their supremum. Replacing I_0 by the separating ideal $I_0 \vee [0, b]$ if necessary, we can, in fact, assume that $\mathscr{B}/I_0$ is dense and hence free. Note that $\mathscr{B}/I_0$ is c.e. The standard step-by-step construction of a free generating sequence for a dense countable boolean algebra produces in the case of $\mathscr{B}/I_0$ an $\varnothing'$-sequence (a_i) such that (a_i/I_0) is a free generating sequence for $\mathscr{B}/I_0$. Now let $\mathscr{F}$ be the boolean algebra of finite or cofinite subsets of $\mathbb{N}$, and consider the natural map $g : \mathscr{B}/I_0 \mapsto \mathscr{F}$ defined by $g(a_i/I_0) = \{i\}$. Clearly, g is computable in $\varnothing'$ if viewed as a map from indices for $\mathscr{B}$ into an effective representation for $\mathscr{F}$. Let C be the ideal $\{x : g(x) = 0\}$, and let D be the ideal generated by the a_i and I_0. Then C, D are Σ^0_3 ideals of $\mathscr{B}$ containing I_0, and the Σ^0_3 ideals X of $\mathscr{B}$ such that $C \subseteq X \subseteq D$ correspond to the Σ^0_3 ideals of $\mathscr{F}$ which are contained in the ideal generated by the atoms. So again $(\Sigma^0_3, \subseteq) \simeq [C, D]_{\mathbf{L}}$. Now, since $(\mathscr{B}, \mathscr{I}(\mathscr{B}))$ can be interpreted in $\mathscr{I}(\mathscr{B})$, this concludes the proof of Theorem 2.1.

3. *Intervals of $\mathscr{E}^*$ and $\mathscr{E}$*

THEOREM 3.1. *Suppose that $D \subseteq A$ and $[D^*, A^*]_{\mathscr{E}^*}$ is not a boolean algebra. Then the elementary theory of $[D^*, A^*]$ is undecidable.*

Proof. We give an interpretation with parameters of the lattice of Σ^0_3 ideals of a $\varnothing''$-effectively dense Σ^0_3 boolean algebra in $[D^*, A^*]_{\mathscr{E}^*}$. Then, by the relativization to $\varnothing''$ of Theorem 2.1, the theory of $[D^*, A^*]_{\mathscr{E}^*}$ is undecidable.

The following argument, due to Lachlan, shows that we can, in fact, assume that $D \subset_m A$. First, we can suppose that A is non-computable, else just replace A by some non-computable set $\tilde{A}$, with $D \subseteq \tilde{A} \subseteq A$, and use $\tilde{A}^*$ as an extra parameter in the interpretation. Now choose a small major subset E of A (see [**15**, **7**]). By the properties of small major subsets, $D \cup E \subset_\infty A$. So replace D by $\tilde{D} = D \cup E$. The advantage of having $D \subset_m A$ is that for each computable $R \subseteq A$, $R \subseteq^* D$. (This is actually equivalent to $D \subset_m A$ if A is non-computable and $D \subseteq_\infty A$.) The desired boolean algebra is

$$\mathscr{B} = \{(A_0 \cup D)^* : A_0 \sqsubset A\},$$

where $X \sqsubset Y$ means that $X \subseteq Y \ \wedge\ Y - X$ is c.e., in which case we call X a *split* of Y. By the reduction principle, $\mathscr{B}$ equals the set of complemented elements in $[D^*, A^*]$. We use a representation of $\mathscr{B}$ as follows. The set $S = \{e : W_e \sqsubset A\}$ is Σ^0_3. Choose a function $f \leq_T \varnothing''$ such that range$(f) = S$, and let k represent the element $(W_{f(k)} \cup D)^*$. Clearly, the induced ordering on indices

$$e \preceq i \Leftrightarrow W_{f(e)} \subseteq^* W_{f(i)} \cup D$$

is Σ^0_3 and $\varnothing''$-computable functions $\vee, \wedge$ can be defined in the appropriate way. Also $\mathscr{B}$, with this representation, is $\varnothing''$-effectively dense, by the Owings Splitting Theorem (see [**15**]). In fact, the Owings Splitting Theorem is effective, but it takes $\varnothing''$ to determine $(W_{f(k)} \cup D)^*$ from k.

We now prove a lemma which allows us to give an interpretation of $\mathscr{I}(\mathscr{B})$ in $[D^*, A^*]$.

LEMMA 3.2. *If $D \subseteq_m A$ and I is a Σ^0_3 ideal of $\mathscr{B}$, then there is C, with $D \subseteq^* C \subseteq^* A$, such that*

$$I = \{j : W_{f(j)} \cap C \subseteq^* D\}. \tag{11}$$

Proof. First we give an effective representation of the filter of complements of elements of I, using the following uniformization fact.

FACT. *If $(W_{g(i)})$ is a sequence of splits of A, $g \leq_T \varnothing''$, then there is a uniformly c.e. sequence of splits (U_i) of A such that $W_{g(i)} \triangle U_i \subseteq^* D$.*

To prove this, choose a u.c.e. sequence (V_k) of initial segments of $\mathbb{N}$ such that $W_p = W_{g(i)} \Leftrightarrow \exists n V_{\langle i,p,n\rangle} = \mathbb{N}$ (this is possible since '$W_p = W_{g(i)}$' is Σ^0_3). The desired u.c.e. sequence is

$$U_i = \{a : \exists s \exists q = \langle i,p,n\rangle [\max \bigcup_{\langle i,p',n'\rangle < q} V_{\langle i,p',n'\rangle,s} < a \leq \max V_{q,s} \wedge a \in W_{p,s}]\}.$$

Given i, let $p = g(i)$ and let $q = \langle p,n\rangle$ be least such that $V_{\langle i,p,n\rangle} = \mathbb{N}$. Then $U_i =^* R \cup W_p$, where R is the computable set $\{a : \exists s\ a \in U_{i,s} \wedge a > \max V_{q,s}\}$. Therefore $W_p \triangle U_i \subseteq^* R \subseteq^* D$. This proves the fact.

Clearly, the indices of c.e. sets which are complements of elements in I,

$$S = \{i : \exists k \in I\ W_i \cap W_{f(k)} \subseteq^* D \wedge W_i \cup W_{f(k)} =^* A\},$$

is Σ^0_3, and therefore S is the range of a function $g \leq_T \varnothing''$. Applying the preceding fact, we obtain a sequence $(\tilde{U}_i)$. Let $U_n = \bigcap_{i \leq n} \tilde{U}_i$. Then the u.c.e. sequence $(U_n \cup D)^*_{n \in \mathbb{N}}$ generates the filter of complements of elements in I.

To build C, we meet for each n the following requirement:

$$P_n : |W_e \cap U_n \cap \overline{D}| = \infty \ \Rightarrow\ |W_e \cap C \cap \overline{D}| \geq k \quad (k = \langle e,n\rangle).$$

The construction of C is the following. Let $C_0 = \varnothing$. At a stage $s+1$, for each $\langle e,k\rangle = n < s$, act as follows. If P_n is currently unsatisfied, namely $|W_{e,s} \cap C_s \cap \overline{D_s}| < k$, and there is an $x \in U_{n,s} - D_s$ such that $x \in W_{e,s}$, then enumerate the least such x into C.

We verify that C satisfies (11). Notice that at most $k+1$ elements which are permanently in $\overline{D}$ are enumerated into C for the sake of $P_{\langle e,k\rangle}$. Therefore $C \subseteq^* D \cup U_m$ for each m. Now, if $j \in I$, then choose an m such that $U_m - (A - W_{f(j)}) \subseteq^* D$, that is, $U_m \cap W_{f(j)} \subseteq^* D$. Since $C \subseteq^* D \cup U_m$, $C \cap W_{f(j)} \subseteq^* D$.

If $j \notin I$, then $(D \cup (A - W_{f(j)}))^*$ is not in the filter dual to I, so $D \cup (A - W_{f(j)})$ does not *-include U_n for any n. Thus if $e = f(j)$, then the hypothesis of all the requirements $P_{\langle e,k\rangle}$ is satisfied. Hence $C \cap W_e \cap \overline{D}$ is infinite.

Clearly (3) holds. So we can now obtain the desired interpretation of $\mathscr{I}(\mathscr{B})$ in $[D^*, A^*]$ as explained in the Introduction; see (4).

REMARK. Instead of building a C satisfying (11), we can also apply the Σ^0_3 case of the ideal definability lemma in [**7**] to produce a $C \in [D, A]$ such that $I = \{k : W_{f(k)} \subseteq^* C\}$. This gives an alternative way to interpret $\mathscr{I}(\mathscr{B})$ in $[D^*, A^*]$. However, we prefer to be as self-contained as possible. In fact, the proof given here is much simpler than the proof in [**7**] for the Σ^0_3 case of the ideal definability lemma, and Lemma 3.2 could be used to simplify that proof.

COROLLARY 3.3. *Suppose that $D \subseteq A$ and $[D, A]_{\mathscr{E}}$ is not a boolean algebra. Then the elementary theory of $[D, A]_{\mathscr{E}}$ is undecidable.*

Proof. As before, we can assume that $D \subset_m A$. But $\mathscr{M}^*$ can be interpreted in $\mathscr{M}$, since for $X, Y \in \mathscr{M}$,

$$X =^* Y \quad \Leftrightarrow \quad [X \cap Y, X \cup Y] \text{ is a boolean algebra.}$$

So another application of the transfer principle (1) concludes the proof.

Note. In recent work [**14**], the author has shown that, in fact, Th($\mathbb{N}$) can be interpreted in Th($\mathscr{I}(\mathscr{B})$), for any effectively dense $\mathscr{B}$. Using Lemma 3.2, this gives a way to interpret Th($\mathbb{N}$) in Th($\mathscr{M}$). Using [**11**], we can now strengthen Theorem 3.1 in a similar way.

4. *Appendix*

The function $n(A)$ was defined in (5).

THEOREM 4.1. *Let $X \subseteq \mathbb{N}$ be an infinite set of even numbers such that for each distinct $n, m \in X$, $(n + m)/2$ is not in X (for example, let $X = \{n \geq 2 : n$ is a power of $2\})$. Then $\{A^* : n(A) \in X\}$ is not definable in $\mathscr{E}^*$.*

REMARK. Notice that $\{A^* : n(A) \in X\}$ is invariant under automorphisms of $\mathscr{E}^*$. Moreover, if X is arithmetical, then this class has an arithmetical index set.

Proof. Let $P = \{A : n(A) \in X\}$. Since P is closed under finite differences, by a result of Lachlan [**10**], it suffices to prove non-definability of P in $\mathscr{E}$. (However, one could also perform some notational changes below to give a direct proof for $\mathscr{E}^*$.) If A is quasimaximal and R is an infinite coinfinite computable set, then $A \cap R$ is quasimaximal in $\mathscr{E}(R) = [\emptyset, R]_{\mathscr{E}}$. Let $n_R(A)$ denote $n(A \cap R)$ (evaluated in $\mathscr{E}(R)$). If B^* is an atom above A^* in $\mathscr{L}^*(A)$, then either $B \subseteq^* A \cup R$, in which case $(B \cap R)^*$ is an atom above $(A \cap R)^*$, or $B \subseteq^* A \cup \overline{R}$, in which case $(B \cap \overline{R})^*$ is an atom above $(A \cap \overline{R})^*$. Conversely, each atom above $(A \cap R)^*$ gives rise to one above A^*, and similarly for atoms above $(A \cap \overline{R})^*$. Thus

$$n(A) = n_R(A) + n_{\overline{R}}(A).$$

As in [7], a special case of a theorem of Feferman and Vaught [5] is used:

if $\mathbf{A}$ *is a structure and* $\varphi(X)$ *is a formula in the language of* $\mathbf{A}$, *then, for each element* $\langle a, b\rangle$ *of* $A \times A$,

$$\mathbf{A} \times \mathbf{A} \models \varphi(\langle a, b\rangle) \Leftrightarrow \bigvee_{\alpha=1,\ldots,r} (\mathbf{A} \models \varphi^\alpha(a) \wedge \mathbf{A} \models \psi^\alpha(b))$$

for some formulas $\varphi^\alpha, \psi^\alpha$ *which depend only on* φ.

If R is an infinite coinfinite computable set, then $\mathscr{E} \cong \mathscr{E} \times \mathscr{E}$ via the map

$$X \mapsto (X \cap R, X \cap \overline{R}).$$

Thus, if P is definable in $\mathscr{E}$ by a formula $\varphi(x)$, then

$$\mathscr{E} \models \varphi(X) \Leftrightarrow \bigvee_{\alpha=1,\ldots,r} (\mathscr{E}(R) \models \varphi^\alpha(X \cap R) \wedge \mathscr{E}(\overline{R}) \models \psi^\alpha(X \cap \overline{R})). \tag{12}$$

Now, for each $C \in P$, choose some computable set R_C such that $n(R_C) = n(C)/2$. By the pigeonhole principle, there are sets $A, B \in P$, $n(A) \neq n(B)$, so that (12) holds via the same α, if R is R_A (R_B, respectively). After applying an appropriate computable permutation, we can assume that $R = R_A = R_B$. Let $D = (A \cap R) \cup (B \cap \overline{R})$. Then $\mathscr{E} \models \varphi(D)$, because

$$\mathscr{E}(R) \models \varphi^\alpha(D \cap R) \wedge \mathscr{E}(\overline{R}) \models \psi^\alpha(D \cap \overline{R}).$$

But $n(D) = (n(A) + n(B))/2 \notin X$, giving a contradiction.

References

1. S. Burris and H. P. Sankappanavar, *A course in universal algebra* (Springer, Heidelberg, 1981).
2. C. C. Chang and H. J. Keisler, *Model theory* (North-Holland, Amsterdam, 1973).
3. P. Cholak and A. Nies, 'Atomless *r*-maximal sets', to appear.
4. R. Downey and A. Nies, 'Undecidability results for low level degree structures', *Proc. 12th Annual IEEE Conf. on Computational Complexity* (IEEE Computer Science Press, Los Alamitos, CA, 1997) 128–132.
5. S. Feferman and A. Vaught, 'The first-order properties of algebraic systems', *Fund. Math.* 47 (1959) 57–103.
6. J.-Y. Girard, *Proof theory and logical complexity* (Bibliopolis, Naples, 1987).
7. L. Harrington and A. Nies, 'Coding in the lattice of enumerable sets', *Adv. Math.*, to appear.
8. E. Herrmann, 'The undecidability of the elementary theory of the lattice of recursively enumerable sets (abstract)', *Proceedings of the 2nd Frege Conference at Schwerin, GDR (1984)*, Math. Forschung 20 (ed. G. Wechsung, Akademie, Berlin, 1984) 66–72.
9. W. Hodges, *Model theory*, Encyclopedia Math. Appl. 42 (Cambridge University Press, 1993).
10. A. Lachlan, 'On the lattice of recursively enumerable sets', *Trans. Amer. Math. Soc.* 130 (1968) 1–37.
11. W. Maass and M. Stob, 'The intervals of the lattice of recursively enumerable sets determined by major subsets', *Ann. Pure Appl. Logic* 24 (1983) 189–212.
12. A. Nerode and J. B. Remmel, 'A survey of lattices of recursively enumerable substructures', *Recursion theory*, Proc. Sympos. Pure Math. 42 (Amer. Math. Soc., Providence, RI, 1985) 322–375.
13. A. Nies, 'Undecidable fragments of elementary theories', *Algebra Universalis* 35 (1996) 8–33.
14. A. Nies, 'Effectively dense boolean algebras and their applications', to appear.
15. R. Soare, *Recursively enumerable sets and degrees* (Springer, Heidelberg, 1987).

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