Global properties of degree structures

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Abstract

We investigate degree structures induced by many-one reducibility and Turing reducibility on the computably enumerable (c.e.), the arithmetical all all subsets of $\mathbb{N}$. We study which subsets of the degree structure automorphism bases: for instance the minimal degrees form an automorphism base for the c.e. many one degrees, but not for the other degree structures based on $\leq_m$. We develop a method to show a subset is an automorphism base and apply it to the c.e. m-degrees and to give a modified proof of a result of Ambos-Spies that initial intervals of the c.e. degrees are automorphism bases. Also, we show that the arithmetical m-degrees form a prime model.

A central topic of computability theory is the study of sets of natural numbers under a notion of relative computability. Specifying a reducibility and an appropriate class of sets gives rise to a degree structure which may have interesting "global" properties. This is the case for the degree structures induced by many-one reducibility (denoted $\leq_m$) and Turing reducibility ($\leq_T$) on all sets, the arithmetical and the computably enumerable (c.e.) sets. For $r \in \{m, T\}$, these structures are denoted by $\mathcal{D}_r$, $\mathcal{A}_r$ and $\mathcal{R}_r$. The least element of a degree structure is denotes by $\mathbf{0}$, and the greatest (if any) by $\mathbf{1}$. In analyzing a degree structure $\boldsymbol{A}$, the following program is frequently carried out:

1. Understanding the basic algebraic properties. In other words, one determines whether $\boldsymbol{A} \models \varphi$, for certain φ of algebraic significance. Examples are: $\mathcal{D}_T, \mathcal{R}_m$ have minimal elements above $\mathbf{0}$, and $\mathcal{R}_T$ is dense.

2. Studying more general algebraic properties of $\boldsymbol{A}$. This includes embedding theorems and extensions of embeddings theorems for partial orders, embeddings of finite lattices (which is most interesting for $\mathcal{R}_T$) and characterizing initial segments (for $\mathcal{D}_T$, $\mathcal{D}_m$, $\mathcal{R}_m$).

3. Analyzing properties of Th($\boldsymbol{A}$). First one proves that Th($\boldsymbol{A}$) is undecidable. This has been done for most degree structures. Next one turns to fragments of the theories. All the Π_2-theories of the degree structures based on $\leq_m$, in the language of u.s.l., are decidable (Degtev [1979]), while, even in the language of p.o., their Π_3-theories are undecidable (Nies [1996]). The Π_3-theory of the p.o. $\mathcal{R}_T$ is undecidable by Lempp et al. [1998]. A further question of interest is whether $\mathrm{Th}(\boldsymbol{A})$ is atomic.

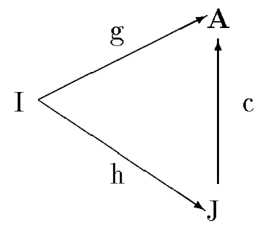
4. Model theoretic properties of $\boldsymbol{A}$. Now one considers the structure itself, not only its theory. Interesting topics are the interaction of subsets of $\boldsymbol{A}$ with the whole (esp. which subsets are automorphism bases), defining a copy of $\mathbb{N}$ without parameters, and whether (in the countable case) $\boldsymbol{A}$ is a prime model of its theory, or even biinterpretable with $\mathbb{N}$ in parameters (i.e. if there is a parameter defined copy M of $(\mathbb{N}, +, \times)$ and a parameter definable 1-1 map $f : \boldsymbol{A} \mapsto M$).

This article contributes mainly to part 4 of the program. The plan is as follows. In the preparatory Section 1 we state the extension property for the structures based on $\leq_m$, use it to recall characterizations of $\mathcal{D}_m$ and $\mathcal{R}_m$ and give a characterization of $\mathcal{A}_m$. A subset B of a structure $\boldsymbol{A}$ is an *automorphism base* if the only automorphism of $\boldsymbol{A}$ fixing B point-wise is the identity. In Section 2 we explore automorphism bases of $\mathcal{D}_m$ and $\mathcal{A}_m$. In Section 3 we consider codings of copies of $(\mathbb{N}, +, \times)$ in these structures and use them to show that $\mathcal{A}_m$ is a prime model and, under the assumption that $V = L$, $\mathrm{Th}(\mathcal{D}_m)$ is atomic. Finally, in Section 4 we develop a general method to show that a subset B of a structure $\boldsymbol{A}$ is an automorphism base. This method is applied firstly to show that each definable $D \subseteq \mathcal{R}_m$, $D \not\subseteq \{\mathbf{o}, \mathbf{1}\}$ is an automorphism base. Secondly we improve the proof of Ambos-Spies' result that each interval $\mathbf{o}, \boldsymbol{c}]$ of $\mathcal{R}_T$, $\boldsymbol{c} \neq \mathbf{o}$, is an automorphism base. As explained in Nies [ta], the different language used here promises to be useful in showing that $\mathcal{R}_T$ is biinterpretable with $\mathbb{N}$ in parameters.

Notation. Let (ψ_e) be an effective list of partial computable functions defined on initial segments of $\mathbb{N}$ which includes all the total computable functions. Given a degree $\boldsymbol{x}$ in degree structure, $\widehat{\boldsymbol{x}}$ denotes $[\mathbf{o}, \boldsymbol{x}]$.

1 The extension property for $\mathcal{D}_m, \mathcal{A}_m$ and $\mathcal{R}_m$

All three structures $\mathcal{D}_m, \mathcal{A}_m$ and $\mathcal{R}_m$ can be characterized as distributive upper semilattices (d.u.s.l.) with certain basic properties and an extension property (EP). The EP for the degree structure $\boldsymbol{A}$ states that the diagram below can be completed by a map c such that $g = c \circ h$, where I, J are ideals of $\mathcal{D}$ and the maps g, h, c are embeddings as ideals, and I, J, g, h, c satisfy the restrictions given in Table 1.



	$\mathcal{D}_m$	$\mathcal{A}_m$	$\mathcal{R}_m$
I,J	D.u.s.l. of size $< 2^\omega$	Arithmetical d.u.s.l. with **1**	Lachlan semilattices
Maps	any embeddings as ideals	arithmetical embeddings	effective embeddings

Table 1

$\mathcal{D}_m$ can be characterized as follows. It is the unique distributive u.s.l. of size 2^ω such that all topped initial segments are countable and the extension property holds for the embeddings described by Table 1 (see Ershov [1975]).
A structure $\boldsymbol{A}$ over a finite symbol set is called *arithmetical* if there is an onto map $\beta : \mathbb{N} \mapsto A$ such that the preimages of the relations of $\boldsymbol{A}$ under β are arithmetical (we call n an *index* for $\beta(n)$). If the structures $\boldsymbol{A}_i$ are arithmetical via β_i ($i = 0, 1$), then we say a map $f : \boldsymbol{A}_0 \to \boldsymbol{A}_1$ is arithmetical if the corresponding relation on indices is.
In the following we discuss $\mathcal{A}_m$. Each arithmetical set C induces an arithmetical presentation of $[\mathbf{o}, \boldsymbol{c}]$, $\boldsymbol{c} = \deg_m(C)$, called the *canonical presentation*, in the following way: Let i be an index for the many–one degree of $\psi_i^{-1}(C)$ (a finite set if ψ_i is partial, see end of the introduction). Clearly $\psi_i^{-1}(C) \leq_m \psi_j^{-1}(C)$ is a $\Sigma_2^0(C \oplus \emptyset')$ relation of i, j.
We say that an u.s.l. W with least element 0 is locally arithmetical if there is an onto map $\alpha : \mathbb{N} \to W$ such that for each n, $[0, \alpha(n)]$ is arithmetical via a map β_n and, if $\alpha(n) \leq \alpha(m)$, then the inclusion map $[0, \alpha(n)] \to [0, \alpha(m)]$ is arithmetical with respect to these presentations. Clearly $\mathcal{A}_m$ itself is locally arithmetical, where $\alpha(n)$ is the many–one degree of the n–th arithmetical set C in some effective listing and β_n is the canonical presentation of $[\mathbf{o}, \boldsymbol{c}]$.

Theorem 1.1 *Up to isomorphism, $\mathcal{A}_m$ is the only locally arithmetical d.u.s.l. satisfying the extension property where I, J are topped arithmetical d.u.s.l. and g, h are arithmetical maps.*

Sketch of proof. By a back and forth argument, any two such d.u.s.l. must be isomorphic. The extension property for $\mathcal{A}_m$ is obtained by analyzing the effective content of Ershov's construction. For simplicity, let us consider the case that $I = \{\mathbf{o}\}$. Thus we have to show that each arithmetical topped d.u.s.l. J is arithmetically isomorphic to an initial segment of $\mathcal{A}_m$. Here it suffices to look at Lachlan's characterization of the topped initial segments of $\mathcal{D}_m$ (Lachlan [1970]), which served as a base for Ershov's work. Effectivizing the proof of Proposition VI.1.12. in Odifreddi [1989], we obtain an arithmetical sequence of strong indices for finite sets (D_i), $D_i \subseteq D_{i+1}$, which are subsets of J containing 0 and closed under join such that $\bigcup_i D_i = J$. Notice that each D_i is a distributive lattice. Now the construction in Lachlan [1970] (with χ_i being the inclusion map $D_i \rightarrow D_{i+1}$) produces an arithmetical set U such that J is arithmetically isomorphic to $[\mathbf{o}, \deg_m(U)]$. $\Diamond$

The characterization of $\mathcal{R}_m$ has been given by Denisov [1978] and is summarized in Nies [2000]. As consequences of the extension properties, we notice that in all cases the degree structure possesses the maximum possible number of automorphisms, that any two minimal degrees are automorphic and that each proper final segment $\{\boldsymbol{x} : \boldsymbol{x} \geq \boldsymbol{e}\}$ is isomorphic to the whole structure.

The EP also leadsto a characterization of the k–*orbits* for $\mathcal{D}_m$ and $\mathcal{A}_m$, i.e. for the orbits under the action of the automorphism group on k –tuples. For a k–tuple $\overline{\boldsymbol{a}}$ let $s_{\overline{\boldsymbol{a}}} = \sup(\{\boldsymbol{a}_i : i < k\})$.

Proposition 1.2 *Suppose $\overline{a}, \overline{b}$ are k tuples in $\mathcal{D}_m$ [$\mathcal{A}_m$]. Then $\overline{\boldsymbol{a}}, \overline{\boldsymbol{b}}$ are in the same k–orbit iff $[\mathbf{o}, s_{\overline{\boldsymbol{a}}}] \cong [\mathbf{o}, s_{\overline{\boldsymbol{b}}}]$ via an [arithmetical] isomorphism p such that $p(\boldsymbol{a}_i) = \boldsymbol{b}_i$.*

Sketch of proof. The direction from left to right is immediate for $\mathcal{D}_m$ and follows from Lemma 3.4 for $\mathcal{A}_m$. For the converse direction, one utilizes the EP in a further back and forth argument. $\Diamond$

Below we need the following concept.

Definition 1.3 *If I is an ideal in an u.s.l. U, b is a* strong minimal cover (s.m.c.) of *I if $I = [0, b)$. In the special case that $I = [0, c]$, we say that b is a s.m.c. of c.*

As a consequence of the EP (using in the arithmetical case techniques from the proof of Lemma 3.3 below), we obtain:

Proposition 1.4 *Each countable ideal of $\mathcal{D}_m$ has a s.m.c. For each $\boldsymbol{c} \in \mathcal{A}_m$, each arithmetical ideal of $\widehat{\boldsymbol{c}} = [\mathbf{o}, \boldsymbol{c}]$ has a s.m.c. in $\mathcal{A}_m$.*

2 Automorphism bases for $\mathcal{D}_m$ and $\mathcal{A}_m$

Various theorems will have a version for $\mathcal{D}_m$ and one for $\mathcal{A}_m$, in which case we will write the changes for $\mathcal{A}_m$ in brackets "[]".

Theorem 2.1 Nies [2000] *Each set $\{\boldsymbol{x} : \boldsymbol{x} \geq \boldsymbol{a}\}$ is an automorphism base of $\mathcal{D}_m$. The same holds for $\mathcal{A}_m$ and (if $\boldsymbol{a} < \mathbf{1}$) for $\mathcal{R}_m$.* $\diamondsuit$

The rest of this section is devoted to proving that many interesting subsets, like the minimal degrees, are *not* automorphism bases of $\mathcal{D}_m$ [$\mathcal{A}_m$]. First we need to develop some algebra of d.u.s.l.

Definition 2.2 *Suppose U is a d.u.s.l., $\pi : I \to J$ is an isomorphism of ideals and M is an ideal of U. We say that π* fixes *M if $\pi(x) = x$ for all $x \in I \cap M$ and $\pi^{-1}(x) = x$ for all $x \in J \cap M$ (and thus $M \cap I = M \cap J$).*

Lemma 2.3 *Suppose π fixes M. Then there is a unique isomorphism ρ between $I \vee M$ and $J \vee M$ which extends π and fixes M. We denote this map ρ by $\pi \vee M$.*

Proof. The map is unique because any such ρ must satisfy $\rho(x \vee m) = \pi(x) \vee m$ (where $x \in I, m \in M$). To see that this equality describes a well–defined map, suppose $x \vee m = y \vee n$, where $x, y \in I, m, n \in M$. Then $x = \widetilde{y} \vee \widetilde{n}$ for some $\widetilde{y} \leq y$ and $\widetilde{n} \leq n$. Hence $\pi(x) \leq \pi(\widetilde{y}) \vee \widetilde{n} \leq \pi(y) \vee n$. By symmetry, $\pi(x) \vee m = \pi(y) \vee n$.
To see that ρ is an isomorphism, define a map $\sigma : J \vee M \to I \vee M$ by $\sigma(z \vee m) = \pi^{-1}(z) \vee m$. Then σ is well–defined and ρ and σ are inverses. $\diamondsuit$
We proceed to the main results of this Section. In the following, an ideal S of $\mathcal{A}_m$ is *locally arithmetical* if $S \cap [\mathbf{o}, \boldsymbol{c}]$ is an arithmetical subset of $[\mathbf{o}, \boldsymbol{c}]$ (with the canonical presentation), for each $\boldsymbol{c} \in \mathcal{A}_m$.

Theorem 2.4 *Suppose M is a proper ideal of $\mathcal{D}_m$[$\mathcal{A}_m$] which is invariant under automorphisms [and locally arithmetical]. Then M is not an automorphism base.*

We begin with the proof for $\mathcal{D}_m$.

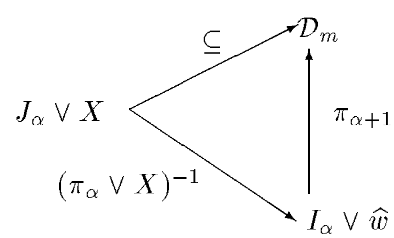
Lemma 2.5 *Suppose that $\pi_0 : I_0 \to J_0$ is an isomorphism of ideals of size $< 2^\omega$ which fixes M. Then π_0 extends to an automorphism π of $\mathcal{D}_m$ which fixes M.*

Theorem 2.4 follows from the Lemma: pick $\boldsymbol{z} \notin M$ and, by the EP, let $\boldsymbol{p}, \boldsymbol{q}$ be distinct strong minimal covers of $\boldsymbol{z}$. Let $I_0 = [\mathbf{o}, \boldsymbol{p}], J_0 = [\mathbf{o}, \boldsymbol{q}]$ and let $\pi_0 : I_0 \rightarrow J_0$ be the isomorphism fixing $[\mathbf{o}, \boldsymbol{z}]$. Then π_0 fixes M. So the automorphism π obtained from Lemma 2.5 shows that M is not an automorphism base.
Before proving the lemma, we give some concrete examples in $\mathcal{D}_m$. Consider the following ideals.

Example 2.6 *For a countable ordinal* α, *let*

$$M_\alpha = [\{\boldsymbol{w} : \exists \beta \leq \alpha \; [\mathbf{o}, \boldsymbol{w}] \cong \beta\}]_{id}.$$

For instance, M_1 is the ideal generated by the minimal degrees. Clearly M_α is automorphism invariant. Moreover, $\alpha < \delta < \omega_1$ implies that M_α is a proper subideal of M_δ, and indeed that there is an automorphism fixing M_α, but not M_δ, pointwise: By Lachlan [1970] pick $\boldsymbol{u} \in \mathcal{D}_m$ such that $[\mathbf{o}, \boldsymbol{u}] \cong \alpha$. As above, let $\boldsymbol{p}, \boldsymbol{q}$ be distinct strong minimal covers of $\boldsymbol{u}$. Then $\boldsymbol{p}, \boldsymbol{q} \in M_\delta - M_\alpha$. Let π_0 be the isomorphism $[\mathbf{o}, \boldsymbol{p}] \rightarrow [\mathbf{o}, \boldsymbol{q}]$ and extend it to an automorphism fixing M_α.
Proof of the Lemma. Fix some wellordering $\triangleleft$ of $\mathcal{D}_m$. We obtain π by a back and forth–construction through the ordinals $< 2^\omega$. Suppose $\pi_\alpha : I_\alpha \rightarrow J_\alpha$ has been defined such that π_α fixes M. If α is even, let w be the $\triangleleft$–least degree $\notin I_\alpha$, and let $X = \widehat{w} \cap M$. By the inductive hypothesis and Lemma 2.2, obtain the isomorphism $\pi_\alpha \vee X : I_\alpha \vee X \rightarrow J_\alpha \vee X$. Now let $I_{\alpha+1} = I_\alpha \vee \widehat{w}$. Apply the EP for the diagram



in order to obtain $\pi_{\alpha+1} : I_{\alpha+1} \rightarrow \mathcal{D}_m$. Let $J_{\alpha+1}$ be the range of $\pi_{\alpha+1}$. We verify that $\pi_{\alpha+1}$ still fixes M. Suppose that $z \in M \cap I_{\alpha+1}$ with the intention to show $\pi_{\alpha+1}(z) = z$. Then $z = w' \vee y$ for some $w' \leq w$ and $y \in I_\alpha \cap M$. Since $\pi_{\alpha+1}(y) = y$ by inductive hypothesis, it suffices to show that $\pi_{\alpha+1}(w') = w'$. But $w' \in X$, so, since the diagram commutes, $w' = \pi_{\alpha+1}((\pi_\alpha \vee X)^{-1}(w')) = \pi_{\alpha+1}(w')$.
To prove the second condition in Definition 2.2, it suffices now to show that $J_{\alpha+1} \cap M \subseteq I_{\alpha+1}$. Let $v = \pi_{\alpha+1}(w)$. Suppose $z \in J_{\alpha+1} \cap M$. Then

$z = v' \vee y$ for some $v' \leq v, y \in J_\alpha$. By the inductive hypothesis $y \in I_\alpha$, so it suffices to show that $v' \in I_{\alpha+1}$. But M is automorphism invariant, so by Proposition 1.2 (with $k = 1$), $\pi_{\alpha+1}(\widehat{w} \cap M) = \widehat{v} \cap M$. So $v' = \pi_{\alpha+1}(w')$ for some $w' \in \widehat{w} \cap M = X$. This implies $w' = v'$, whence $v' \in X \subseteq I_{\alpha+1}$.
If α is odd we proceed as before but with I_α, J_α interchanged. At limit ordinals λ we take unions of partial isomorphisms. Note that $I_\lambda = \bigcup_{\alpha<\lambda} I_\alpha$ has cardinality $< 2^\omega$ since it is generated by $|\lambda|$ many principal ideals. $\diamondsuit$
To prove the Theorem for $\mathcal{A}_m$, we have to adapt Lemma 2.5 and its proof.

Lemma 2.7 *Suppose that $\pi_0 : I_0 \rightarrow J_0$ is an arithmetical isomorphism of initial intervals of $\mathcal{A}_m$ which fixes M. Then π_0 extends to an automorphism π of $\mathcal{D}_m$ which fixes M.*

The Theorem for $\mathcal{A}_m$ is now obtained as before. To prove the Lemma, go through a back an forth–construction of ω steps, using the EP for $\mathcal{A}_m$ and that M is locally arithmetical. $\diamondsuit$
Example 2.6 has an application to $\mathrm{Aut}(\mathcal{D}_m)$. Note that if S is an automorphism invariant subset of a structure $\boldsymbol{A}$, then the pointwise stabilizer of S is a normal subgroup of $\mathrm{Aut}(\boldsymbol{A})$.

Theorem 2.8 *The automorphism group of $\mathcal{D}_m$ possesses a strictly descending chain $(U_\alpha)_{\alpha<\omega_1}$ of normal subgroups.*

Proof. Let U_α be the stabilizer of M_α. Ths sequence is strictly descending by the remarks after Example 2.6. $\diamondsuit$
In a similar way, one obtains a descending chain of length ω_1^{AR} of normal subgroups for $\mathrm{Aut}(\mathcal{A}_m)$. Here ω_1^{AR} is the first ordinal not given by an arithmetical wellordering.

3 Atomic theories and prime models

Recall that a structure $\boldsymbol{A}$ is a *prime model* if $\boldsymbol{A}$ is an elementary substructure of each $\boldsymbol{B} \equiv \boldsymbol{A}$. For countable $\boldsymbol{A}$, being a prime model is equivalent to each realized k-type being principal, or again each k–orbit being $\emptyset$–definable. Thus, a proof that a countable $\boldsymbol{A}$ is prime helps us to understand the orbits. A complete theory T in a countable language has a prime model iff all it Lindenbaum algebras are atomic (such a T is itself called *atomic*).

3.1 Schemes

The following is discussed in more detail in Nies et al. [1998]. A *scheme* for coding in an L-structure $\boldsymbol{A}$ is given by a list of L-formulas $\varphi_1, \ldots, \varphi_n$ with a shared parameter list $\overline{p}$, together with a correctness condition $\alpha(\overline{p})$. If a scheme S_X is given, $X, X_0, X_1, \ldots$ denote objects coded via S_X by a list of parameters satisfying the correctness condition.

Example 3.1 *A scheme S_N for coding models of some finitely axiomatized fragment PA^- of Peano arithmetic (in the language $L(+, \times)$) is given by the formulas $\varphi_{num}(x, \overline{p}), \varphi_+(x, y, z; \overline{p}), \varphi_\times(x, y, z; \overline{p})$ and a correctness condition $\alpha_0(\overline{p})$ which expresses, among others things, that φ_+ and $\varphi_\times$ define binary operations on the nonempty set $\{x : \varphi_{num}(x; \overline{p})\}$, and that $\{x : \varphi_{num}(x; \overline{p})\}$ with the corresponding operations satisfies the finitely many axioms of PA^-.*

For our applications it is sufficient to work with Robinson arithmetic Q. Then all coded models N have a standard part isomorphic to $\mathbb{N}$. If this standard part equals N we say that N is *standard*.

Example 3.2 *A scheme S_g for defining a function g is given by a formula $\varphi_1(x, y; \overline{p})$ defining the relation between arguments and values, and a correctness condition $\alpha(x, y; \overline{p})$ which says that a function is defined.*

3.2 Coded copies of $(\mathbb{N}, +, \times)$

In the following "ar." abbreviates "arithmetical". First we need some facts on distributive upper semilattices. Recall that a sequence (u_i) in an u.s.l. U is *independent* if no u_i is below a finite supremum of other elements in the sequence. In the following, when we say the d.u.s.l. U is *embedded* into V, we mean hat U is isomorphic to an ideal of V.

Lemma 3.3 *Suppose U is a countable [ar.] d.u.s.l. and (a_i) is an [ar.] sequence in U. Then U can be embedded into a countable [ar.] d.u.s.l. V with greatest element which contains an [ar.] independent sequence (b_i) such that, for all i, b_i is a s.m.c. of a_i.*

Proof. Each countable [ar.] d.u.s.l. can be embedded into a countable [ar.] Boolean algebra $\mathcal{B}$ preserving $0, \vee$. By the Stone representation theorem we can suppose that elements of $\mathcal{B}$ are [ar.] subsets of T where $T \subseteq 2^{<\omega}$ is an [ar.] tree, and the operations of $\mathcal{B}$ are the usual operations on sets. Coding the elements of T by odd numbers, we can assume that U is a subsemilattice of $(\mathcal{P}(\{2n + 1 : n \in \mathbb{N}\}, \cup, \emptyset)$.

First we add only one s.m.c. b_0 to a_0. Let $b_0 = a_0 \cup \{0\}$, and let V_0 be the collection of sets generated under finite unions by $U \cup \{b_0\}$. Clearly, each element of $V_0 - U$ has the form $x \cup b_0$ for somce $x \in U$. Thus U is an ideal of V_0 and b_0 is a s.m.c. of a_0. It remains to be shown that V_0 is distributive. Given $x \subseteq y \cup z$, we want to find $y' \subseteq y, z' \subseteq z$ such that $x = y' \cup z'$. If $y \cup z \in U$, this is possible by the distributivity of U. Otherwise, say $z \notin U$ so that $z = \widetilde{z} \cup a_0 \cup \{0\}$. Then $y \cup z = \widetilde{y} \cup \widetilde{z} \cup a_0 \cup \{0\}$ where $\widetilde{y} \in U$, $\widetilde{y} \subseteq y$. If $x \in U$, then $x \subseteq \widetilde{y} \cup \widetilde{z} \cup a_0$, and we obtain x', y' by distributivity of U. Otherwise $x = \widetilde{x} \cup a_0 \cup \{0\}$, where $\widetilde{x} \in U$. Then $\widetilde{x} \subseteq \widetilde{y} \cup \widetilde{z} \cup a_0$ implies $\widetilde{x} = y' \cup z''$ for some $y' \subseteq \widetilde{y}$, $z'' \subseteq \widetilde{z} \cup a_0$. Let $z' = z'' \cup a \cup \{0\} \in V$, then $x = y' \cup z'$.
Now continue this process, adding for $i > 0$ a s.m.c. $b_i = a_i \cup \{2i\}$ in order to embed V_{i-1} into V_i, and let $V = \{\mathbb{N}\} \cup \bigcup_i V_i$. Clearly (b_i) is independent and V has a greatest element. Finally, in the arithmetical setting, V is arithmetical. $\diamondsuit$
The following Lemma on coding of copies of $(\mathbb{N}, +, \times)$ has again versions for $\mathcal{D}_m$ and $\mathcal{A}_m$. The coding schemes used, which will be kept fixed from now on, do not change from one version to the other.

Lemma 3.4 *There are schemes S_M, S_f as in Example 3.1 and Example 3.2 with the following property. For any sequence $(\boldsymbol{a}_i)_{i \in \mathbb{N}}$ [any ar. sequence of degrees in an initial interval of $\mathcal{A}_m$], there is a standard M and a map f such that $\boldsymbol{a}_i = f(i^M)$. Moreover, the domain of M is an independent set. [Finally, S_M can be evaluated in an initial interval of $\mathcal{A}_m$.]*

Proof. We first consider $\mathcal{D}_m$. Let $U = [\{\boldsymbol{a}_i : i \in \mathbb{N}\}]_{id}$. We will embed U into a d.u.s.l. V encoding M, f as desired, and then apply the EP. The first step is to apply Lemma 3.3 to the sequence $\boldsymbol{a}_0, \mathbf{o}, \boldsymbol{a}_1, \mathbf{o}, \ldots$ in order to obtain an extension V' and a sequence of s.m.c. $(\boldsymbol{b}_i)$. The domain of M will be $\{\boldsymbol{b}_i : i \text{ even }\}$. By the proof of Thm 4.2 in Nies [1996], there is a scheme without parameters via which that a copy M of $(\mathbb{N}, +, \times)$ can be encoded into a symmetric graph $(\mathbb{N}, E)$, where the even numbers serve as the domain of the coded model. Embed V' into a d.u.s.l. V by applying Lemma 3.3 to a sequence listing $\{\boldsymbol{b}_i \vee \boldsymbol{b}_j : Eij\}$ (notice that, by independence, all these suprema are distinct). Let $(\boldsymbol{c}_k)$ be the sequence of s.m.c. obtained in this way. Now apply the EP to U, V and identify the elements of V with many-one degrees. By Proposition 1.4, there are s.m.c. $\boldsymbol{b}, \boldsymbol{c}$ for the ideals $[\{\boldsymbol{b}_i : i \in \mathbb{N}\}]_{id}$ and $[\{\boldsymbol{c}_k : i \in \mathbb{N}\}]_{id}$, respectively. Then the sets $\{\boldsymbol{b}_i : i \in \mathbb{N}\}$ and $\{\boldsymbol{c}_k : k \in \mathbb{N}\}$ are definable in $[\mathbf{o}, \boldsymbol{b})$ and $[\mathbf{o}, \boldsymbol{c})$ as the maximal join-irreducible elements. Since we can now recover a copy of $(\mathbb{N}, E)$ in a first–order way, this gives a scheme S_M involving parameters $\boldsymbol{b}, \boldsymbol{c}$ to code M. The

scheme S_f in the same parameters is determined by $f(\boldsymbol{x}) = \boldsymbol{a} \;\Leftrightarrow\; \boldsymbol{x} \in M$ is a s.m.c. of $\boldsymbol{a}$.
All the constructions involved are arithmetical, and V has a greatest element. This gives the Lemma for $\mathcal{A}_m$. $\diamondsuit$
As a consequence we can quantfy over countable subsets [arithmetical subsets of initial segments]. Then we are able enrich the scheme S_M by a correctness condition implying that all coded M are isomorphic to $\mathbb{N}$: the standard part of any M is a set of the required type, so we can express in a first-order way that the standard part equals M. In the following we assume this additional correctness condition.
In the case of $\mathcal{D}_m$, arbitrary subsets Y of M can be represented by elements $\boldsymbol{d}$ of $\mathcal{D}_m$, because by the independence of the domain of M and the existence of a s.m.c. for $[Y]_{id}$, $Y = M \cap [\mathbf{o}, \boldsymbol{d}]$ for some $\boldsymbol{d}$. Thus, for each second–order formula $\psi(X)$ in the language of arithmetic, there is a formula $\beta(d, \widetilde{q})$ such that, if M is coded by parameters $\widetilde{\boldsymbol{q}}$, then $X \subseteq M$ satisfies ψ in M iff $\mathcal{D}_m \models \beta(\boldsymbol{d}, \widetilde{\boldsymbol{q}})$ for any $\boldsymbol{d}$ representing X.

3.3 Results for $\mathcal{D}_m$ and $\mathcal{A}_m$

The following Theorem connects the three classical areas of Mathematical Logic.

Theorem 3.5 *If there is an analytical wellordering on $\mathcal{P}(\omega)$ (for instance if $V = L$), then* $\mathrm{Th}(\mathcal{D}_m)$ *is atomic.*

Proof. Suppose the set theoretical hypothesis holds. We must show that each nonempty definable k–ary relation R on $\mathcal{D}_m$ contains a $\emptyset$–definable k–orbit.
The functions ψ_i were defined at the end of the introduction. From here on, let us assume also that, for all i, ψ_{2i} is a total 1-1 map onto $\mathbb{N}^{[i]} = \{\langle n, i\rangle : n \in \mathbb{N}\}$. For $A \subseteq \mathbb{N}$, the following set of numbers describes $[\mathbf{o}, \boldsymbol{a}]$:

$$(1) \qquad \mathrm{Diag}(A) = \{\langle i, j\rangle : \psi_i^{-1}(A) \leq_m \psi_j^{-1}(A)\}.$$

In the following, we call a set $X \subseteq \mathbb{N}$ second-order definable if $\{X\}$ is projective. $X_0 \oplus \ldots \oplus X_{k-1}$ denotes the set $\{\langle n, j\rangle : n \in X_j \;\&\; j < k\}$.

Claim 3.6 *Suppose A_i $(i < k)$ are sets, $A = A_0 \oplus \ldots \oplus A_{k-1}$ and Diag(A) is second-order definable. Then the k–orbit of $(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{k-1})$ is $\emptyset$–definable.*

To prove the Claim, by Lemma 3.4, choose a standard M and f such that $f(i^M) = \deg_m(\psi_i^{-1}(A))$ and in particular, for $i < k$, $f(2i^M) = \boldsymbol{a}_i$.
By the remark after Lemma 3.4, replacing set quantifiers by degree quantifiers in a second–order definition of Diag(A), one obtains a formula $\beta(\boldsymbol{c}, \widetilde{\boldsymbol{q}})$ such that, given $M_{\widetilde{\boldsymbol{q}}}$ (this notation indicates that M is coded by the tuple $\widetilde{\boldsymbol{q}}$), $\beta(\boldsymbol{c}, \widetilde{\boldsymbol{q}})$ holds iff $\widehat{\boldsymbol{c}} \cap M_{\widetilde{\boldsymbol{q}}}$ is Diag(A) (viewed as a subset of M). Then $\overline{\boldsymbol{b}}$ is automorphic to $\overline{\boldsymbol{a}}$ iff

$\exists \widetilde{\boldsymbol{q}}\ \exists \widetilde{f} : M_{\widetilde{\boldsymbol{q}}} \to [\mathbf{o}, s_{\overline{\boldsymbol{b}}}]$ onto $\exists \boldsymbol{c}\ [\beta(\boldsymbol{c}, \widetilde{\boldsymbol{q}})\ \&\ \forall i < k\ \widetilde{f}(2i^{M_{\widetilde{\boldsymbol{q}}}}) = \boldsymbol{b}_i\ \&$
$\quad \{\langle i, j\rangle \in M_{\widetilde{\boldsymbol{q}}} : \widetilde{f}(i) \leq \widetilde{f}(j)\} = \widehat{\boldsymbol{c}} \cap M_{\widetilde{\boldsymbol{q}}}]$

(for the direction from right to left one uses Proposition 1.2). Since the second statement can be formulated in the first–order language of u.s.l., this proves the Claim. Now, if S is a nonempty projective k–ary relation, then the class of sets $\{\mathrm{Diag}(A_0 \oplus \ldots \oplus A_{k-1}) : (A_0, \ldots, A_{k-1}) \in S\}$ is projective. By the set theoretic hypothesis, this class contains a 2nd–order definable element. If we let S be the (projective) relation on subsets of $\mathbb{N}$ corresponding to R, by the Claim we obtain a definable k–orbit in R. $\diamondsuit$
G. Hjorth and the author have proved that Th($\mathcal{D}_m$) is not atomic in a Cohen extension of the universe.

Theorem 3.7 *$\mathcal{A}_m$ is a prime model.*

Proof. We must show that each k–orbit is $\emptyset$–definable. Suppose $\overline{\boldsymbol{a}}$ is a k–tuple in $\mathcal{A}_m$, and that A_i is a set in $\boldsymbol{a}_i$. Let $A = A_0 \oplus \ldots \oplus A_{k-1}$. Then the set Diag($A$) given by (1) can be defined in $(\mathbb{N}, +, \times)$ by a formula $\delta(n)$. We conclude that $\overline{\boldsymbol{b}}$ is automorphic to $\overline{\boldsymbol{a}}$ iff

$\exists \widetilde{M}\ \exists \widetilde{f} : \widetilde{M} \to [\mathbf{o}, s_{\overline{\boldsymbol{b}}}]$ onto $[\forall i < k\ \widetilde{f}(2i^{\widetilde{M}}) = \boldsymbol{b}_i\ \&$
$\quad \{\langle \boldsymbol{i}, \boldsymbol{j}\rangle \in \widetilde{M} : \widetilde{f}(\boldsymbol{i}) \leq \widetilde{f}(\boldsymbol{j})\} = \{\boldsymbol{n} \in \widetilde{M} : \widetilde{M} \models \delta(\boldsymbol{n})\}]$

(once again, we use Proposition 1.2). $\diamondsuit$

4 Proving sets are automorphism bases

To prove a subset B of a structure $\boldsymbol{A}$ is an automorphism base, one might try to find a 1-1 B–definable map $H : \boldsymbol{A} \to B$. Then, if the automorphism F fixes B pointwise, it must be the identity. Our method uses this idea in a more general setting: a 1-1 map H maps elements of $\boldsymbol{A}$ (or, more generally, of an automorphism invariant subset C which is already known to be an automorphism base) to complex objects of one and the same "B–type" τ, which are constructed from elements of B. Such a map is first–order definable in a more general sense described below. The proofs that such a map H is 1-1 are described by finite games. Given distinct elements $\boldsymbol{x}, \boldsymbol{y} \in C$, we try to show $H(\boldsymbol{x}) \neq H(\boldsymbol{y})$ while the opponent claims $H(\boldsymbol{x}) = H(\boldsymbol{y})$. We begin with an example with the structure $\mathcal{R}_T$, where $C = \mathcal{R}_T$ and $B = \mathrm{NC}$ is the set of noncappable degrees. For this we need the following splitting theorem, which can be readily proved using the finitary techniques in Soare [1987].

Proposition 4.1 *If $\boldsymbol{p}$ is low and $\boldsymbol{c} \not\leq \boldsymbol{p}$, then for each $\boldsymbol{x}$ there are low degrees $\boldsymbol{x}_0, \boldsymbol{x}_1$ such that $\boldsymbol{x} = \boldsymbol{x}_0 \vee \boldsymbol{x}_1$ and $\boldsymbol{c} \not\leq \boldsymbol{p} \vee \boldsymbol{x}_i$ for $i = 0, 1$. Moreover, if $\boldsymbol{x} \in NC$, one can ensure that $\boldsymbol{x}_0, \boldsymbol{x}_1 \in NC$.* $\diamondsuit$

Proposition 4.2 *The map H given by $H(\boldsymbol{x}) = \{[\boldsymbol{p}, \mathbf{1}] \cap NC, [\boldsymbol{q}, \mathbf{1}] \cap NC : \boldsymbol{p} \vee \boldsymbol{q} = \boldsymbol{x}\}$ is 1-1. Hence NC is an automorphism base.*

Proof. Suppose $\boldsymbol{x} \neq \boldsymbol{y}$. We play the following game. Say $\boldsymbol{x} \not\leq \boldsymbol{y}$.

Round 1. We provide low degrees $\boldsymbol{p}, \boldsymbol{q}$ such that $\boldsymbol{y} = \boldsymbol{p} \vee \boldsymbol{q}$. To support his claim, the opponent has to provide $\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}}$ such that $\boldsymbol{x} = \widetilde{\boldsymbol{p}} \vee \widetilde{\boldsymbol{q}}$, $[\widetilde{\boldsymbol{p}}, \mathbf{1}] \cap \mathrm{NC} = [\boldsymbol{p}, \mathbf{1}] \cap \mathrm{NC}$ and $[\widetilde{\boldsymbol{q}}, \mathbf{1}] \cap \mathrm{NC} = [\boldsymbol{q}, \mathbf{1}] \cap \mathrm{NC}$.
Round 2. Since $\boldsymbol{x} \not\leq \boldsymbol{y}$, $\widetilde{\boldsymbol{p}} \not\leq \boldsymbol{p}$ or $\widetilde{\boldsymbol{q}} \not\leq \boldsymbol{q}$, say the first. By Proposition 4.1, we choose $\boldsymbol{u} \in \mathrm{NC}$ such that $\boldsymbol{p} \leq \boldsymbol{u}$ but $\widetilde{\boldsymbol{p}} \not\leq \boldsymbol{u}$. (Let $\boldsymbol{x} = \mathbf{1}, \boldsymbol{c} = \widetilde{\boldsymbol{p}}$. At least one $\boldsymbol{x}_i$ is noncappable, so let $\boldsymbol{u} = \boldsymbol{p} \vee \boldsymbol{x}_i$ for this i.) The opponent loses. $\diamondsuit$
Next we introduce the formalism of B–types and the corresponding maps.

Definition 4.3 *Suppose $\boldsymbol{A}$ is a structure and $B \subseteq \boldsymbol{A}$.*

(i) Each formula $\varphi(x, y)$ with parameters in B is a B–type.

(ii) If $\tau_1, \ldots, \tau_n$ are B–types and $\varphi(x, y_1, \ldots, y_n)$ is a formula with parameters in B, then $\tau = (\tau_1, \ldots, \tau_n; \varphi)$ is a B–type.

Definition 4.4 *Suppose $B \subseteq \boldsymbol{A}$ and τ is a B–type. We define a map $H_{\tau B}$ by induction on τ.*

(i) If τ is $\varphi(x, y)$, let $H_{\tau B}(\boldsymbol{x}) = \{\boldsymbol{y} \in B : \varphi(\boldsymbol{x}, \boldsymbol{y})\}$.

(ii) If $\tau = (\tau_1, \ldots, \tau_n; \varphi)$, then let $H_{\tau B}(\boldsymbol{x}) = \{(H_{\tau_1 B}(\boldsymbol{y}_1), \ldots, H_{\tau_n B}(\boldsymbol{y}_n)) : \varphi(\boldsymbol{x}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)\}$.

To see how this applies to the preceding example, for $i = 0, 1$ let $\varphi_i(x_i, y)$ be $y \in \mathrm{NC}$ & $x_i \leq y$. Moreover, let $\psi(x, x_0, x_1)$ be $x = x_0 \vee x_1$. Then for the NC–type $\tau = (\varphi_0, \varphi_1; \psi)$, $H_{\tau \mathrm{NC}}$ is the map introduced in Proposition 4.2.

Suppose $F : A \to A$. We define $F(H_{\tau B}(\boldsymbol{x}))$ by applying F to the atomic components of $H_{\tau B}(\boldsymbol{x})$:
In Case (i) of Definition 4.3, let $F(H_{\tau B}(\boldsymbol{x})) = \{F(\boldsymbol{y}) : \varphi(\boldsymbol{x}, \boldsymbol{y}) \ \& \ \boldsymbol{y} \in B\}$.
In Case (ii), let $F(H_{\tau B}(\boldsymbol{x})) = \{(F(H_{\tau_1 B}(\boldsymbol{y}_1)), \ldots, F(H_{\tau_n B}(\boldsymbol{y}_n))) : \varphi(\boldsymbol{x}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)\}$. Clearly, if F fixes B pointwise, then F fixes $H_{\tau B}(\boldsymbol{x})$. Moreover, by induction over the length of τ, one verifies that, whenever F is also an automorphism, then $H_{\tau B}(F(\boldsymbol{x})) = F(H_{\tau B}(\boldsymbol{x}))$.

Proposition 4.5 *Suppose C is an automorphism invariant automorphism base and $H_{\tau B}\lceil C$ is 1-1. Then B is an automorphism base.*

Proof. Suppose the automorphism F fixes B pointwise. Then, for each $\boldsymbol{x} \in C$, $H(F(\boldsymbol{x})) = F(H(\boldsymbol{x})) = H(\boldsymbol{x})$. Since $F(\boldsymbol{x}) \in C$, by hypothesis on H this implies that $F(\boldsymbol{x}) = \boldsymbol{x}$. Thus F fixes C pointwise. ♢
Our first application of the method is to $\mathcal{R}_m$. Recall that $\widehat{\boldsymbol{x}}$ denotes $[\mathbf{o}, \boldsymbol{x}]$.

Theorem 4.6 *Suppose $D \subseteq \mathcal{R}_m$ is $\emptyset$–definable and $D \not\subseteq \{\mathbf{o}, \mathbf{1}\}$. Then D is an automorphism base of $\mathcal{R}_m$.*

As an example, for each finite distributive lattice L, the set $\{\boldsymbol{x} : [\mathbf{o}, \boldsymbol{x}] \cong L\}$ is an automorphism base. In Nies [2000], this was proved for the special case of $L = \{0, 1\}$. A similar result holds for $L = (\omega^n, \leq)$, $n < \omega$ (compare this to Example 2.6!).
Proof. Consider the first–order property

$$\gamma_D(y) \equiv \forall q[[0, y) \not\subseteq \widehat{q} \Rightarrow \exists d \in D \ (d < y \ \& \ d \not\leq q)]. \tag{2}$$

Many degrees enjoy this property:

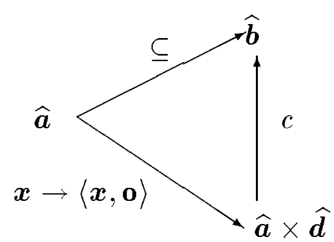
Claim 4.7 $\forall \boldsymbol{z} \forall \boldsymbol{x}[\boldsymbol{z} \not\leq \boldsymbol{x} \Rightarrow \exists \boldsymbol{e} > \boldsymbol{x}(\boldsymbol{z} \not\leq \boldsymbol{e} \ \& \ \gamma_D(\boldsymbol{e}))]$.

Before proving the Claim, we apply it to provide an appropriate τD–map showing that D is an automorphism base: let

$$H_{\tau D}(\boldsymbol{x}) = \{[\mathbf{o}, \boldsymbol{y}) \cap D : \gamma_D(\boldsymbol{y}) \ \& \ \boldsymbol{y} \geq \boldsymbol{x}\}.$$

To show that this map is 1-1 on $C = \mathcal{R}_m$, given $\boldsymbol{z}, \boldsymbol{x}$ such that $\boldsymbol{z} \not\leq \boldsymbol{x}$, using the Claim we play $\boldsymbol{e} > \boldsymbol{x}$ such that $\boldsymbol{z} \not\leq \boldsymbol{e}$ and $\gamma_D(\boldsymbol{e})$. The opponent answers with $\boldsymbol{u} > \boldsymbol{z}$ such that $\gamma_D(\boldsymbol{u})$ and claims that $[\mathbf{o}, \boldsymbol{u}) \cap D = [\mathbf{o}, \boldsymbol{e}) \cap D$. But $[\mathbf{o}, \boldsymbol{u}) \not\subseteq \widehat{\boldsymbol{e}}$, so this contradicts $\gamma_D(\boldsymbol{u})$.

Proof of the Claim. We can assume that D is a $\emptyset$-definable subset of $\mathcal{R}_m^- = \mathcal{R}_m$-$\{\mathbf{1}\}$ and $D \not\subseteq \{\mathbf{0}\}$. We make use of a technique which was introduced in Nies [2000] to prove that for some $\boldsymbol{e} < \mathbf{1}$, $[\mathbf{0}, \boldsymbol{e})$ is an elementary submodel of $\mathcal{R}_m^-$. By recursion over $k \geq 0$, increasing functions F_k on indices of c.e. many–one degrees were defined with the property that, for each $\boldsymbol{a} < \mathbf{1}$, $[\mathbf{0}, F_k(\boldsymbol{a})) \prec_k \mathcal{R}_m^-$. (i.e., the inclusion map preserves Σ_k-formulas). In addition, $[\mathbf{0}, F_0(\boldsymbol{a})]$ is effectively isomorphic to $\mathcal{R}_m$.
Given $\boldsymbol{z}, \boldsymbol{x}$ as in the Claim, we will let $\boldsymbol{e} = F_k(\boldsymbol{x})$, where $k \geq 1$ is a number such that D can be defined by a Σ_k-formula. Since we need $\boldsymbol{z} \not\leq \boldsymbol{e}$, we have to slightly extend the construction in Nies [2000]. For each r we want to achieve that $\boldsymbol{z} \not\leq \boldsymbol{a} \Rightarrow \boldsymbol{z} \not\leq F_r(\boldsymbol{a})$.
For $r = 0$ this is possible by Remark 2.4 (2) Nies [2000]. $F_{r+1}(\boldsymbol{a})$ is a degree $\boldsymbol{y}$ such that $[\mathbf{0}, \boldsymbol{y}) = \bigcup_i [\mathbf{0}, F_r^{(i)}(\boldsymbol{a}))$. By Lemma 2.1 there, which goes back to Ershov and Lavrov, this strong minimal cover of can be chosen distinct from $\boldsymbol{z}$. Thus $\boldsymbol{z} \leq \boldsymbol{y}$ would imply $\boldsymbol{z} \leq F_r^{(i)}(\boldsymbol{a})$ for some i.
It remains to show $\gamma_D(\boldsymbol{e})$. Suppose $[\mathbf{0}, \boldsymbol{e}) \not\subseteq \widehat{\boldsymbol{q}}$. Since $[\mathbf{0}, \boldsymbol{e}) \prec_k \mathcal{R}_m^-$, there exists $\boldsymbol{d} \in (\mathbf{0}, \boldsymbol{e}) \cap D$. If $\boldsymbol{d} \not\leq \boldsymbol{q}$, we are done. Otherwise we replace $\boldsymbol{d}$ by an automorphic image $\boldsymbol{d}'$ in $\widehat{\boldsymbol{e}}$ such that $\boldsymbol{d}' \not\leq \boldsymbol{q}$: Let $I = \widehat{\boldsymbol{e}} \cap \widehat{\boldsymbol{q}}$. By hypothesis on $\boldsymbol{q}$, $I \neq [\mathbf{0}, \boldsymbol{e})$. Therefore, for some $i > 0$, $\boldsymbol{b} = F_0(F_{k-1}^{(i)}(\boldsymbol{x})) \notin I$ and $\boldsymbol{b} < \boldsymbol{e}$. By the definition of F_0, $\widehat{\boldsymbol{b}}$ is effectively isomorphic to $\mathcal{R}_m$. We apply Lemma 2.1 in Nies [2000] to the Σ_3^0–ideal $I \cap \widehat{\boldsymbol{b}}$ of $\widehat{\boldsymbol{b}}$ and obtain an $\boldsymbol{a} < \boldsymbol{b}$ such that $I \cap \widehat{\boldsymbol{b}} \subseteq \widehat{\boldsymbol{a}}$. Now we apply the EP for $\mathcal{R}_m$, again with $\widehat{\boldsymbol{b}}$ in place of $\mathcal{R}_m$, to the following diagram:



in order to obtain the map c. Let $\boldsymbol{d}' = c(\langle \mathbf{0}, \boldsymbol{d} \rangle)$, then $\boldsymbol{d}' \leq \boldsymbol{b}$, $\boldsymbol{d}' \not\leq \boldsymbol{a}$ and therefore $\boldsymbol{d}' \not\leq \boldsymbol{q}$. By a back and forth–construction within $\widehat{\boldsymbol{e}}$, the map c can be extended to an automorphism of $\widehat{\boldsymbol{e}}$. Since $[\mathbf{0}, \boldsymbol{e}) \prec_k \mathcal{R}_m^-$, this implies $\boldsymbol{d}' \in D$. $\diamondsuit$
Little is known about the structure of $\mathrm{Aut}(\mathcal{R}_m)$. However, since

$$\mathcal{R}_m - \{\mathbf{1}\} \cong$$

is isomorphic to the Δ^0_2 m-degrees (Denisov [1978]), one can infer the existence of an automorphism of order 2 which corresponds to complementation of Δ^0_2–sets.
Our second application is to $\mathcal{R}_T$.

Theorem 4.8 Ambos-Spies [ta] *For each* $\boldsymbol{c} \in \mathcal{R}_T$-$\{\mathbf{o}\}$, $[\mathbf{o}, \boldsymbol{c}]$ *is an automorphism base.*

Proof. We will apply Proposition 4.5 with C = NC (a $\emptyset$-definable set which is an automorphism base by Proposition 4.2) and $B = \widehat{\boldsymbol{c}}$. We will introduce an appropriate $H_{\sigma\widehat{\boldsymbol{c}}}$–map, where σ is a B-type. The map is based on Ambos-Spies' "downward splitting property": let

$$\mathrm{DSP} = \{\boldsymbol{a} : \forall \boldsymbol{b} \not\leq \boldsymbol{a}\ \forall \boldsymbol{d} \not\leq \boldsymbol{b} \vee \boldsymbol{a}[\widehat{\boldsymbol{d}} \cap \widehat{\boldsymbol{a}} \neq \widehat{\boldsymbol{d}} \cap \widehat{\boldsymbol{b}}]\}$$

This property corresponds to γ_D in the previous proof. Note that DSP $\subseteq$ NC. The next Lemma shows that sufficiently many degrees are in DSP.

Lemma 4.9 Ambos-Spies [ta] *Suppose* $\boldsymbol{p} \not\leq \boldsymbol{q}$, $\boldsymbol{p} \in$ *NC and* $\boldsymbol{q}$ *is low. Then there is an* $\boldsymbol{a}$ *such that* $\boldsymbol{a} \in$ *DSP,* $\boldsymbol{a} \leq \boldsymbol{p}$ *and* $\boldsymbol{a} \not\leq \boldsymbol{q}$.

$H_{\sigma\widehat{\boldsymbol{c}}}(\boldsymbol{x})$ is a complex object constructed from ideals $\widehat{\boldsymbol{c}} \cap \widehat{\boldsymbol{a}}$, where $\boldsymbol{a} \in$ DSP: let $H_{\sigma\widehat{\boldsymbol{c}}}(\boldsymbol{x}) =$
$\{\{\langle\{\widehat{\boldsymbol{c}}\cap\widehat{\boldsymbol{a}} : \boldsymbol{a} \in \widehat{\boldsymbol{p}}_0 \cap \mathrm{DSP}\}, \{\widehat{\boldsymbol{c}}\cap\widehat{\boldsymbol{a}} : \boldsymbol{a} \in \widehat{\boldsymbol{p}}_1 \cap \mathrm{DSP}\}\rangle : \boldsymbol{p}_0 \vee \boldsymbol{p}_1 = \boldsymbol{p}\ \&\ \boldsymbol{p}_i \in \mathrm{NC}\},$
$\{\langle\{\widehat{\boldsymbol{c}}\cap\widehat{\boldsymbol{a}} : \boldsymbol{a} \in \widehat{\boldsymbol{q}}_0 \cap \mathrm{DSP}\}, \{\widehat{\boldsymbol{c}}\cap\widehat{\boldsymbol{a}} : \boldsymbol{a} \in \widehat{\boldsymbol{q}}_1 \cap \mathrm{DSP}\}\rangle : \boldsymbol{q}_0 \vee \boldsymbol{q}_1 = \boldsymbol{q}\ \&\ \boldsymbol{q}_i \in \mathrm{NC}\} :$
$\boldsymbol{p} \vee \boldsymbol{q} = \boldsymbol{x}\ \&\ \boldsymbol{p}, \boldsymbol{q} \not\leq \boldsymbol{c}\ \&\ \boldsymbol{p}, \boldsymbol{q} \in \mathrm{NC}\}$
We must show that $H_{\sigma\widehat{\boldsymbol{c}}}\lceil$NC ia 1-1. Suppose $\boldsymbol{x}, \boldsymbol{y} \in$ NC are given, and $\boldsymbol{x} \not\leq \boldsymbol{y}$. The game is as follows.

Round 1. By Proposition 4.1, we play a split $\boldsymbol{y} = \boldsymbol{p} \vee \boldsymbol{q}$, where $\boldsymbol{p}, \boldsymbol{q} \in$ NC $\cap$ Low and $\boldsymbol{c} \not\leq \boldsymbol{p}, \boldsymbol{q}$. The opponent responds with a split $\boldsymbol{x} = \widetilde{\boldsymbol{p}} \vee \widetilde{\boldsymbol{q}}$ where $\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}} \in$ NC and $\boldsymbol{c} \not\leq \widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{q}}$.
Round 2. Say $\widetilde{\boldsymbol{p}} \not\leq \boldsymbol{p}$. Again by Proposition 4.1, we split $\widetilde{\boldsymbol{p}} = \boldsymbol{d}_0 \vee \boldsymbol{d}_1$, where $\boldsymbol{d}_i \in$ NC and $\boldsymbol{c} \not\leq \boldsymbol{d}_i \vee \boldsymbol{p}$. The opponent responds with a split $\boldsymbol{p} = \boldsymbol{e}_0 \vee \boldsymbol{e}_1$, where $\boldsymbol{e}_i \in$ NC. Then $\boldsymbol{e}_i \in$ Low.
Round 3. Say $\boldsymbol{d}_0 \not\leq \boldsymbol{e}_0$. Using Lemma 4.9, we play $\boldsymbol{b}$ such that $\boldsymbol{b} \in$ DSP, $\boldsymbol{b} \leq \boldsymbol{d}_0$ and $\boldsymbol{b} \not\leq \boldsymbol{e}_0$. The opponent responds with $\boldsymbol{a}$ such that $\boldsymbol{a} \in$ DSP and $\boldsymbol{a} \leq \boldsymbol{e}_0$, and claims that $\widehat{\boldsymbol{a}} \cap \widehat{\boldsymbol{c}} = \widehat{\boldsymbol{b}} \cap \widehat{\boldsymbol{c}}$.
But his claim is wrong: $\boldsymbol{b} \not\leq \boldsymbol{a}$ since $\boldsymbol{a} \leq \boldsymbol{e}_0$ while $\boldsymbol{b} \not\leq \boldsymbol{e}_0$. Moreover, $\boldsymbol{c} \not\leq \boldsymbol{a} \vee \boldsymbol{b}$ since $\boldsymbol{a} \vee \boldsymbol{b} \leq \boldsymbol{d}_0 \vee \boldsymbol{e}_0$ while $\boldsymbol{c} \not\leq \boldsymbol{d}_0 \vee \boldsymbol{p}$. Thus his claim contradicts $\boldsymbol{a} \in$ DSP.

Proof of Lemma 4.9 We sketch Ambos-Spies' construction of a c.e. set A such that $\boldsymbol{a} \in \mathrm{DSP}$. By the finitary character of the strategies, the construction can be easily expanded in order to prove the Lemma.
Given c.e. sets B, C, we enumerate sets $F \leq_T A, C$ and G_e ($e \in \mathbb{N}$) such that, under the assumption that $B \not\leq_T A$ and $C \not\leq_T B \oplus A$, the following requirements are met:

$P_e: \quad F \neq \{e\}^B \ \vee \ G_e \leq_T B, C$

$Q_{e,i}: \quad F = \{e\}^B \ \Rightarrow \ G_e \neq \{i\}^A$,

where P_e has higher priority than any $Q_{e,i}$. The reductions $F \leq_T A, C$ are by direct permitting, $G_e \leq_T C$ (if applicable) by delayed permitting, while $G_e \leq_T B$ is is a proper T-reduction (but with no delay in correcting). We do not name these reductions and write e.g. $G_e \leq_T C(y)$ for the output on input y.
A *set-up* for $Q_{e,i}$ at stage s is a pair $x \in \omega^{[e]}, y \in \omega^{[\langle e,i \rangle]}$ such that $\{e\}^B(x) = \{i\}^A(y) = 0[s]$ and $u(A; i, y, s) < x < \gamma_B(y, s)$, where $\gamma_B(y, s)$ is the "standard marker" $max\{t \leq s : B_t \lceil y \neq B_{t+1} \lceil y\}$. The number x is targeted for F, while y is targeted for G_e. The strategy for $Q_{e,i}$ is the following: if the requirement is unsatisfied, a set-up x, y exists at stage s and $C_s \lceil y \neq C_{s+1} \lceil y$, then we say the strategy requires attention. At this stage s, attempt to meet P_e by putting x into F and A (note that this A-enumeration does not destroy $\{i\}^A(y)$). Moreover, initialize the strategies of lower priority than $Q_{e,i}$. This is a win for the higher priority requirement P_e (and hence all $Q_{e,i}$) unless, at a stage $t > s$, for the first time B changes below the use of $\{e\}^B(x)$. The use of the computation $G_e \leq_T B(y)$ at a stage r is defined to be $u(B, e, \gamma_B(y))[r]$ (where we assume that the use of $\{e\}^B$ is monotonic in the input). So this B–change makes $G_e \leq_T B(y)$ undefined. Also, at stage s we used the C–change to make $G_e \leq_T C(y)$ undefined, and redefine it only now at stage t. Thus we are allowed to put y into G_e and declare the requirement satisfied. If $F = \{e\}^B$, then the B–change must occur, so we always redefine $G_e \leq_T C(y)$ (if $Q_{e,i}$ is initialized before, we also redefine the reduction).
In the *construction* we start the highest priority strategy $Q_{e,i}$ which requires attention, and later do the G_e enumeration (which does not injure other strategies) when necessary. Also we update the functionals as described above.
The *verification* is as follows. Clearly the action of the $Q_{e,i}$ is finitary after it is no more initialized. Suppose that $F = \{e\}^B$. The reduction $G_e \leq C$ is total since it is always redefined and its use equals the input. The reduction $G_e \leq_T B$ is total since $\{e\}^B$ is. Now suppose $Q_{e,i}$ is no met, i.e. $G_e = \{i\}^A$. If $B \not\leq_T A$, then infinitely many set-ups x, y for $Q_{e,i}$ must appear at stages s

such that $\{e\}^B(x)$ and $\{i\}^A(y)$ are stable, and thus the set-up is permanent. Since $A \oplus B$ can recognize such permanent set-ups, $C \not\leq_T A \oplus B$ implies that infinitely often a $C\lceil y$ change must occur when a set-up x, y exists. Thus, after $Q_{e,i}$ is no more injured, we win the requirement. $\diamondsuit$

One can define composition of maps as above: for instance, if H is the map from Proposition 4.2, $H_{\rho\widehat{\mathbf{c}}} = H \circ H_{\sigma\widehat{\mathbf{c}}}(\boldsymbol{x})$ is $\{\{H_{\sigma\widehat{\mathbf{c}}}(\boldsymbol{y}_1) : \boldsymbol{y}_1 \in [\boldsymbol{p}, \mathbf{1}] \cap \mathrm{NC}\}, \{H_{\sigma\widehat{\mathbf{c}}}(\boldsymbol{y}_2) : \boldsymbol{y}_2 \in [\boldsymbol{q}, \mathbf{1}] \cap \mathrm{NC}\} : \boldsymbol{p} \vee \boldsymbol{q} = \boldsymbol{x}\}$. This is a 1-1 map from the whole structure into objects of type $\rho\widehat{\mathbf{c}}$. It follows that $\mathcal{R}_T$ is biinterpretable with $\mathbb{N}$ in parameters if, for some $\boldsymbol{c} \neq \mathbf{o}$, there is a parameter defined copy M of $(\mathbb{N}, +, \times)$ and a parameter definable 1-1 map $g : \widehat{\boldsymbol{c}} \mapsto M$.

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