The first order theories of the Medvedev and the Muchnik lattice

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Abstract

We show that the first order theories of the Medevdev lattice and the Muchnik lattice are both computably isomorphic to the third order theory of true arithmetic.

1 Introduction

A major theme in the study of computability theoretic reducibilities has been the question of how complicated the first order theories of the corresponding degree structures are. A computability theoretic reducibility is usually a preordering relation on sets of numbers, or on number-theoretic functions. If $\leq_r$ is a reducibility (i.e. a preordering relation) on, say, functions, then $f \leq_r g$, with f, g functions, has usually an arithmetical definition. Therefore, if $(\mathcal{P}, \leq_r)$ is the degree structure corresponding to $\leq_r$, then first order statements about the poset $(P, \leq_r)$ can be translated into second order arithmetical statements, allowing for quantification over functions. This usually establishes that $\mathrm{Th}(P, \leq_r) \leq_m \mathrm{Th}_2(\mathbb{N})$. (Here $\leq_m$ denotes m-reducibility, and by $\mathrm{Th}_n(\mathbb{N})$ we denote the set of n-th order arithmetical sentences that are true in the set of natural numbers $\mathbb{N}$: precise definitions for $n = 2, 3$ will be given later. $\mathrm{Th}_n(\mathfrak{A})$ is usually called the *n-th order theory* of $\mathbb{N}$.)

For instance, if one considers the Turing degrees $\mathfrak{D}_T = (\mathcal{D}_T, \leq_T)$, it immediately follows from the above that $\mathrm{Th}(\mathfrak{D}_T) \leq_m \mathrm{Th}_2(\mathbb{N})$. On the other hand a

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classical result due to Simpson, [8] (see also [9]) shows that $\mathrm{Th}_2(\mathbb{N}) \leq_m \mathrm{Th}(\mathfrak{D}_T)$. Thus the first order theory of the Turing degrees $\mathfrak{D}_T = (\mathcal{D}_T, \leq_T)$ is as complicated as it can be, i.e. computably isomorphic to second order arithmetic $\mathrm{Th}_2(\mathbb{N})$. For an updated survey on this subject and related topics, we refer the reader to the recent survey by R. Shore, [7].

An interesting, although not much studied, computability theoretic reducibility, is Medvedev reducibility. Here the story is completely different, since Medvedev reducibility is a preordering relation defined on sets of functions. Therefore we need quantification over sets of functions to express first order statements about the corresponding degree structure, which is a bounded distributive lattice called the Medvedev lattice. This suggests that in order to find un upper bound to the complexity of the first order theory of the Medvedev lattice, one has to turn to third order arithmetic. The purpose of this note is to show that third order arithmetic indeed the exact level: we will show that the first order theories of the Medvedev lattice, and of its nonuniform version, called the Muchnik lattice, are in fact computably isomorphic to third order arithmetic.

2 Basics

We briefly review the basic definitions about the Medvedev lattice, and the Muchnik lattice. For more detail the reader is referred to [6], and [10].

A *mass problem* is a subset of $\mathbb{N}^{\mathbb{N}}$. On mass problems one can define the following preordering relation: $\mathcal{A} \leq_M \mathcal{B}$ if there is a Turing functional Ψ such that for all $f \in \mathcal{B}$, $\Psi(f)$ is total, and $\Psi(f) \in \mathcal{A}$. The relation $\leq_M$ induces an equivalence relation on mass problems: $\mathcal{A} \equiv_M \mathcal{B}$ if $\mathcal{A} \leq_M \mathcal{B}$ and $\mathcal{B} \leq_M \mathcal{A}$. The equivalence class of $\mathcal{A}$ is denoted by $\deg_M(\mathcal{A})$ and is called the *Medvedev degree* of $\mathcal{A}$ (or, following Medvedev [3], the *degree of difficulty* of $\mathcal{A}$). The collection of all Medvedev degrees is denoted by $\mathcal{M}$, partially ordered by $\deg_M(\mathcal{A}) \leq_M \deg_M(\mathcal{B})$ if $\mathcal{A} \leq_M \mathcal{B}$. Note that there is a smallest Medvedev degree $\mathbf{0}$, namely the degree of any mass problem containing a computable function. There is also a largest degree $\mathbf{1}$, the degree of the empty mass problem. For functions f and g, as usual we define the function $f \oplus g$ by $f \oplus g(2x) = f(x)$ and $f \oplus g(2x+1) = g(x)$. Let $\langle n \rangle \hat{} \mathcal{A} = \{\langle n \rangle \hat{} f : f \in \mathcal{A}\}$, where $n \hat{} f$ is the function such that $n \hat{} f(0) = n$, and for $x > 0$ $n \hat{} f(x) = f(x-1)$. The join operator

$$\mathcal{A} \vee \mathcal{B} = \{f \oplus g : f \in \mathcal{A} \wedge g \in B\},$$

and the meet operator

$$\mathcal{A} \wedge \mathcal{B} = 0 \hat{} \mathcal{A} \cup 1 \hat{} \mathcal{B}$$

on mass problems originate well defined operations on Medvedev degrees that make $\mathcal{M}$ a bounded distributive lattice $\mathfrak{M} = (\mathcal{M}, \vee, \wedge, \mathbf{0}, \mathbf{1})$, called the *Medvedev lattice*. Henceforth, when talking about the first order theory of the Medvedev lattice, denoted $\mathrm{Th}(\mathfrak{M})$, we will refer to $\mathrm{Th}(\mathcal{M}, \leq_M)$. Clearly $\mathrm{Th}(\mathcal{M}, \leq_M) \equiv \mathrm{Th}(\mathcal{M}, \vee, \wedge, \mathbf{0}, \mathbf{1})$, where the symbol $\equiv$ denotes computable isomorphism.

One can consider a nonuniform variant of the Medvedev lattice, the *Muchnik lattice* $\mathfrak{M}_w = (\mathcal{M}_w, \leq_w)$, introduced and studied in [4]. This is the structure resulting from the reduction relation on mass problems defined by

$$\mathcal{A} \leq_w \mathcal{B} \Leftrightarrow (\forall g \in \mathcal{B})(\exists f \in \mathcal{A})[f \leq_T g],$$

where $\leq_T$ denotes Turing reducibility. Again, $\leq_w$ generates an equivalence relation $\equiv_w$ on mass problems. The equivalence class of $\mathcal{A}$ is called the *Muchnik degree* of $\mathcal{A}$, denoted by $\deg_w(\mathcal{A})$. The above displayed operations on mass problems turn $\mathcal{M}_w$ in a lattice too, denoted by $\mathfrak{M}_w = (\mathcal{M}_w, \vee, \wedge, \mathbf{0}_w, \mathbf{1}_w)$, where $\mathbf{0}_w$ is the Muchnik degree of any mass problem containing a computable function, and $\mathbf{1}_w = \deg_w(\emptyset)$. The first order theory of the Muchnik lattice, in the language of partial orders, will be denoted by $\mathrm{Th}(\mathfrak{M}_w)$.

It is well known that the Turing degrees can be embedded into both $\mathcal{M}$ and $\mathcal{M}_w$. Indeed, the mappings

$$\begin{aligned} i(\deg_T(A)) &= \deg_M(\{c_A\}), \\ i_w(\deg_T(A)) &= \deg_w(\{c_A\}) \end{aligned}$$

(where, given a set A, we denote by c_A its characteristic function), are well defined embeddings of $(\mathcal{D}_T, \leq_T)$ into $(\mathcal{M}, \leq_M)$ and $(\mathcal{M}_w, \leq_w)$, respectively. Moreover, i and i_w preserve least element, and the join operation. Henceforth, we will often identify the Turing degrees with the range of i, or i_w according to the case. Thus, we say that a Medvedev degree (respectively, a Muchnik degree) $\mathbf{X}$ is a Turing degree if it is in the range of i (respectively, i_w). It is easy to see that $\mathbf{X} \in \mathfrak{M}$ (respectively, $\mathbf{X} \in \mathfrak{M}_w$) is a Turing degree if and only if $\mathbf{X} = \deg(\{f\})$ (respectively, $\mathbf{X} = \deg_w(\{f\})$) for some function f. When thinking of a Turing degree $\mathbf{X}$ within $\mathfrak{M}$, or $\mathfrak{M}_w$, we will always choose a mass problem that is a singleton as a representative of $\mathbf{X}$.

Lemma 2.1 *The Turing degrees are first order definable in both* $(\mathcal{M}, \leq_M)$ *and* $(\mathcal{M}_w, \leq_w)$ *via the formula*

$$\varphi(u) =^{def} \exists v[u < v \wedge \forall w[u < w \rightarrow v \leq w]].$$

Proof. See [1]. □

It is perhaps worth observing that the Medvedev lattice and the Muchnik lattice are not elementarily equivalent:

Theorem 2.2 $\mathrm{Th}(\mathfrak{M}) \neq \mathrm{Th}(\mathfrak{M}_w)$.

Proof. We exhibit an explicit first order difference. Let

$$\begin{aligned} \mathbf{0}' &= \deg_M(\{f : f \text{ non computable}\}), \\ \mathbf{0}'_w &= \deg_w(\{f : f \text{ non computable}\}). \end{aligned}$$

Notice that $\mathbf{0}'$ and $\mathbf{0}'_w$ are definable in the respective structures by the same first order formula, expressing that $\mathbf{0}'$ is the least element amongst the nonzero Medvedev degrees, and $\mathbf{0}'_w$ is the least element amongst the nonzero Muchnik degrees. (Notice that $\mathbf{0}'$ is the element v witnessing the existential quantifier in the above formula $\varphi(u)$ when u is interpreted as the least Turing degree in the Medvedev lattice; similarly $\mathbf{0}'_w$ is the element v witnessing the existential quantifier in the above formula $\varphi(u)$ when u is interpreted as the least Turing degree in the Muchnik lattice.) It is now easy to notice an elementary difference between the Medvedev and the Muchnik lattice, as it can be shown that $\mathbf{0}'$ is meet-irreducible in $\mathfrak{M}$: this follows from the characterization of meet-irreducible elements of $\mathfrak{M}$ given in [1], see also [10, Theorem 5.1]. On the other hand, let f be a function of minimal Turing degree, and let $\mathbf{A} = \deg_w(\{f\})$, $\mathbf{B} = \deg_w(\{g : f \not\leq_T g\})$. Then, in $\mathfrak{M}_w$, $\mathbf{0}'_w = \mathbf{A} \wedge \mathbf{B}$, i.e. $\mathbf{0}'_w$ is meet-reducible. $\square$

3 The complexity of the first order theory

We will show that the first order theories of the Medevdev lattice and the Muchnik lattice are both computably isomorphic to third order arithmetic.

3.1 Some logical systems

We now introduce second and third order arithmetic and some useful related logical systems.

Third order arithmetic Third order arithmetic is the logical system defined as follows. The language, with equality, consists of: The basic symbols $+, \times, 0, 1, <$ of elementary arithmetic; first order variables $x_0, x_1, \dots$ (for numbers); second order variables $p_0, p_1, \dots$ (for unary functions on numbers); third order variables $X_0, X_1, \dots$ (for sets of functions, i.e. mass problems). Terms and formulas are built up as usual, but similarly to function symbols, second order variables are allowed to form terms: thus if t is a term and p is a second order variable, then $p(t)$ is a term; if p a second order variable and X is a third order variable then $p \in X$ is allowed as atomic formula. Finally, we are allowed for quantification also on second order variables, and on third order variables. Sentences are formulas in which all variables are quantified. A sentence is *true* if its standard interpretation in the natural numbers is true (with first order variables being interpreted by numbers; second order variables being interpreted by unary functions from $\mathbb{N}$ to $\mathbb{N}$; third order variables being interpreted by mass problems. The symbol $\in$, here and in the following systems, is interpreted as membership). The collection of all true sentences, under this interpretation, is called *third order arithmetic*, denoted by $\mathrm{Th}_3(\mathbb{N})$. Notice that by limiting ourselves to adding to elementary arithmetic only variables for functions, we get what is known as *second order arithmetic*, denoted by $\mathrm{Th}_2(\mathbb{N})$.

Second order theory of the real numbers The second order theory of the field $\mathbb{R}$ of the real numbers is the logical system (with equality) defined as follows. The language, with equality, consists of the basic symbols $+, \times, 0, 1, <$. We have first order variables $r_0, r_1, \dots$ (for real numbers); second order variables $X_0, X_1, \dots$ (for sets of real numbers). Terms, atomic formulas and formulas are built in the usual way, where we regard $r \in Y$ as an atomic formula if r is a first order variable, and X is a second order variable. Quantification on both first and second order variables is allowed. Sentences are formulas in which all variables are quantified. A sentence is *true* if the standard interpretation of the sentence in the field of real numbers is true (where first order variables are interpreted by real numbers; second order variables are interpreted by sets of real numbers). By *second order theory of the field* $\mathbb{R}$, denoted by $\mathrm{Th}_2(\mathbb{R})$, we mean the collection of all such true sentences.

We are now ready to give a useful, although simple, characterization of third order arithmetic $\mathrm{Th}_3(\mathbb{N})$.

Lemma 3.1 $\mathrm{Th}_3(\mathbb{N})$ *is computably isomorphic to* $\mathrm{Th}_2(\mathbb{R})$.

Proof. Let EA_1 be the logical system obtained from elementary arithmetic as follows. The language, with equality, consists of the basic elementary symbols od arithmetic $+, \times, 0, 1, <$; we have first order *numerical variables* $x_0, x_1, \ldots$, and in addition we have first order variables of a different sort, $r_0, r_1, \ldots$, called *real variables.* Then the system EA_1 is obtained by taking all sentences which are true under interpreting numerical variables with numbers, real variables with real numbers, and interpreting $+$ and $\times$ accordingly. This system is known as *elementary analysis.* It is known (see for instance [6, Theorem 16.XIII], for a proof) that $\mathrm{Th}_2(\mathbb{N}) \equiv EA_1$. Let now EA_2 be the logical system obtained by adding to the language of EA_1 second order variables $R_0, R_1, \dots$ (for sets of reals); and by adding atomic formulas of the form $r \in R$, where r is a real variable and R is a second order variable. Then EA_2 is the collection of all sentences in this language that are true under interpretation of second order variables as sets of real numbers. Following up the argument in Rogers, [6], it is now easy to show that $\mathrm{Th}_3(\mathbb{N}) \equiv EA_2$. It is then sufficient to show that $EA_2 \equiv \mathrm{Th}_2(\mathbb{R})$. Indeed, $EA_2 \leq_1 \mathrm{Th}_2(\mathbb{R})$ follows from the fact that $\mathbb{N} \subseteq \mathbb{R}$ is second-order definable in $(\mathbb{R})$, being the smallest inductive set. On the other hand, it is clear that $\mathrm{Th}_2(\mathbb{R}) \leq_1 EA_2$. $\square$

Lemma 3.2 *Let* $\mathfrak{A} \subseteq \mathfrak{D}_T$ *be an antichain, let* $\mathfrak{B} \subseteq \mathfrak{A}$*, and via the embedding of the Turing degrees into* $\mathfrak{M}$ *directly regard* $\mathfrak{A}$ *as a subset of* $\mathfrak{M}$*. For every* $\mathbf{X} \in \mathfrak{A}$ *let* f_x *be a function such that* $\mathbf{X} = \deg_M(\{f_x\})$*. Let* $\mathbf{C}$ *be the Medvedev degree of the mass problem* $\mathcal{C} = \{f_y : \mathbf{Y} \in \mathfrak{B}\}$*. Then*

$$(\forall \mathbf{X} \in \mathfrak{A})[\mathbf{C} \leq_M \mathbf{X} \Leftrightarrow \mathbf{X} \in \mathfrak{B}].$$

A similar result applies to the Muchnik lattice. In this latter case, we of course regard each $\mathbf{X} \in \mathfrak{A}$ *as the Muchnik degree* $\deg_w(\{f_x\})$ *of some function* f_x*, and we work with* $\leq_w$ *instead of* $\leq_M$*.*

Proof. Suppose we work with the Medvedev lattice, and let $\mathfrak{A} \subseteq \mathfrak{D}_T$ be an antichain, viewed as an antichain in $\mathfrak{M}$ via the embedding of the Turing degrees. Let $\mathfrak{B} \subseteq \mathfrak{A}$, f_x, $\mathcal{C}$ and $\mathbf{C}$ be defined as in the statement of the lemma.

If $\mathbf{C} \leq_M \mathbf{X}$ then $\mathcal{C} \leq_M \{f_x\}$, which implies that $f_x \in \mathcal{C}$, hence $\mathbf{X} \in \mathfrak{B}$. For the other direction, if $\mathbf{X} \in \mathfrak{B}$ then $f_x \in \mathcal{C}$, implying $\mathcal{C} \leq_M \{f_x\}$, i.e. $\mathbf{C} \leq_M \mathbf{X}$.

The case of the Muchnik lattice is similar. $\square$

3.2 The complexity of the theory

We now show that the first order theories of the Medvedev lattice and the Muchnik lattice have the same m-degree as $\mathrm{Th}_3(\mathbb{N})$.

One direction is trivial:

Lemma 3.3 $\mathrm{Th}(\mathfrak{M}), \mathrm{Th}(\mathfrak{M}_w) \leq_m \mathrm{Th}_3(\mathbb{N})$.

Proof. Trivially, the first order theories of $\mathfrak{M}$ and $\mathfrak{M}_w$ can be interpreted in third order arithmetic. $\square$

For the converse, we first need a computability theoretic result. All unexplained computability notions which are used in this section can be found in [2], see in particular Chapter V. Following [2], we say that a *tree* T is a function from binary strings to binary strings such that for every binary string σ, $T(\sigma\hat{\ }0)$ and $T(\sigma\hat{\ }1)$ are incomparable extensions of $T(\sigma)$. Here the symbol $\hat{\ }$ denotes concatenation of strings. If σ is a string and n is a number then we let $\sigma\hat{\ }n = \sigma\hat{\ }\langle n\rangle$: a similar convention holds of $n\hat{\ }\sigma$. The length of a string σ is denoted by $|\sigma|$. A tree T is *computable* if T is computable as a function. We also say that T' is a *subtree of* T if $\mathrm{range}(T') \subseteq \mathrm{range}(T)$. Given a tree T, the collection of all infinite paths in T will be denoted by $[T]$.

Lemma 3.4 *There is a tree T such that, for any Turing degrees $\mathbf{x}, \mathbf{y}, \mathbf{z}$ of distinct paths of T, the following hold:*

(i) $\mathbf{x}$ *is minimal;*

(ii) $\mathbf{x} \not\leq \mathbf{y} \vee \mathbf{z}$.

Proof. Given any tree T and any σ, σ' let $\mathrm{Ext}(T,\sigma)(\sigma') = T(\sigma\hat{\ }\sigma')$: if d is a number then $\mathrm{Ext}(T,d) = \mathrm{Ext}(T,\langle d\rangle)$. For every computable tree T and every n let $\mathrm{Min}(T,n)$ be a computable subtree of T such that if $A \in [\mathrm{Min}(T,n)]$ then φ_n^A total $\Rightarrow [\varphi_n^A$ computable $\vee\, A \leq_T \varphi_n^A]$: see for instance [2] for the details of the construction of $\mathrm{Min}(T,e)$ starting from T.

For any computable trees T_0, T_1, T_2 and any n, let for each $i < 2$,

$$\mathrm{Diag}_n^i(T_0,T_1,T_2) = \hat{T}_i$$

for some computable $\hat{T}_i \subseteq T_i$ such that if $A_0 \in \hat{T}_0$, $A_1 \in \hat{T}_1$ and $A_2 \in \hat{T}_2$ then $A_0 \neq \varphi_n^{A_1 \oplus A_2}$. That $\mathrm{Diag}_n^i(T_0,T_1,T_2)$ exists can be seen as follows: Let T_0, T_1, T_2 and n be given. We distinguish the following cases:

Case 1) $(\exists x)(\exists \rho_1)(\exists \rho_2)(\forall \tau_1 \supseteq \rho\_1)(\forall \tau_2 \supseteq \rho_2)[\varphi_n^{T_1(\tau_1)\oplus T_2(\tau_2)}(x)\uparrow]$: in this case choose ρ_1 and ρ_2 and define

$$\hat{T}_0 = T_0 \qquad \hat{T}_1 = \mathrm{Ext}(T_1, \rho_1) \qquad \hat{T}_2 = \mathrm{Ext}(T_2, \rho_2)$$

Case 2) Otherwise, we can find strings τ_1 and τ_2 such that

$$(\forall x < |T_0(0)|, |T_0(1)|)[\varphi_n^{T_1(\tau_1)\oplus T_2(\tau_2)}(x)\downarrow].$$

Since $T_0(0) \neq T_1(1)$ we can choose $j \in \{0, 1\}$ such that

$$T_0(j)(x) \neq \varphi_n^{T_1(\tau_1)\oplus T_2(\tau_2)}(x),$$

for some $x < |T_0(0)|, |T_0(1)|$, and define

$$\hat{T}_0 = \mathrm{Ext}(T_0, j) \qquad \hat{T}_1 = \mathrm{Ext}(T_1, \tau_1) \qquad \hat{T}_2 = \mathrm{Ext}(T_2, \tau_2).$$

We now define T which satisfies the hypothesis of the lemma in stages, defining $T(\sigma)$ for all σ of length n at stage n. For each σ we also define an auxiliary value T_σ.

Stage 0. Let $T(\lambda) = \lambda$ and define $T_\lambda = \mathrm{Id}$.

Stage $n+1$. Let $\{T_i^0 : i < 2^{n+1}\}$ be the set of all values $\mathrm{Ext}(T_\sigma, d)$ such that σ is of length n and $d \in \{0, 1\}$ and for each $i < 2^{n+1}$ let $\sigma_i = \sigma\hat{\ }d$ for σ and d such that $T_i^0 = \mathrm{Ext}(T_\sigma, d)$. Let r be the number of triples (k, l, m) with $k, l, m < 2^{n+1}$ and $k \neq l \neq m$.

Step (1). Fixing any order on the set of all such triples, proceed as follows for each such triple in turn. For the j^{th} triple (k, l, m), given T_k^{j-1}, T_l^{j-1} and T_m^{j-1}, let $T_k^j = \mathrm{Diag}_n^0(T_k^{j-1}, T_l^{j-1}, T_m^{j-1})$, $T_l^j = \mathrm{Diag}_n^1(T_k^{j-1}, T_l^{j-1}, T_m^{j-1})$ and $T_m^j = \mathrm{Diag}_n^2(T_k^{j-1}, T_l^{j-1}, T_m^{j-1})$. For each $i < 2^{n+1}$ such that $i \notin \{k, l, m\}$ define $T_i^j = T_i^{j-1}$.

Step (2). For each $i < 2^{n+1}$, define $T_{\sigma_i} = \mathrm{Min}(T_i^r, n)$ and $T(\sigma_i) = T_{\sigma_i}(\lambda)$.

□

Lemma 3.5 $\mathrm{Th}_2(\mathbb{R})$ *can be interpreted in both* $\mathrm{Th}(\mathfrak{M})$ *and* $\mathrm{Th}(\mathfrak{M}_w)$.

Proof. Again the proof is given for $\mathfrak{M}$, but *mutatis mutandis* it works for $\mathfrak{M}_w$ too. By the usual coding methods, see e.g. [5], the ordered field $\mathbb{R}$ can be first-order defined in a symmetric graph (V, E), where we may assume $V = 2^{\mathbb{N}}$, the Cantor space. Since T as in Lemma 3.4 is homeomorphic to $2^{\mathbb{N}}$, we may assume that in fact V is the set of paths of T. We can now obtain a coding scheme $\mathbf{R}_{A,B}$ to code with two appropriate parameters A, B a copy of the ordered field $\mathbb{R}$ into $\mathfrak{M}$. Let $\mathfrak{B}$ be the collection of Turing degrees of the paths of T (viewed inside $\mathfrak{M}$). The parameter A picks up $\mathfrak{B}$ among the minimal Turing degrees, that are first order definable in $\mathfrak{M}$, via Lemma 3.2. The parameter B picks the edge relation

$$\{\mathbf{x} \vee \mathbf{y} : Exy\},$$

for $x, y \in V$, obtained by applying Lemma 3.2 to the antichain

$$\{\mathbf{x} \vee \mathbf{y} : \mathbf{x} \neq \mathbf{y} \wedge \mathbf{x}, \mathbf{y} \in \mathfrak{B}\}.$$

Applying Lemma 3.2, we may now quantify over subsets of the coded copy of $\mathbb{R}$. It is clear how to translate each second order sentence Φ in the language of $\mathrm{Th}_2(\mathbb{R}, +, \times)$ into a formula $\widehat{\Phi}_{A,B}$ with parameters A, B, according to this coding scheme of $\mathbb{R}$ into $\mathfrak{M}$.

We obtain a correctness condition on parameters, $\alpha(A, B)$, saying that the coded model $\mathbf{R}_{A,B}$ is isomorphic to $\mathbb{R}$, by requiring the second order axioms of a complete ordered field (i.e. each bounded nonempty subset has a supremum). So

$$\Phi \in \mathrm{Th}_2(\mathbb{R}) \Leftrightarrow \mathfrak{M} \models \exists A, B[\alpha(A, B) \wedge \widehat{\Phi}_{A,B}].$$

□

Theorem 3.6 $\mathrm{Th}(\mathfrak{M}), \mathrm{Th}(\mathfrak{M}_w) \equiv \mathrm{Th}_3(\mathbb{N})$.

Proof. By Lemma 3.3, we get $\mathrm{Th}(\mathfrak{M}), \mathrm{Th}(\mathfrak{M}_w) \leq_1 \mathrm{Th}_3(\mathbb{N})$. On the other hand, by Lemma 3.5, we get $\mathrm{Th}_2(\mathbb{R}) \leq_1 \mathrm{Th}(\mathfrak{M}), \mathrm{Th}(\mathfrak{M}_w)$, and thus by Lemma 3.1, $\mathrm{Th}_3(\mathbb{N}) \leq_1 \mathrm{Th}(\mathfrak{M}), \mathrm{Th}(\mathfrak{M}_w)$. □

References

[1] E.Z. Dyment. Certain properties of the Medvedev lattice. *Mathematics of the USSR Sbornik*, 30:321–340, 1976. English Translation.

[2] M. Lerman. *Degrees of Unsolvability.* Perspectives in Mathematical Logic. Springer–Verlag, Heidelberg, 1983.

[3] Y. T. Medevdev. Degrees of difficulty of the mass problems. *Dokl. Nauk. SSSR*, 104:501–504, 1955.

[4] A.A. Muchnik. On strong and weak reducibility of algorithmic problems. *Sibirskii Matematicheskii Zhurnal*, 4:1328–1341, 1963. Russian.

[5] A. Nies. Undecidable fragments of elementary theories. *Algebra Universalis*, 35:8–33, 1996.

[6] H. Rogers, Jr. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill, New York, 1967.

[7] Richard A. Shore. Degree structures: local and global investigations. *Bull. Symbolic Logic*, 12(3):369–389, 2006.

[8] Stephen G. Simpson. First order theory of the degrees of recursive unsolvability. *Ann. of Math.*, 105:121–139, 1977.

[9] T. A. Slaman and W. H. Woodin. Definability in the Turing degrees. *Illinois J. Math.*, 30:320–334, 1986.

[10] A. Sorbi. The Medvedev lattice of degrees of difficulty. In Cooper S.B., Slaman T.A., and Wainer S.S., editors, *Computability, Enumerability, Unsolvability - Directions in Recursion theory*, London Mathematical Society Lecture Notes Series, pages 289–312. Cambridge University Press, New York, 1996.