From Automatic Structures to Borel Structures

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Abstract

We study the classes of Büchi and Rabin automatic structures. For Büchi (Rabin) automatic structures their domains consist of infinite strings (trees), and the basic relations, including the equality relation, and graphs of operations are recognized by Büchi (Rabin) automata. A Büchi (Rabin) automatic structure is injective if different infinite strings (trees) represent different elements of the structure. The first part of the paper is devoted to understanding the automata-theoretic content of the well-known Löwenheim-Skolem theorem in model theory. We provide automata-theoretic versions of Löwenheim-Skolem theorem for Rabin and Büchi automatic structures. In the second part, we address the following two well-known open problems in the theory of automatic structures: Does every Büchi automatic structure have an injective Büchi presentation? Does every Rabin automatic structure have an injective Rabin presentation? We provide examples of Büchi structures without injective Büchi and Rabin presentations. To answer these questions we introduce Borel structures and use some of the basic properties of Borel sets and isomorphisms. Finally, in the last part of the paper we study the isomorphism problem for Büchi automatic structures.

Contents

1. Introduction

We study the classes of structures that can be recognized by Büchi and Rabin automata and compare them with other classes of structures. By a *structure* we mean a set that has finitely many relations and operations defined on it like, for example, a ring, an ordered group, or a Boolean algebra, etc. Structures presented by finite word and tree automata have been studied extensively over the last several years (e.g. [4, 13, 15]), and we refer to these structures as *word automatic* and *tree automatic structures*. However, there has been very little work on structures presented by Büchi and Rabin automata. Therefore, several foundational questions that have relatively simple solutions for word and tree automatic structures are still outstanding for the classes of

Büchi and Rabin automatic structures. We address some of these questions in this paper. We mention that Blumensath and Grädel studied Büchi automatic structures in [4]. They proved that the class of Büchi automatic structures has a complete structure, that is a Büchi automatic structure in which every Büchi automatic structure is interpretable [4]. Also, Kuske and Lohrey in [16] studied the model checking problem for Büchi automatic structures in some extensions of the first order logic. We also refer to [3] for some results on Rabin automatic structures.

For Büchi and Rabin automatic structures their domains consist of infinite strings or infinite trees, and the basic relations, including the equality relation, and graphs of operations are recognized by Büchi and Rabin automata. All these structures, as shown in [4, 13, 15, 16], have strong closure and decidability properties. For example, the first order theory of these structures are decidable. Moreover, these classes of structures are closed under first order interpretations.

The paper consists of three parts. The first part is devoted to understanding the automata-theoretic content of the well-known Löwenheim-Skolem theorem in model theory. The theorem states that every uncountable structure on a countable language has a countable elementary substructure (see for example [17]). In order to investigate an automatic version of the Löwenheim-Skolem theorem, we study Büchi and Rabin automatic structures from a finitistic view point. We define a new notion of finite automata, that we call *loop-automata* (see Section 3). These automata run on a certain type of finite strings that have loop shape in a sense we specify later. The idea is that these loops represent infinite eventually periodic strings. We then look at the class of structures that can be recognized by these automata. We call them loop-automatic structures. This class of countable structures has strong closure and decidability property like the automatic structures, but it is a larger class. Every automatic structure is loop-automatic. But, we will see that, for instance, the atomless Boolean algebra is loop-automatic and we know it is not automatic (as shown by Khoussainov, Rubin, and Stephan [15]). Also, the ordered group of the rational numbers $(\mathbb{Q}, +, \leq)$ is loop-automatic but not known to be automatic. The loop automatic structures provide an automata-theoretic content to the Löwenheim-Skolem theorem. Namely, in Theorem 4.2, we show that every Büchi structure has an elementary loop-automatic substructure. We also define a new notion of finite tree automata that we call *looped tree automata*. For looped tree-automatic structures we also get strong closure and decidability properties as we get for automatic structures. This also gives us a version of Löwenheim-Skolem for Rabin structures (Theorem 4.2).

In the second part of the paper we address two well-known open problems in the theory of automatic structures:

1. *Does every Büchi automatic structure have an injective Büchi presentation?*

2. *Does every Rabin automatic structure have an injective Rabin presentation?*

An injective Büchi presentation of a structure is one where different infinite strings represent different elements of the structure, while in a Büchi (and Rabin) presentation we only require the equality relation to be a Büchi (Rabin) equivalence relation on the set of infinite strings (trees) (see Definition 2.5 for details). We call these two problems the *injectivity problems* for Büchi and Rabin automatic structures.

In order to investigate the questions above, we compare the classes of Büchi and Rabin automatic structures with the class of Borel structures. Borel structures are uncountable structures whose domains are subsets of the set of infinite strings. For Borel structures we require the domain, the equality relation and the graph of the operations and relations to be Borel sets (see Section 5). All Büchi automatic structures are examples of Borel structures.

We answer both of the injectivity problems negatively in Section 6, see Theorem 6.4 and Theorem 6.6 , by building a Büchi structure that does not even have a Borel injective presentation. Interestingly, the structure built is also an example of a structure without injective Rabin presentation. We point out that these results use notions from descriptive set theory. It would be interesting to prove these theorems using purely automata-theoretic methods. We should mention that the corresponding counterparts for word and tree automatic structures have positive solutions [13] [4] [6]. In fact, for tree automatic structures the positive solution, given in [6], involves non-trivial technical details that correct the proof from [3]. For countable Büchi automatic structures the injectivity problem has a positive solution. The solution uses an automata-theoretic analysis of the state space of automata representing the equality relation [10]. In addition, the paper [4] addresses the injectivity problem for Büchi automatic structures and claims that Büchi automatic structures have injective presentations (see Proposition 5.2 in [4]).

More generally, we prove that the inclusions between classes of structures showed in the diagram below are the all proper. Two of these inclusions solve the injectivity problems stated above. The diagram also depicts the fact, proved in Theorem 6.3, that there is a Rabin injective structure that does not have a Borel presentation. The coding of this structure is based on the well-known result of D. Niwinski (see [18]) showing that there exists a Rabin recognizable language that is not a Borel set. The readers can consult the papers [19, 21, 18] on relationship between Rabin recognizable languages and Borel hierarchy.

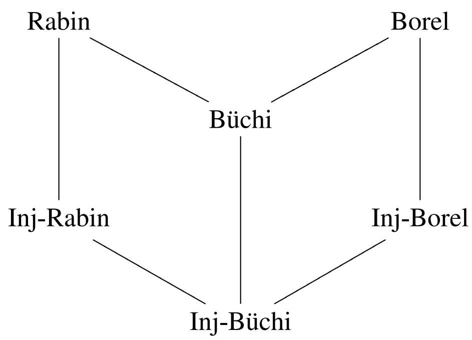


Figure 1. Classes of structures

2. Background on Büchi and Rabin automatic structures

In this section we review the basic facts and definitions about Büchi and Rabin automatic structures. Büchi recognizable languages were first introduced by Büchi in [5] with the intention of proving that monadic second order of one successor has a decidable theory. Rabin automata were used to prove that the monadic second order theory of two successor functions is decidable [20]. These have applications in many other areas of computer science.

2.1. Büchi and Rabin automata

Let $\mathbb{B}^\star$ be the set of finite strings and $\mathbb{B}^\omega$ be all infinite words over finite alphabet $\mathbb{B}$. We denote these infinite words by symbols $\alpha, \beta, \ldots$.

Definition 2.1. A *Büchi automaton* $\mathbf{M}$ is a quadruple (S, ι, Δ, F), where

- S is a finite set of *states*,
- $\iota \in S$ is the *initial state*,
- $\Delta \subset S \times \mathbb{B} \times S$ is the *transition table*,
- and $F \subset S$ is the set of *accepting states*.

A *run* of $\mathbf{M}$ on $\alpha = \sigma_0\sigma_1 \ldots$ is a sequence of states $\mathbf{r} = s_0, s_1, \cdots \in \mathbb{B}^\omega$ such that $s_0 = \iota$ and $(s_i, \sigma_i, s_{i+1}) \in \Delta$ for all $i \in \omega$. The run is *accepting* if the set $\mathrm{In}(\mathbf{r}) = \{s : \exists^\infty i(q_i = s)\}$ has a state from F. The automaton accepts the string α if it has an accepting run on it. The *language* accepted by the automaton $\mathbf{M}$, denoted $L(\mathbf{M})$, is the set of all infinite words accepted by $\mathbf{M}$.

Büchi automata can also recognize n-tuples of infinite strings. For this we need a simple definition. The *convolution of a tuple* $(\alpha_1, \cdots, \alpha_n) \in (\mathbb{B}^\omega)^n$ is the infinite word $\mathbf{c}(\alpha_1, \cdots, \alpha_n) \in (\mathbb{B}^n)^\omega$ whose k'th symbol is $(\alpha_1(k), \ldots, \alpha_n(k)) \in \mathbb{B}^n$. Note that the size of the alphabet of the convoluted word has increased. The *convolution of a relation* $R \subset (\mathbb{B}^\omega)^n$, denoted by $\mathbf{c}(R)$, is the language formed as the set of convolutions of all the tuples in R. Say that R is *Büchi recognizable* if $\mathbf{c}(R)$ is a Büchi recognizable language. This definition can be generalized to tuples of infinite strings over different alphabets in an obvious way.

Example 2.2. 1. The lexicographic relation $\{(\alpha, \beta) \mid \alpha, \beta \in \{0,1\}^\omega,\ \alpha \leq_{lex} \beta\}$ is a binary Büchi recognizable relation.

2. The equivalence relation $=^*$ on $\{0,1\}^\omega$, defined by $\alpha =^* \beta$ if $\exists n \forall m \geq n(\alpha(m) = \beta(m))$ is also Büchi recognizable.

For a language $S \subseteq \mathbb{B}_1^\omega \times \mathbb{B}_2^\omega$ its projection (to the first component) is the language $\{\alpha \in \mathbb{B}_1^\omega : \exists \beta \left((\alpha, \beta) \in S\right)\}$. Büchi proved the following:

Theorem 2.3. *([5]) The class of all Büchi recognizable languages is closed under the operations of union, intersection, projection, and complementation. Moreover, there is an algorithm that, given a Büchi automaton* $\mathbf{M}$*, decides whether* $L(\mathbf{M})$ *is empty.*

We now define Rabin automata. Let $\mathcal{T}$ be the binary tree $(\{0,1\}^\star; L, R)$ called *two successor structure* where $L(x) = x0$ and $R(x) = x1$ for all $x \in \{0,1\}^\star$. Let $\mathbb{B}$ be a finite alphabet. Let $Tree(\mathbb{B})$ be all the $\mathbb{B}$-labeled trees $(\mathcal{T}, v)$, where $v : \mathcal{T} \to \mathbb{B}$. A *Rabin automaton* $\mathbf{M}$ is $(S, \iota, \Delta, \mathcal{F})$, where

- S is a set of *states*,
- $\iota \in S$ is the *initial state*,
- $\Delta : S \times \mathbb{B} \to P(S \times S)$ is the *transition table*, and
- $\mathcal{F} \subset P(S)$ is the set of *designated subsets*.

A *run* of $\mathbf{M}$ on $(\mathcal{T}, v)$ is a mapping $\mathbf{r} : \mathcal{T} \to S$ such that such that $\mathbf{r}(root) = \iota$, and for each $x \in \mathcal{T}$ we have $(\mathbf{r}(L(x)), \mathbf{r}(R(x))) \in \Delta(\mathbf{r}(x), v(x))$. The run is *accepting* if for every path η in $\mathcal{T}$ the set $\{s \mid s$ appears on the run $\mathbf{r}$ along η infinitely many times$\}$ belongs to $\mathcal{F}$. The *language* accepted by the automaton $\mathbf{M}$, denoted $L(\mathbf{M})$, is the set of all trees $(\mathcal{T}, v)$ for which there is an accepting run of $\mathbf{M}$. We call these *Rabin recognizable languages*. In 1968, Rabin extended Büchi's theorem to tree languages.

Theorem 2.4. *([20]) The class of all Rabin recognizable tree languages is closed under the operations of union, intersection, projection, and complementation. Moreover, there is an algorithm that, given a Rabin automaton* $\mathbf{M}$*, decides whether the automaton accepts some* $\mathbb{B}$*-tree or not.*

One can extend (in a straightforward way) the definition of Rabin recognizable language to Rabin recognizable relations on the set $Tree(\mathbb{B})$ of all $\mathbb{B}$-labeled trees.

2.2. Büchi and Rabin automatic structures

Now we use the automata defined above to represent structures. We start with the definition of Büchi presentation:

Definition 2.5. We say the tuple $\mathcal{S} = (D; E, R_1, \ldots, R_n)$ is a *Büchi representation* of a structure $\mathcal{A}$ if

1. All D, E, $R_1, \ldots$, R_n are Büchi recognizable (D is called the domain of the representation).

2. All E, $R_1, \ldots, R_n$ are relations on D.

3. E is an equivalence relation on the domain D such that E is compatible with $R_1, \ldots, R_n$.

4. The quotient structure $\mathcal{S}/E$ is isomorphic to $\mathcal{A}$.

In this case we say that $\mathcal{A}$ is a *Büchi automatic structure*. In case when E is the equality relation on D, then $\mathcal{S}$ is an *injective presentation of* $\mathcal{A}$, and $\mathcal{A}$ is *injective Büchi automatic structure*.

Example 2.6. 1. For $X, Y \subseteq \mathbb{N}$, we write $X =^* Y$ if the symmetric difference of X and Y is finite and $X \subseteq^* Y$ if $X - Y$ is finite. The partial order $\mathcal{B}^*$ defined as $(\mathcal{P}(\mathbb{N})/ =^*, \subseteq^*)$ is a Büchi automatic structure.

2. The ordered group $(\mathbb{R}, +, \leq)$. This is also a Büchi automatic structure.

3. The linear ordered set $(\mathbb{B}^\omega, \leq_{lex})$ is Büchi automatic.

To explain the first example, we represent sets in the structure $\mathcal{B}^\star$ as infinite binary strings. The domain $P(\mathbb{N})$, the equivalence relation $=^\star$, and the relation $\subseteq^*$ are all Büchi recognizable. Hence $\mathcal{B}^*$ is Büchi automatic. In the second example we can represent reals in binary in a way that the graph of the addition operation is Büchi recognizable. The last example follows from Example 2.2 (1).

Definition 2.7. We say that a countable structure $\mathcal{A} = (A, R_1, \ldots, R_n)$ is *decidable* if its domain is computable and there is an algorithm that, given a tuple $\bar{a} \in A$ and a first order logic formula $\varphi(\bar{x})$, decides whether $\mathcal{A} \models \varphi(\bar{a})$.

Recall that *the model checking problem* for a structure $\mathcal{A}$ is formulated as follows. Design an algorithm that given a first order formula $\varphi(\bar{x})$ and a tuple $\bar{a}$ in $\mathcal{A}$ tells if $\mathcal{A} \models \varphi(\bar{a})$. Thus decidable models are the ones for which the model checking problem has a positive solution. For example, all word and tree automatic structures are decidable. The notion of decidable structure does not make sense for structures of size $2^{\aleph_0}$.

A structure $\mathcal{A}$ *has a decidable theory* if there is an algorithm that, given a sentence φ, decides whether $\mathcal{A} \models \varphi$.

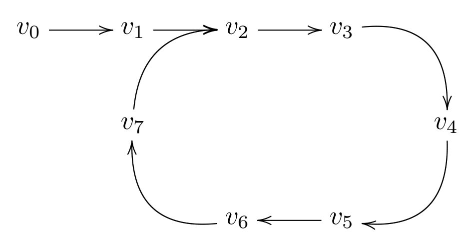
One can naturally generalize the concept of Büchi automatic structure to Rabin automatic structure. As in Definition 2.5 one defines the notion of *Rabin automatic structure* and *Rabin presentation* of structures. One just needs to replace Büchi recognizable languages and relations with Rabin recognizable languages and relations. Note that in Büchi automatic structures elements of the structures are represented as infinite strings over an alphabet $\mathbb{B}$, while in Rabin automatic structures elements are $\mathbb{B}$-labeled trees.

Theorem 2.8. *([4] [3]) There exists an algorithm that given a Büchi (Rabin) presentation of a structure and a first order formula $\varphi(\bar{x})$ computes a Büchi (Rabin) automaton $\mathbf{M}_{\varphi(\bar{x})}$ such that $L(\mathbf{M}_{\varphi(\bar{x})})$ consists of all tuples $\bar{\alpha}$ at which $\varphi(\bar{x})$ is true in the structure. In particular, every Büchi (Rabin) automatic structure has a decidable theory.*

3. Loop-automata

An *infinite eventually periodic word* (or just a *periodic word*) is a string of the form $uvvv\ldots \in \mathbb{B}^\omega$ where $u, v \in \mathbb{B}^\star$. Below we will code infinite periodic words in $\mathbb{B}^\omega$ by finite objects that we call $\mathbb{B}$-*loops*. Of course, one could code the periodic word $uvvvv\ldots$ by the pair (u, v). However, we will see that using $\mathbb{B}$-loops will make definitions and proofs smother.

Definition 3.1. A $\mathbb{B}$-*loop* is a tuple $G = (V, v_0, E, l)$, where V is a finite set, $v_0 \in V$ is the *initial vertex*, E is the *edge function* $E \colon V \to V$, and l is the *labeling function* $l \colon V \to \mathbb{B}$. Let $\mathcal{L}o(\mathbb{B})$ be the set of $\mathbb{B}$-loops.



The picture represents a $\mathbb{B}$-loop where $V = \{v_0, \ldots, v_7\}$, for $i = 0, ...6$, $E(v_i) = v_{i+1}$ and $E(v_7) = v_2$.

To each $\mathbb{B}$-loop corresponds an infinite eventually periodic word in $\mathbb{B}^\omega$ in a natural way as follows. Given a $\mathbb{B}$-loop G let $v^G \colon \omega \to V$ be defined by $v^G(0) = v_0$, and $v^G(n+1) = E(v^G(n))$. Let

$$\alpha_G = l \circ v^G \colon \omega \to \mathbb{B}.$$

Note that α_G is a periodic word. Conversely, every periodic word in $\mathbb{B}^\omega$ is of the form α_G for some $\mathbb{B}$-loop G. We say that two $\mathbb{B}$-loops G_1 and G_2 are *equivalent* if $\alpha_{G_1} = \alpha_{G_2}$.

Now we define how Büchi automata run on these $\mathbb{B}$-loops. The idea is that at each step in a run, we move to the next vertex in the $\mathbb{B}$-loop and to a next state given by the transition table. At some stage we will be in a vertex and a state that we have been before. At that point we halt the computation. We accept the run if we went through an accepting state in between the two stages when we had repeated vertex and state. Here is a more detailed definition.

Definition 3.2. Let $\mathbf{M} = (S, \iota, \Delta, F)$ be a Büchi automaton. A *run* of $\mathbf{M}$ on a $\mathbb{B}$-loop G is a sequence $\mathbf{r} = v_0, s_0, v_1, \ldots, v_k, s_k$ such that $v_i \in V$, $s_i \in S$, $s_0 = \iota$, and for every $i < k$

$$v_{i+1} = E(v_i) \quad \text{and} \quad (s_i, l(v_i), s_{i+1}) \in \Delta,$$

or, in other words, $v_i = v^G(i)$ and $(s_i, \alpha_G(i), s_{i+1}) \in \Delta$. A run is *simple* if, for every $i < j < k$, $(v_i, s_i) \neq (v_j, s_j)$. Say that a simple run is *complete* if there is a (necessarily unique) $t < k$ such that $(v_t, s_t) = (v_k, s_k)$. A complete run is *accepting* if the set of states $\text{In}(\mathbf{r}) = \{s_t, s_{t+1}, ..., s_k\}$ has a state from F. The set of $\mathbb{B}$-loops accepted by the automaton $\mathbf{M}$ is denoted by $L^{\mathcal{L}o}(\mathbf{M})$. A set $S \subseteq \mathcal{L}o(\mathbb{B})$ is *loop automata-recognizable* if $S = L^{\mathcal{L}o}(\mathbf{M})$ for some Büchi automaton $\mathbf{M}$.

3.1. $\mathbb{B}$-loops versus $\mathbb{B}$-infinite strings

We study the relationship between loop automata-recognizable and Büchi automata recognizable sets. Given $S \subseteq \mathbb{B}^\omega$, let $\text{lo}(S) = \{G \in \mathcal{L}o(\mathbb{B}) : \alpha_G \in S\}$.

Lemma 3.3. *Let* $\mathbf{M}$ *be a Büchi automaton and* G *a* $\mathbb{B}$*-loop. Then* $G \in L^{\mathcal{L}o}(\mathbf{M}) \iff \alpha_G \in L(\mathbf{M})$. *Therefore,* $L^{\mathcal{L}o}(\mathbf{M}) = \text{lo}(L(\mathbf{M}))$.

Proof. The proof in the direction from left to right is clear. For the reversal consider an accepting run $\mathbf{r} = s_0, s_1, \ldots$ of $\mathbf{M}$ on α_G; we want to build a simple complete accepting run on G. Apply the following stepwise process to $\mathbf{r}$.

1. Let t be the least number such that there is a $k > t$ with $v^G(t) = v^G(k)$ and $s_t = s_k$. Let k be the least such.
2. If there exists an accepting state $s \in F$ such that for some i with $t \leq i < k$ we have $s_i = s$, then we have that $v^G(0), s_0, ..., v^G(k), s_k$ is a complete accepting run on G as wanted. In this case we are done.
3. Otherwise, $\mathbf{r}' = s_0, s_1, ..., s_{t-1}, s_t, s_{k+1}, s_{k+2}...$ is an accepting run on α_G. Set $\mathbf{r} = \mathbf{r}'$, and go to step (1).

The process stops and builds an accepting run on G. □

Corollary 3.4. *If* G_1 *and* G_2 *are equivalent* $\mathbb{B}$*-loops, then* $\mathbf{M}$ *accepts* G_1 *if and only if it accepts* G_2.

From Büchi's theorem 2.3 and the lemma above we have:

Corollary 3.5. *The class of loop-automaton recognizable subsets of* $\mathcal{L}o(\mathbb{B})$ *is closed under the Boolean operations.*

The next corollary follows from the fact that every non-empty Büchi language contains a periodic word.

Corollary 3.6. *There is an algorithm that given* $\mathbf{M}$ *decides whether* $L^{\mathcal{L}o}(\mathbf{M}) \subseteq \mathcal{L}o(\mathbb{B})$ *is empty or not.*

Corollary 3.7. *Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be Büchi automata. Then*

$$L(\mathbf{M}_1) = L(\mathbf{M}_2) \iff L^{\mathcal{L}o}(\mathbf{M}_1) = L^{\mathcal{L}o}(\mathbf{M}_2)$$

Proof. The direction from left to right follows immediately from the lemma above. For the other direction assume that $L(\mathbf{M}_1) \not\subseteq L(\mathbf{M}_2)$, and hence $L(\mathbf{M}_1) \setminus L(\mathbf{M}_2)$ is not empty. By Büchi's theorem 2.3, there exists a Büchi automaton $\mathbf{M}$ recognizing the language $L(\mathbf{M}_1) \setminus L(\mathbf{M}_2)$. Since $L(\mathbf{M}) \neq \emptyset$, $L^{\mathcal{L}o}(\mathbf{M}) \neq \emptyset$. Hence, there is some $G \in \mathcal{L}o(\mathbb{B})$ such that α_G is accepted by $\mathbf{M}_1$ but not by $\mathbf{M}_2$. Thus, we have $L^{\mathcal{L}o}(\mathbf{M}_1) \not\subseteq L^{\mathcal{L}o}(\mathbf{M}_2)$ which is a contradiction. □

3.2. Products and projections

We want to consider sets of n-tuples of $\mathbb{B}$-loops. The convolution of loops is defined in a natural way by using the Cartesian product operation.

Definition 3.8. Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be finite alphabets. Define the map $\mathbf{c}\colon \mathcal{L}o(\mathbb{B}_1) \times \mathcal{L}o(\mathbb{B}_2) \to \mathcal{L}o(\mathbb{B}_1 \times \mathbb{B}_2)$ as follows. The *convolution* of a $\mathbb{B}_1$-loop $G_1 = (V_1, v_{1,0}, E_1, l_1)$ and a $\mathbb{B}_2$-loop $G_2 = (V_2, v_{2,0}, E_2, l_2)$ is the $(\mathbb{B}_1 \times \mathbb{B}_2)$-loop $\mathbf{c}(G_1, G_2) = (V, v_0, E, l)$ where

- $V = V_1 \times V_2$ with $v_0 = (v_{1,0}, v_{2,0})$,
- $E(v, w) = (E_1(v), E_2(w))$ and
- $l(v, w) = (l_1(v), l_2(w))$.

One similarly defines

$$\mathbf{c}\colon \mathcal{L}o(\mathbb{B}_1) \times \ldots \times \mathcal{L}o(\mathbb{B}_1) \to \mathcal{L}o(\mathbb{B}_1 \times \ldots \times \mathbb{B}_n),$$

and in particular $\mathbf{c}\colon (\mathcal{L}o(\mathbb{B}))^n \to \mathcal{L}o(\mathbb{B}^n)$. The *convolution* of a relation R is $\mathbf{c}(R)$, the image of R under $\mathbf{c}$. Say that R is *loop automata-recognizable* if $\mathbf{c}(R)$ is.

Observation 3.9. Note that $\alpha_{(\mathbf{c}(G_1,G_2))} = \mathbf{c}(\alpha_{G_1}, \alpha_{G_2})$, where the $\mathbf{c}$ in the right-hand-side refers to the convolution function on infinite strings defined in Subsection 2.1.

Lemma 3.10. *Let* $\mathbf{M}$ *be a Büchi automaton on* $\mathbb{B}_1 \times \mathbb{B}_2$ *and* $G_2 \in \mathcal{L}o(\mathbb{B}_2)$. *There is a Büchi automaton* $\mathbf{M}_{G_2}$ *on* $\mathbb{B}_1$ *such that* $L(\mathbf{M}_{G_2}) = \{\beta \in \mathbb{B}_1^\omega : \mathbf{c}(\beta, \alpha_{G_2}) \in L(\mathbf{M})\}$, *and*

$$L^{\mathcal{L}o}(\mathbf{M}_{G_2}) = \{G_1 \in \mathcal{L}o(\mathbb{B}_1) : \mathbf{c}(G_1, G_2) \in L^{\mathcal{L}o}(\mathbf{M})\}.$$

Proof. Let $\mathbf{M} = (S, s_0, \Delta, \mathcal{F})$ and $G_2 = (V_2, v_{2,0}, E_2, l_2)$. The automaton $\mathbf{M}_{G_2} = (S_1, s_{1,0}, \Delta_1, F_1)$ is defined essentially as the cartesian product of $\mathbf{M}$ and G_2. Formally,

- $S_1 = S \times V_2$ with $s_{1,0} = (s_0, v_{2,0})$,
- $((s, v), \sigma, (s', v')) \in \Delta_1 \iff$
  $v' = E_2(v)$ & $(s, (\sigma, l_2(v)), s') \in \Delta$, and
- $F_1 = F \times V_2$.

It is easy to show, using the lemma, its corollaries and the observation above, that $\mathbf{M}_{G_2}$ is a desired automaton. $\square$

Lemma 3.11. *The class of loop automaton-recognizable sets of* $\mathbb{B}$*-loops is closed under projections.*

Moreover, for every Büchi automaton $\mathbf{M}$ *on* $\mathbb{B}_1 \times \mathbb{B}_2$*, there is a Büchi automaton* $\mathbf{M}_1$ *on* $\mathbb{B}_1$ *such that*

$$L(\mathbf{M}_1) = \{\alpha \in \mathbb{B}_1^\omega : (\exists \beta \in \mathbb{B}_2^\omega)\, \mathbf{c}(\alpha, \beta) \in L(M)\}, \quad (1)$$

and $G_1 \in L^{\mathcal{L}o}(\mathbf{M}_1)$ *if and only if*

$$(\exists G_2 \in \mathcal{L}o(\mathbb{B}_2))\, \mathbf{c}(G_1, G_2) \in L^{\mathcal{L}o}(M).$$

Proof. By Büchi's theorem 2.3 there is a Büchi automaton $\mathbf{M}_1$ satisfying (1). We claim that $\mathbf{M}_1$ is as wanted. Let G_1 be a $\mathbb{B}_1$-loop. Suppose first that $(\exists G_2 \in \mathcal{L}o(\mathbb{B}_2))\, \mathbf{c}(G_1, G_2) \in L^{\mathcal{L}o}(M)$. Then for α_{G_2} we have $\mathbf{c}(\alpha_{G_1}, \alpha_{G_2}) \in L(M)$, and hence $\alpha_{G_1} \in L(\mathbf{M}_1)$. Thus $G_1 \in L^{\mathcal{L}o}(\mathbf{M}_1)$. Suppose now that $G_1 \in L^{\mathcal{L}o}(\mathbf{M}_1)$ and hence that $\alpha_{G_1} \in L(\mathbf{M}_1)$. Then, the set

$$\{\beta \in \mathbb{B}_2^\omega : \mathbf{c}(\alpha_{G_1}, \beta) \in L(M)\}$$

is not empty. We proved in the previous lemma that this set is recognized by the automaton $\mathbf{M}_{G_1}$. Then $L^{\mathcal{L}o}(\mathbf{M}_{G_1})$ is also non empty, and hence there exists $G_2 \in \mathcal{L}o(\mathbb{B}_2)$ such that $\mathbf{c}(G_1, G_2) \in L^{\mathcal{L}o}(M)$. $\square$

3.3. Looped-tree automata

The notion of periodic infinite strings can be extended to $\mathbb{B}$-labeled trees as follows. Let x be a node in the binary $\mathbb{B}$ tree $(\mathcal{T}, v)$. Consider the $\mathbb{B}$-tree $(\mathcal{T}, v_x)$, where $v_x(y) = v(xy)$ for all $y \in \mathbb{B}^\star$. We call this a *subtree* of $(\mathcal{T}, v)$. Say that $\mathbb{B}$-labeled tree $(\mathcal{T}, v)$ is *regular* if there are only finitely many different $\mathbb{B}$-labeled subtrees of $(\mathcal{T}, v)$.

Now we consider a notion of automata that is to Rabin automata as loop-automata is for Büchi automata.

Definition 3.12. A $\mathbb{B}$*-looped tree* is a tuple $G = (V, v_0, E_L, E_R, l)$, where V is a finite set, $v_0 \in V$ is the *initial vertex*, E_L and E_R are unary functions $: V \to V$, called *edge functions*, and l is the *labeling function* $l\colon V \to \mathbb{B}$. Let $\mathcal{L}o\mathcal{T}r(\mathbb{B})$ be the set of $\mathbb{B}$-looped tree s.

Now we define how Rabin automata run on these objects.

Definition 3.13. Let $\mathbf{M} = (S, \iota, \Delta, \mathcal{F})$ be a Rabin automaton. A *complete run* of $\mathbf{M}$ on an $\mathbb{B}$-looped tree G is a $(V \times S)$-looped tree

$$\mathbf{r} = (W, w_0, E_R^W, E_L^W, l^W)$$

such that $l^W(w_0) = (v_0, \iota)$, and, for every $w \in W$, if $l^W(w) = (v, s)$, there is a pair $(s_L, s_R) \in \Delta(s, l(v))$ such that

$$l^W(E_L^W(w)) = (E_L(v), s_L),\ l^W(E_R^W(w)) = (E_R(v), s_R).$$

Say that $w_1, ..., w_k \in W$ is a loop of $\mathbf{r}$ if $w_1 = w_k$ and for every $i = 1, .., k-1$, either $w_{i+1} = E_R^W(w_i)$ or $w_{i+1} = E_L^W(w_i)$. Say that the complete run is *accepting* if for every loop $(v_1, s_1), ..., (v_k, s_k)$ of $\mathbf{r}$, the set of states $\{s_1, ..., s_k\}$ is in $\mathcal{F}$. The automaton *accepts the graph* G if there is an accepting complete run on G. The set of $\mathbb{B}$-looped tree s G accepted by the automaton $\mathbf{M}$ is denoted by $L^{\mathcal{L}o}(\mathbf{M})$.

Each $\mathbb{B}$-looped tree codes a regular tree in a natural way as follows. Given a $\mathbb{B}$-looped tree G, there is a unique functions $v^G\colon \mathcal{T} \to V$ such that $v^G(\emptyset) = v_0$, and for every $\sigma \in \mathcal{T}$ and $D \in \{L, R\}$, $v^G(D(\sigma)) = E_D(v^G(\sigma))$. Let $\mathcal{T}_G = l \circ v^G\colon \mathcal{T} \to \mathbb{B}$. Clearly, $\mathcal{T}_G$ is a regular tree and every regular tree is of the form $\mathcal{T}_G$ for some $\mathbb{B}$-looped tree G.

Lemma 3.14. *Let* $\mathbf{M}$ *be a Rabin automaton and* G *a* $\mathbb{B}$*-looped tree. Then*

$$G \in L^{\mathcal{L}o}(\mathbf{M}) \iff \mathcal{T}_G \in L(\mathbf{M}).$$

All the results about loop-automata are also true about looped tree -automata replacing Büchi by Rabin.

4. Löwenheim and Skolem go automatic

Loop-automatic and *Looped tree automatic representations of structures* are defined exactly as in Definition 2.5 changing condition (1) appropriately.

The well-known Löwenheim-Skolem theorem in model theory states that for any infinite structure $\mathcal{A}$ contains a countable elementary substructure. The goal of this section is to investigate this theorem for Büchi and Rabin automatic structures. Recall that a substructure $\mathcal{B}$ of $\mathcal{A}$ is an elementary substructure, written $\mathcal{A} \preceq \mathcal{B}$, if for every tuple $\bar{a} \in A$ and a formula $\varphi(\bar{x})$ of the first order language of $\mathcal{A}$, we have that $\varphi(\bar{a})$ is true in $\mathcal{A}$ if and only if it is true in $\mathcal{B}$.

The theorem below follows from Corollary 3.5 and Lemma 3.11. For the theorem recall Definition 2.7:

Theorem 4.1. *Every loop-automatic (looped tree-automatic) structure is decidable.*

The main result of this section is the following one. For the proof of the theorem, recall that for a given $S \subseteq \mathbb{B}^\omega$, the notation $\mathrm{lo}(S)$ denotes the set $\{G \in \mathcal{L}o(\mathbb{B}) : \alpha_G \in S\}$.

Theorem 4.2. *1. Every Büchi presentable structure has a loop-automatic elementary substructure.*

2. Every Rabin presentable structure has a looped tree-automatic elementary substructure.

Proof. We sketch the proof of the first part. The proof for Rabin automatic structures is similar.

Let $\mathcal{S} = (D; E, R_1, ..., R_n)$ be a Büchi presentation of structure $\mathcal{A}$. Set $D^{\mathcal{L}o} = \mathrm{lo}(D)$, $E^{\mathcal{L}o} = \mathrm{lo}(E)$,$R_1^{\mathcal{L}o} = \mathrm{lo}(R_1)$, $\ldots$, $R_n^{\mathcal{L}o} = \mathrm{lo}(R_n)$ and $\mathcal{S}^{\mathcal{L}o} = (D^{\mathcal{L}o}; E^{\mathcal{L}o}, R_1^{\mathcal{L}o}, \ldots, R_n^{\mathcal{L}o})$. Let $f\colon D^{\mathcal{L}o} \to D$ be given by $f(G) = \alpha_G$. The mapping f is an embedding of structures and the image of f is the restriction of D to the set of infinite periodic words. We claim that f is an elementary embedding.

By Theorem 2.8, for each formula $\varphi(x_1, ..., x_k)$ in the language of $\mathcal{S}$, there is a Büchi automata $\mathbf{M}_\varphi$ such

$$L(\mathbf{M}_\varphi) = \{(\alpha_1, ..., \alpha_k) \in D^k : \mathcal{S} \models \varphi(\alpha_1, ..., \alpha_k)\}.$$

Using the ideas in the proofs of Corollary 3.5 and Lemma 3.11 one can show by induction on the size of φ that

$$L^{\mathcal{L}o}(\mathbf{M}_\varphi) = \{G_1, ..., G_k \in D^{\mathcal{L}o} : \mathcal{S}^{\mathcal{L}o} \models \varphi(G_1, ..., G_k)\}.$$

Moreover, using the construction of Lemma 3.10, we get that given $H_1, ..., H_h \in D^{\mathcal{L}o}$ with $h < k$, we get that

$$(\alpha_{h+1}, ..., \alpha_k) \in L(\mathbf{M}_{\varphi, H_1, ..., H_h}) \iff \mathcal{S} \models \varphi(f(H_1), ..., f(H_h), \alpha_{h+1}, ..., \alpha_k),$$

and that

$$(G_{h+1}, ..., G_k) \in L^{\mathcal{L}o}(\mathbf{M}_{\varphi, H_1, ..., H_h}) \iff \mathcal{S}^{\mathcal{L}o} \models \varphi(H_1, ..., H_h, G_{h+1}, ..., G_k),$$

By Corollary 3.7, using that $L(\mathbf{M}_{\varphi, H_1, ..., H_h})$ is empty if and only if $L^{\mathcal{L}o}(\mathbf{M}_{\varphi, H_1, ..., H_h})$, one can show that f is an elementary embedding. $\square$

The main algorithmic property of Büchi and Rabin automatic structures is expressed in the following corollary. The corollary is slightly more general than Theorem 2.8 ([4]) because of special parameters involved:

Corollary 4.3. *There exists an algorithm that, given a Büchi (Rabin) automatic structure $\mathcal{A}$, a formula of first order logic $\varphi(\bar{x})$ and a tuple $\vec{p}$ of eventually periodic words (regular trees) given as $\mathbb{B}$-loops ($\mathbb{B}$-looped trees), decides whether $\mathcal{A} \models \varphi(\vec{p})$.*

The elementary substructures for Büchi automata built in the theorem above are not necessarily finite word automatic. This is explained by the following proposition.

Proposition 4.4. *There exists a Büchi automatic structure that does not have elementary word automatic substructures.*

Proof. The Boolean algebra $\mathcal{B} = (\mathcal{P}(\mathbb{N})/ =^*; \cup, \cap, \neg)$, from Example 2.6, is Büchi automatic. In [15] it is proved that an infinite countable Boolean algebra is automatic if and only if it is isomorphic to a finite Cartesian product of $\mathcal{B}_\omega$, where $\mathcal{B}_\omega$ is the Boolean algebra of all finite and co-finite subsets of $\mathbb{N}$. None of these automatic Boolean algebras is elementary equivalent to $(P(\mathbb{N})/ =^*; \cup, \cap, \neg)$ because they have atoms and $(P(\mathbb{N})/ =^*; \cup, \cap, \neg)$ is atomless. Hence, no countable elementary substructure of $(P(\mathbb{N})/ =^*; \cup, \cap, \neg)$ is word automatic. $\square$

4.1. Two examples of loop automatic structures

The first example is not known to be word automatic. The second one is a structure that has no word automatic presentation. So, we get that the class of loop-automatic structures is strictly larger than the well-studied class of word automatic structures.

Example 4.5. The structure $(\mathbb{Q}, +, \leq)$ is loop-automatic. This structures is the one we get if we apply Theorem 4.2 to $(\mathbb{R}, +, \leq)$ (see Example 2.6), where we code the real numbers by their dyadic presentation. The reals with periodic dyadic presentations are exactly the rational ones. It is still an open question whether the group $(\mathbb{Q}, +)$ is automatic. It is also not know if $(\mathbb{Q}, +)$ is tree automatic.

Example 4.6. The countable atomless Boolean algebra is loop-automatic. Let the domain of this structure be $D = \mathcal{L}o(\{0, 1\})$. Let E be the equivalence relation such that $E(G_1, G_2)$ if and only if α_{G_1} and α_{G_2} are equal everywhere except for finitely many places. The Boolean operations are defined in the obvious way using the standard Boolean operations in the set $\{0, 1\}$. This is the structures we get if we apply the Löwenheim-Skolem theorem for Büchi structures to the Büchi structure in Proposition 4.4. It is not hard to see that it is isomorphic to the atomless Boolean algebra. It was shown in [15] that the atomless Boolean algebra is not automatic. One can show that the atomless Boolean algebra is tree automatic.

If we let E be the identity on periodic words (i.e. $E(G_1, G_2)$ if and only if $\alpha_{G_1} = \alpha_{G_2}$, then we get the atomic Boolean algebra that when quotiented by the ideal generated by the atoms, one gets the atomless Boolean algebra.

5. Borel structures

We now look at the class of Borel subsets of $\mathbb{B}^\omega$. The space $\mathbb{B}^\omega$ is usually called the Cantor space. This space has the following natural metric d associated. If $\alpha \neq \beta$ then $distance(\alpha, \beta) = 2^{-n}$, where n is the first position at which α and β are distinguishable (that is $\alpha(n) \neq \beta(n)$); if $\alpha = \beta$, then $distance(\alpha, \beta) = 0$. This defines a *topology* on $\mathbb{B}^\omega$ generated by the family of *basic open sets* $\{\alpha \in \mathbb{B}^\omega : \tau \text{ is an initial segment of } \alpha\}$, where $\tau \in \mathbb{B}^\star$. The class of *Borel* sets is the smallest σ-algebra containing the basic open sets. In other words, the class of *Borel sets of* $\mathbb{B}^\omega$ is the smallest class of subsets of $\mathbb{B}^\omega$ which contains the basic open sets and is closed under countable unions and complementation. A standard reference on Borel sets is Kechris [11] that we will often use in this section.

A set $A \subseteq \mathbb{B}_1^\omega$ is said to be $\mathbf{\Sigma_1^1}$ if there is a Borel set $B \subseteq (\mathbb{B}_1^\omega) \times (\mathbb{B}_2^\omega)$ such that $A = \{\alpha : \exists \beta : (\alpha, \beta) \in B\}$. Note that here we identify $(\mathbb{B}_1^\omega) \times (\mathbb{B}_2^\omega)$ with $(\mathbb{B}_1 \times \mathbb{B}_2)^\omega)$ via the convolution map. Suslin proved the following result that we state as a lemma; for the proof see [11].

Lemma 5.1. *A set $A \subseteq \mathbb{B}^\omega$ is Borel if and only if both A and its complement are $\mathbf{\Sigma_1^1}$.*

We are interested in Borel structures and their basic properties because we will use them in the analysis of the injectivity problems explained in the introduction. For completeness' sake we define Borel structures:

Definition 5.2. We say the tuple $\mathcal{S} = (D; E, R_1, \ldots, R_n)$ is a *Borel representation* of a structure $\mathcal{A}$ if

1. All D, E, R_1,..., R_n are Borel sets.
2. All E, R_1, ..., R_n are relations on D.
3. E is an equivalence relation on the domain D such that E is compatible with R_1, ..., R_n.
4. The quotient structure $\mathcal{S}/E$ is isomorphic to $\mathcal{A}$.

In this case we say that $\mathcal{A}$ is a *Borel structure*. In case when E is the equality relation on D, then $\mathcal{S}$ is an *injective Borel presentation of* $\mathcal{A}$, and $\mathcal{A}$ is *injective Borel structure*.

Example 5.3. Here are some examples of Borel structures:

1. All Büchi automatic structures are Borel structures. In fact, Büchi automatic structures are languages that belong to a Boolean combination of Σ_2^0-languages in Borel hierarchy.
2. The fields $(\mathbb{R}, +, \times)$ and $(\mathbb{C}, +, \times)$ are Borel structures.
3. The Boolean algebra $(\mathcal{P}(\mathbb{N}), \subseteq)$ is a Borel structure.
4. The structure $(\mathbb{N}, \mathcal{P}(\mathbb{N}), 0, 1, +, \times)$ is also a Borel structure.

The first example suggests that techniques of descriptive set theory could be used in the study of Büchi automatic structures. In the next section we will show that this is indeed the case in answering the injectivity problems. For the second example, we comment that it is not known if the fields $(\mathbb{R}, +, \times)$ and $(\mathbb{C}, +, \times)$ are Büchi or Rabin automatic structures. Borel structures do not need to have decidable theories as Büchi automatic ones do. The third example will be essentially used in the next section. The last example is the second order arithmetic. It is an example of a Borel structure whose first order theory is not decidable.

The following is an example of a structure which is not Borel but that has decidable first order theory:

Proposition 5.4. *The well ordered set $(\omega_1, \leq)$, where ω_1 is the first uncountable ordinal, is not Borel. Hence, this structure is not Büchi automatic either.*

Proof. Suppose $(\omega_1, \leq)$ is a Borel structure. Let $\mathcal{B}$ be a Borel presentation of $(\omega_1, \leq)$. Therefore, the class of linear orderings of $\mathbb{N}$ which embed in $\mathcal{B}$ is $\mathbf{\Sigma_1^1}$. The boundedness theorem for WF [11, Thm 31.2] implies that every $\mathbf{\Sigma_1^1}$ set of well-orderings is bounded by some ordinal $\gamma < \omega_1$. Hence, $(\omega_1, \leq)$ cant have a Borel presentation. □

A stronger result of Harrington and Shelah [7] states that no Borel presentable linear order has a subset of order type ω_1.

We need some basic notions about Borel sets and equivalence relations. A function $F : X \to Y$ where X, Y are standard Borel spaces is a *Borel function* if $F^{-1}(S)$ is Borel for each open set $S \subseteq Y$. The next lemma is from [11, Thm 14.12]). We will also use this lemma in the next section:

Lemma 5.5. *The mapping $F : X \to Y$ is Borel if and only if the graph $\{(x, F(x)) : x \in X\}$ is Borel as a subset of $X \times Y$.*

Finally, we will use the following well-known theorem in Descriptive Set Theory that will be key for our proofs. See Example 1.6 in [8].

Theorem 5.6. *There is no Borel function $F : \mathcal{P}(\mathbb{N}) \to \mathbb{B}^\omega$ such that $X =^* Y \Leftrightarrow F(X) = F(Y)$ for each $X, Y \subseteq \mathbb{N}$.*

6. Separation of classes of structures

Our objective is to separate the classes of structures in Figure 1, where a line between two classes of structures corresponds to inclusion. Our separation results will show that all the inclusions are proper. Some of the proper inclusions are immediate. The class of injective Borel structures is not included in the class of Rabin structures; we saw that in Example 5.3 that second order arithmetic is a Borel structure but not Rabin automatic. For the remaining separations we will show that there is an Injective Rabin automatic structure without a Borel presentation and that there is a Büchi structure which has neither an injective Borel presentation nor an injective Rabin presentation, and of course no injective Büchi presentation.

Our separation results rely on a lemma which states that the Boolean algebra $(\mathcal{P}(\mathbb{N}), \subseteq)$ is Borel stable in the sense that *all* isomorphisms between any two Borel presentations of the algebra are Borel.

Lemma 6.1. *Let $\mathcal{S} = (A, E, \leq)$ and $\mathcal{S}' = (B, F, \leq')$ be Borel presentations of $\mathcal{B} = (\mathcal{P}(\mathbb{N}), \subseteq)$ and let $\Phi : \mathcal{S}/E \mapsto \mathcal{S}'/F$ be an isomorphism. Then the graph of Φ,*

$$\{\langle x, y\rangle \in A \times B : \Phi([x]_E) = [y]_F\}$$

is Borel.

Proof. Recall that an element x is an atom of a Boolean algebra if $x \neq 0$ and no element y exists such that $0 < y < x$. Thus, atoms of $(\mathcal{P}(\mathbb{N}), \subseteq)$ are the sets of the form $\{n\}$, where n is a natural number. Let $\{[a_n]_E : n \in \mathbb{N}\}$ be a listing of the atoms of $\mathcal{S}/E$. Let $b_n \in B$ be such that $\Phi([a_n]_E) = [b_n]_F$. Then

$$\Phi([x]_E) = [y]_F \text{ iff } \forall n\,(a_n \leq x \leftrightarrow b_n \leq' y).$$

Thus the graph of Φ is a countable intersection of Borel relations. Borel sets are closed under countable intersections. Hence the graph is Borel itself. □

The next lemma shows that being Borel is an intrinsic property of relations in the Boolean algebra $(\mathcal{P}(\mathbb{N}), \subseteq)$:

Lemma 6.2. *Suppose C is a countable set and $U \subseteq \mathcal{P}(C)^m$ is not Borel. Then the structure $(\mathcal{P}(C), \subseteq, U)$ has no Borel presentation.*

Note that identifying C with $\mathbb{N}$, we can identify the set $\mathcal{P}(C)$ with $\{0,1\}^\omega$ in a natural way and hence talk about Borel relations of $\mathcal{P}(C)$.

Proof. Without loss of generality assume that U is a unary relation. Suppose Ψ is an isomorphism from $(\mathcal{P}(C), \subseteq, U)$ to a Borel presentation $(A, E, \leq, V)/E$. Then for each $X \subseteq \mathbb{N}$, we have $X \in U \Leftrightarrow [\Psi(X)]_E \in V/E \Leftrightarrow \exists b \in V\, \Psi(X) \in [b]_E$. Also, $X \in \mathcal{P}(C) \setminus U \Leftrightarrow [\Psi(X)]_E \in (A \setminus V)/E \Leftrightarrow \exists b \in A \setminus V\, \Psi(X) \in [b]_E$. So both U and $\mathcal{P}(C) \setminus U$ are $\mathbf{\Sigma^1_1}$. Hence by Lemma 5.1 the set U is Borel which is a contradiction. □

As a corollary we prove the following theorem:

Theorem 6.3. *There exists an injective Rabin presentable structure that is not Borel presentable.*

Proof. For the proof recall that a set is $\mathbf{\Pi^1_1}$ if and only if its complement is $\mathbf{\Sigma^1_1}$. Also, $\mathbf{\Pi^1_1}$-complete sets are *not* Borel [11].

Let $C = \{0,1\}^\star$. Consider the set

$$U = \{B \subseteq 2^{<\omega} : \forall \pi \in 2^\omega\, |\{n : \pi \upharpoonright n \in B\}| < \infty\}.$$

For the reader U can be thought of as the collection of all $\{0,1\}$-labeled trees $(\mathcal{T}, v)$ such that along every path η in the tree the number of 1s appear finitely often. It is not hard to see that the set U is Rabin recognizable. We now invoke a result of Niwinsky stating that the set U is $\mathbf{\Pi^1_1}$-complete, and hence not Borel [18] (also see [1]). Indeed, in order to show that U is $\mathbf{\Pi^1_1}$-complete, consider the embedding from $\omega^\star$ into $2^\star$ given by $(n_0, \ldots, n_k) \rightarrow 0^{n_0}1\ldots0^{n_k}1$. The pre-image of U under this embedding is the class of well-founded trees. It is well-known that the class of well-founded trees is $\mathbf{\Pi^1_1}$-complete [11].

The desired structure is $(\mathcal{P}(C), \subseteq, U)$. It is clear that the structure is Rabin automatic. By Lemma 6.2 the structure has no Borel presentation. □

We answer the first injectivity problem formulated in the introduction:

Theorem 6.4. *There exists a Büchi automatic structure $\mathcal{A}$ without an injective Borel presentation.*

Proof. We use the signature consisting of three symbols $\leq$, U and R, where U is a unary predicate symbol, and $\leq$ and R both are binary relation symbols. Let $\mathcal{B} = (\mathcal{P}(\mathbb{N}), \subseteq)$ and $\mathcal{B}^* = (\mathcal{P}(\mathbb{N})/=^*, \leq)$. The structure $\mathcal{A}$ is the disjoint union of the partial orders $\mathcal{B}, \mathcal{B}^*$, where U holds for the elements of $\mathcal{B}$, and R is the canonical projection $\mathcal{B} \rightarrow \mathcal{B}^*$.

First we give a Büchi presentation $\mathcal{S} = (D, E, \leq, U, R)$ of $\mathcal{A}$. Let $(D, \leq) = \mathcal{B}_0 \sqcup \mathcal{B}_1$ where $\mathcal{B}_0, \mathcal{B}_1$ are disjoint copies of $\mathcal{B}$. Let E be the equivalence relation on D which is the identity on $\mathcal{B}_0$, and $=^*$ on $\mathcal{B}_1$. Let U be the domain of $\mathcal{B}_0$, and let R be the bijection $\mathcal{B}_0 \mapsto \mathcal{B}_1$ given by the identity. Clearly, all the relations can be recognized by Büchi automata.

Now assume that $\mathcal{S}' = (D', \leq', U', R')$ is an injective Borel presentation of $\mathcal{A}$. Let Φ be an isomorphism $\mathcal{S} \mapsto \mathcal{S}'$ and let G be the restriction of Φ to $\mathcal{B}_0$, which is Borel by Lemma 6.1. Then the map $F = R' \circ G \colon \mathcal{P}(\mathbb{N}) \rightarrow D'$ is Borel and satisfies that

$$\begin{aligned} X =^* Y \quad &\Leftrightarrow \quad R^{\mathcal{A}}(X) = R^{\mathcal{A}}(Y) \\ &\Leftrightarrow \quad \Phi(R^{\mathcal{A}}(X)) = \Phi(R^{\mathcal{A}}(Y)) \\ &\Leftrightarrow \quad R'(G(X)) = R'(G(Y)), \end{aligned}$$

contrary to Theorem 5.6. □

Corollary 6.5. *There exists a Büchi automatic structure which does not have an injective Büchi automatic presentation.* □

Now we answer the second injectivity problem:

Theorem 6.6. *There exists a Büchi automatic structure $\mathcal{A}$ without an injective Rabin presentation.*

Proof. Our goal is to show that the structure obtained in the proof of Theorem 6.4 has no injective Rabin presentation.

This proof requires the use of a technical notion in set theory, namely the one of *absolutely* Δ^1_2 *sets*. We first outline the idea. If there were an injective Rabin presentation of the structure obtained in the proof of Theorem 6.4, then, as in that proof, we would be able to obtain an absolutely Δ^1_2 function $F\colon \mathcal{P}(\mathbb{N}) \to \mathbb{B}^\omega$ for some $\mathbb{B}$ such that $X =^* Y \Leftrightarrow F(X) = F(Y)$ for each $X, Y \subseteq \mathbb{N}$. An extension of Theorem 5.6 says that no such a function F exists.

We now give the details. A relation R on $\mathbb{B}^\omega$ is called absolutely Δ^1_2 if there are descriptions of R and its complement as (lightface) Σ^1_2 relations that yield the complementary relations in each generic extension of the set-theoretical universe. If R is given by a Rabin automaton then by the proof of Rabin's complementation theorem [20] there is a computable function g taking us from the automaton for R to an automaton for its complement. This form of Rabin's Theorem involving g can of course be proved in ZFC. Therefore each Rabin recognizable relation is absolutely Δ^1_2.

Each absolutely Δ^1_2 set has the property of Baire, and hence each absolutely Δ^1_2 function is Baire measurable [9]. Since each Baire measurable function is continuous on a comeager set, we can strengthen Theorem 5.6 to the effect that F cannot be absolutely Δ^1_2.

If the structure had a Rabin presentation then as in the proof of Theorem 6.4 we would obtain such an F that is absolutely Δ^1_2. This is a contradiction. (Here we used a modified version of Lemma 6.1: if the given presentations of $\mathcal{B}$ are lightface Δ^1_2 then so is the graph of the isomorphism.) □

7. Complexity of isomorphism and Borel Categoricity

A natural question when studying a certain class of structures is how to recognize when two structures from the class are isomorphic, and how complex the isomorphism can be. We think of presentations as being encoded in some effective way by natural numbers. For the case of automatic structures, Khoussainov, Nies, Rubin and Stephan [14] have shown that the problem of deciding whether two presentations describe isomorphic structures is Σ^1_1-complete. This tells us that the problem is as hard as an isomorphism problem can be in a class of computable structures. The complexity of the isomorphism problem for word automatic structures has also been investigated in Khoussainov and Minnes [12]. From these results it follows that the isomorphism problem for loop-automatic structures is also Σ^1_1-complete.

We give an example of a structure that has two Büchi presentations which are not Borel isomorphic.

We start with the following definition that singles out the structures with exactly one Borel isomorphism type:

Definition 7.1. We say that Borel presentations $\mathcal{S}_1, \mathcal{S}_2$ are **Borel isomorphic** if there is a Borel mapping $f : D_1 \to D_2$ that induces an isomorphism between the presentations. A structure $\mathcal{A}$ is **Borel categorical** if any two Borel presentations of $\mathcal{A}$ are Borel isomorphic.

Example 7.2. Examples of Borel categorical structures are:

1. The Boolean algebra $(\mathcal{P}(\mathbb{N}), \subseteq)$.
2. The linearly ordered set $(\mathbb{R}, \leq)$.
3. The field $(\mathbb{R}, +, \times)$.

The first example follows from Lemma 6.1. The second and third examples also follow from the fact that the structures have isomorphic countable dense substructures. The isomorphisms between these substructures can naturally be extended to the main structures.

Note that the first two examples above are Büchi automatic structures. They may suggest that automaticity of the structure would imply Borel categoricity. This is refuted in the following theorem:

Theorem 7.3. *There are two Büchi automatic presentations of* $(\mathbb{R}, +)$ *that are not Borel isomorphic.*

Proof. The structures $(\mathbb{R}, +)$ and $(\mathbb{R}, +) \times (\mathbb{R}, +)$ are both Büchi presentable as we have seen before. These structures are isomorphic because they are both $\mathbb{Q}$-vector spaces of dimension $2^{\aleph_0}$. However, they are not Borel isomorphic since any Borel isomorphism between Polish groups must be a homeomorphism (see for instance Section 1.2 of [2]. □

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